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A NOTE ON RELATION BETWEEN COMPLETE GRAPHS TO DIRECTED HAMILTONIAN PATH

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ABSTRACT

A Hamiltonian Path is a spanning path in a graph, i.e., the path passing through every vertex of the graph. In this paper we study and giving a condition for a Relation Between Complete Graph to Directed Hamiltonian Path and given some related examples.

Key Words: *Graph, complete graph, path, Hamiltonian path, Tree, Spanning tree.*

1. INTRODUCTION

The origin of graph theory started with the problem of Koinsber bridge, in 1735. This problem lead to the concept of Eulerian Graph. Euler studied the problem of Koinsberg bridge and constructed a structure to solve the problem called Eulerian graph. In 1840, A.F Mobius gave the idea of complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems. The concept of tree, (a connected graph without cycles was implemented by Gustav Kirchhoff in 1845, and he employed graph theoretical ideas in the calculation of currents in electrical networks or circuits.

In 1852, Thomas Gutherie found the famous four color problem. Then in 1856, Thomas. P. Kirkman and William R.Hamilton studied cycles on polyhydra and invented the concept called Hamiltonian graph by studying trips that visited certain sites exactly once. In 1913, H.Dudeney mentioned a puzzle problem. Eventhough the four color problem was invented it was solved only after a century by Kenneth Appel and Wolfgang Haken. This time is considered as the birth of Graph Theory. Caley studied particular analytical forms from differential calculus to study the trees. This had many implications in theoretical chemistry. This lead to the invention of enumerative graph theory. Any how the term "Graph" was introduced by Sylvester in 1878 where he drew an analogy between "Quantic invariants" and covariants of algebra and molecular diagrams. In 1941, Ramsey worked on colorations which lead to the identification of another branch of graph theory called extremel graph theory. In 1969, the four color problem was solved using computers by Heinrich. The study of asymptotic graph connectivity gave rise to random graph theory.

In this paper we study the Hamiltonian graphs, Hamiltonian paths and given a relations betwen Complete Graphs to Directed Hamiltonian Path.

1.1 Definition: A graph – usually denoted G (V, E) or G = (V, E) – consists of set of vertices V together with a set of edges E. The number of vertices in a graph is usually denoted *n* while the number of edges is usually denoted *m*.

1.2 Definition: Vertices are also known as nodes, points and (in social networks) as actors, agents or players.

1.3 Definition: Edges are also known as lines and (in social networks) as ties or links. An edge e = (u,v) is defined by the unordered pair of vertices that serve as its end points.

1.4 Example: The graph depicted in Figure 1 has vertex set V= {a, b, c, d, e. f} and edge set E = {(a, b), (b, c), (c, d), (c, e),(d, e),(e, f)}.

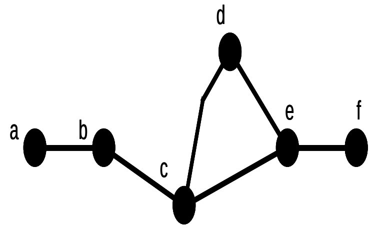

Figure 1.

1. 5 Definition: Two vertices *u* and *v* are *adjacent* if there exists an edge (*u*, *v*) that connects them.

1.6 Definition: An edge (u, v) is said to be *incident* upon nodes u and v.

1.7 Definition: An edge e = (u, u) that links a vertex to itself is known as a *self-loop* or *reflexive* tie.

1.8 Definition: Every graph has associated with it an *adjacency matrix*, which is a binary $n \times n$ matrix A in which $a_{ij} = 1$ and $a_{ji} = 1$ if vertex vi is adjacent to vertex vj, and aij = 0 and aji = 0 otherwise. The natural graphical representation of an adjacency matrix is a table, such as shown below.

	a	b	c	d	e	f
a	0	1	0	0	0	0
b	1	0	1	0	0	0
c	0	1	0	1	1	0
d	0	0	1	0	1	0
e	0	0	1	1	0	1
f	0	0	0	0	1	0

Adjacency matrix for graph in Figure 1.

1.9 Definition: Examining either Figure 1 or given adjacency Matrix, we can see that not every vertex is adjacent to every other. A graph in which all vertices are adjacent to all others is said to be *complete*.

1.10 Definition: A *subgraph* of a graph *G* is a graph whose points and lines are contained in *G*. A complete subgraph of *G* is a section of *G* that is complete

1.11 Definition: While not every vertex in the graph in Figure 1 is adjacent, one can construct a sequence of adjacent vertices from any vertex to any other. Graphs with this property are called *connected*.

1.12 Note: Reachability. Similarly, any pair of vertices in which one vertex can reach the other via a sequence of adjacent vertices is called *reachable*. If we determine reachability for every pair of vertices, we can construct a reachability matrix R such as depicted in Figure 2. The matrix R can be thought of as the result of applying transitive closure to the adjacency matrix A.

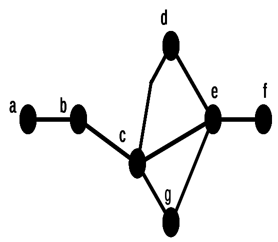

Figure: 2

1.13 Definition: A *component* of a graph is defined as a maximal subgraph in which a path exists from every node to every other (i.e., they are mutually reachable). The size of a component is defined as the number of nodes it contains. A connected graph has only one component.

1.14 Definition: A sequence of adjacent vertices $v_0, v_1, \ldots, v_n$ is known as a *walk*. In Figure 3, the sequence *a, b, c, b, c, g* is a walk. A walk can also be seen as a sequence of *incident* edges, where two edges are said to be incident if they share exactly one vertex.

1.15 Definition: A walk is closed if $v_o = v_n$.

1.16 Definition: A walk in which no vertex occurs more than once is known as a *path*. In Figure 3, the sequence *a, b, c, d, e, f* is a path.

1.17 Definition: A walk in which no edge occurs more than once is known as a *trail*. In Figure 3, the sequence *a, b, c, e, d, c, g* is a trail but not a path. Every path is a trail, and every trail is a walk.

1.18 Definition: A *cycle* can be defined as a closed path in which *n* >= 3. The sequence *c, e, d* in Figure 3 is a cycle.

1.19 Definition: A *tree* is a connected graph that contains no cycles. In a tree, every pair of points is connected by a unique path. That is, there is only one way to get from A to B.

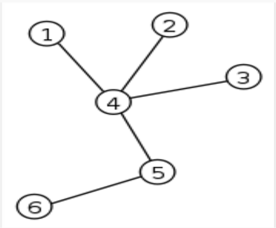

Figure 3: A labeled tree
With 6 vertices and 5 edges

1.20 Definition: A *spanning tree* for a graph *G* is a sub-graph of *G* which is a tree that includes every vertex of *G*.

1.21 Definition: The length of a walk (and therefore a path or trail) is defined as the number of edges it contains. For example, in Figure 3, the path *a,b,c,d,e* has length 4.

1.22 Definition: The number of vertices adjacent to a given vertex is called the *degree* of the vertex and is denoted d(v).

1.23 Definition: In the mathematical field of graph theory, a bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

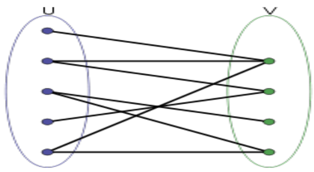

Figure 4: Example of a bipartite graph.

1.24 Definition: An Eulerian circuit in a graph G is circuit which includes every vertex and every edge of G. It may pass through a vertex more than once, but because it is a circuit it traverse each edge exactly once. A graph which has an Eulerian circuit is called an Eulerian graph. An Eulerian path in a graph G is a walk which passes through every vertex of G and which traverses each edge of G exactly once

1.25 Example: Königsberg bridge problem: The city of Königsberg (now Kaliningrad) had seven bridges on the Pregel River. People were wondering whether it would be possible to take a walk through the city passing exactly once on each bridge. Euler built the representative graph, observed that it had vertices of odd degree, and proved that this made such a walk impossible. Does there exist a walk crossing each of the seven bridges of Königsberg exactly once?

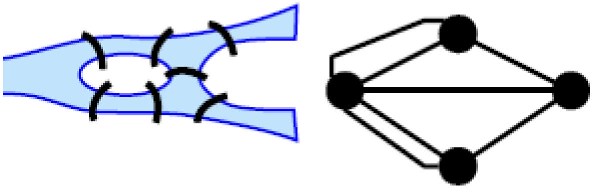

Figure 5: K¨onigsberg problem

2. HAMILTONIAN PATH AND HAMILTONIAN CIRCUIT

In this section we proved main result related to the condition of complete graph to directed Hamiltonian Path.

2.1 Definition: Another closely related problem is finding a Hamilton path in the graph (named after an Irish mathematician, Sir William Rowan Hamilton). Whereas an Euler path is a path that visits every edge exactly once, a Hamilton path is a path that visits every vertex in the graph exactly once.

A Hamilton circuit is a path that visits every vertex in the graph exactly once and return to the starting vertex. Determining whether such paths or circuits exist is an NP-complete problem. In the diagram below, an example Hamilton Circuit would be

2.2 Example:

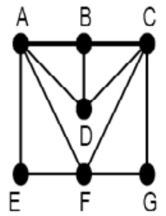

Figure 6: Hamilton Circuit would be AEFGCDBA.

2.3 Theorem: Every complete graph has a directed Hamiltonian path

Proof: This theorem is proved by mathematical induction on the number of vertices.

By the actual sketch the theorem can be shown for all complete graphs of 3, and 4 vertices.

Let us take the inductive assumption that the theorem is true for all complete graphs of 'm' vertices and prove that it also holds for all graphs of 'm+1' vertices.

Let 'G' be any complete graph of 'm+1 'vertices.

Let 'g' be an m-vertex complete sub graph of 'G'.

By inductive assumption, 'g' has a directed Hamiltonian path.

Let that path be $v_1, v_2, \ldots v_m$.

Let the vertex present in "G" but not in 'g' be called v_{m+1}.

Since 'G' is complete graph of m+1 vertices, the vertex v_{m+1} in 'G' has a directed edge either to or from each of the other vertices $v_1, v_2, v_3, v_4 \ldots\ldots v_m$.

The following three cases are possible.

Case 1: The edge between v_{m+1} and v_1 is directed toward v_1. Then we have a Hamiltonian path $v_{m+1}, v_1, v_2, \ldots v_m$ in 'G" and this is complete the proof.

Case 2: There is an edge directed from v_m to v_{m+1}.

Then also we have a Hamiltonian path in 'G' which is $v_1, v_2, v_3, \ldots v_{m+1}$.

Hence the proof is complete.

Case 3: Instead, both these edges are directed from v_1 to v_{m+1} and from v_{m+1} to v_m.

In this case, as we move from v_1 to v_m, we encounter a reversal of direction in the edges incident on v_{m+1}.

This reversal must occur because edge(v_1,v_{m+1}) is directed towards v_{m+1}, but edge (v_m,v_{m+1}) is directed away from v_{m+1}. Call the vertex at which the first reversal occurs v_i(v_i may be v_m itself).

Then the edge (v_{i-1},v_{m+1}) must be directed toward v_{m+1}.

In this case we have a directed Hamiltonian path $v_1,v_2,v_3 \ldots v_{i-1},v_{m+1},v_i,v_{i+1} \ldots v_m$ in G.

Hence the theorem.

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