International Journal of Mathematical Archive-3(8), 2012, 3118-3121

Available online through www.ijma.info ISSN 2229 – 5046

ON ASSOCIATIVE FILTERS OF LATTICE IMPLICATION ALGEBRAS

M. SAMBASIVA RAO*

Department of Mathematics, MVGR College of Engineering, Chintalavalasa
Vizianagaram, Andhra Pradesh, India-535005

(Received on: 30-07-12; Accepted on: 24-08-12)

ABSTRACT

The properties of associative filters are studied. The notion of relative annihilators, induced by an associative filter, are introduced and their prop-erties are studied. The prime filter theorem is finally extended to the case of associative filters and it is proved with the help od relative annihilators of lattice implication algebras.

***2000 Mathematics Subject Classification**: 03G10, 06B10.*

***Keywords**: Lattice implication algebra, filter, lattice filter, associative filter, relative annihilator.*

INTRODUCTION

In order to research the logical system whose propositional value is given in a lattice from the semantic viewpoint, Xu [4] proposed the concept of lattice implication algebras, and discussed some of their properties in [3, 4]. Xu and Qin [5] introduced the notion of filters in a lattice implication algebra, and investigated their properties. Later, Y.B. Jun[1, 2], introduced various types of filters like implicative filter, positive implicative filter and fantastic filters e.t.c. in lattice implication algebras and studied their properties. In [2], Y.B. Jun, Y. Xu and K.Y. Qin introduced the notion of associative filters and studied their properties.

In this paper, the concept of relative annihilators induced by an associative filter is introduced. Some useful properties of these relative annihilators are then studied. Finally, the famous prime filter theorem of lattice implication algebra is extended to the case of prime associative filters.

1. PRELIMINARIES

In this section, we cite some elementary aspects which have taken mostly from [2] and [4] those will be used in the sequel of this paper as well as for ready reference.

Definition 1.1. [4] By a lattice implication algebra we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution $'$ and a binary operation $\rightarrow$ satisfying the following axioms:

(1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$
(2) $x \rightarrow x = 1$
(3) $x \rightarrow y = y' \rightarrow x'$
(4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$
(5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$
(6) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$
(7) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$ for all $x, y, z \in L$.

Note that the conditions (6) and (7) are equivalent to the conditions

(6$'$) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$ and
(7$'$) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$, respectively.

***Corresponding author:** M. SAMBASIVA RAO**
Department of Mathematics, MVGR College of Engineering, Chintalavalasa
Vizianagaram, Andhra Pradesh, India-535005

Theorem 1.2. [4] In a lattice implication algebra L, the following hold for all $x, y, z \in L$:

(1) $0 \to x = 1$; $1 \to x = x$ and $x \to 1 = 1$
(2) $x \leq y$ implies $y \to z \leq x \to z$ and $z \to x \leq z \to y$
(3) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$
(4) $x \to ((x \to y) \to y) = 1$.

In what follows, *L* would mean a lattice implication algebra unless otherwise specified. In [5], Y. Xu and K. Y. Qin defined the notions of a filter and an implicative filter in a lattice implication algebra.

Definition 1.3. [5] Let $(L, \vee, \wedge, \to, 0, 1)$ be a lattice implication algebra. A subset F of L is called a filter of L if it satisfies for all $x, y \in L$.

(F1) $1 \in F$,
(F1) $x \in F$ and $x \to y \in F$ imply $y \in F$.]

Lemma 1.4. [1] Every filter F of L has the following property: $x \leq y$ and $x \in F$ imply $y \in F$ for all $x; y \in L$

Definition 1.5. [2] Let x be a fixed element of L. A subset F of L is called an associative filter of L with respect to x if it satisfies the following:

(A1) $1 \in F$
(A1) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $z \in F$.

for all *y, z* $\in L$. An associative filter of *L* with respect to all $x \neq 0$ is called an associative filter of *L*.

Theorem 1.6. [2] Every associative filter is a filter.

Theorem 1.7. [2] Let F be a filter of L. Then F is an associative filter if and only if it satisfies the following property:

$$x \to (y \to z) \in F \text{ implies } (x \to y) \to z \in F \quad \text{for all } x, y, z \in L$$

Definition 1.8. A proper filter P of a lattice implication algebra L is called prime if, for all $x, y \in L$, $x \vee y \in P$ implies either $x \in P$ or $y \in P$.

2. MAIN RESULTS

In this section, the concept of annulets is introduced in lattice implication algebras. Some interesting properties of annulets are studied. Finally the prime filter theorem is extended to the case of associative filters.

Definition 2.1. A non-empty subset F of a lattice implication algebra *L* is called an associative filter if it satisfies the following condition, for all $x, y, z \in L$.

(A1) $1 \in F$
(A2) $x \to (y \to z) \in F$ implies $(x \to y) \to z \in F$

The following Lemma is a routine verification.

Lemma 2.2. Let F *be an* associative filter of L. Then we have the following:

(A3) $x \in F$ and $x \leq y$ imply that $y \in F$

(A4) $x \in F$ and $x \to y$ imply that $y \in F$

Theorem 2.3. Every associative filter is a lattice filter.

Proof. Let *F* be an associative filter of a lattice implication algebra *L*. Let $x, y \in F$. Then $y \leq x \to y = 1 \wedge (x \to y) = (x \to x) \wedge (x \to y) = x \to (x \wedge y)$. Hence by the property (A3), we get $x \to (x \wedge y) \in F$. Again by the property

(A4), it yields that $x \wedge y \in F$. Therefore F is a lattice filter.

Corollary 2.4. Let F be an associative filter of a lattice implication algebra. Then F is closed under $\wedge$.

Theorem 2.5. [2] A lattice filter F of L is an associative filter if and only if it satisfies the following condition for all $x, y, z \in L$

$$x \to (y \to z) \in F \text{ and } x \to y \in F \text{ imply } z \in F$$

In the following, we now introduce the notion of relative annihilators of lattice implication algebras induced by the associative filters of *L*.

Definition 2.6. For any associative filter *F* of *L* and $a \in L$, the relative annihi-lator of *a* induced by *F* is defined as follows

$$(a; F) = \{x \in L \mid a \to x \in F\}$$

It is obvious that $(0; F) = L$ and $(1; F) = F$.

Proposition 2.7. If F is an associative filter of L and $a \in L$, then (a; F) is an associative filter of L containing both a and F.

Proof. Clearly $1 \in a; F$. Let $x, y \in (a; F)$. Then $a \to x \in F$ and $a \to y \in F$. Now $a \to (x \wedge y) = (a \to x) \wedge (a \to y) \in F$. Hence $x \wedge y \in (a; F)$. Again, let $x \to (y \to z) \in (a; F)$. Then we get

$$\begin{aligned} a \to [x \to (y \to z)] \in F &\Rightarrow x \to [a \to (y \to z)] \in F \\ &\Rightarrow x \to [y \to (a \to z)] \in F \\ &\Rightarrow (x \to y) \to (a \to z) \in F \\ &\Rightarrow a \to [(x \to y) \to z] \in F \end{aligned}$$

Thus $(x \to y) \to z \in (a; F)$, which yields that (a; F) is an associative filter of *L*.

We now observe the remaining fact. Clearly $a \in (a; F)$. Let $x \in F$. Since $x \le a \to x$, we can get $a \to x \in F$. Hence $x \in (a; F)$, which yields that $x \in (a; F)$. Therefore (a; F) is an associative filter containing both *a* and *F*.

Some properties of above relative annihilators can be observed in the following.

Lemma 2.8. Let F; G be two associative filters of L. Then for any $a, b \in L$, we have the following:

1) $(a; L) = L$
2) $a \in F$ if and only if $(a; F) = F$
3) $a \le b$ implies $(b; F) \subseteq (a; F)$
4) $F \subseteq G$ implies $(a; F) \subseteq (a; G)$
5) $(a; F) \cap (a; G) = (a; F \cap G)$
6) $(a; F) \cap (b; F) = (a \vee b; F)$
7) $(a; F) = (b; F)$ implies $(a \vee c; F) = (b \vee c; F)$ for any $c \in L$

Proof*:*
1). It is clear.

2). Assume that $a \in F$. By the above Proposition 2.5, we get $F \subseteq (a; F)$. Now for any $x \in (a; F)$, we get that $a \to x \in F$ and hence $x \in F$. Hence $(a; F) \subseteq F$. Therefore $(a; F) = F$. Conversely assume that $(a; F) = F$. Then $a \in (a; F) = F$.

3). Suppose $a \le b$. Let $x \in (b; F)$. Then $b \to x \in F$. Since $a \le b$, we get $b \to x \le a \to x$. Hence $a \to x \in F$.

Thus $x \in (a; F)$. Therefore $(b; F) \subseteq (a; F)$.

4). Suppose that $F \subseteq G$. Let $x \in (a; F)$. Then we get $a \to x \in F \subseteq G$. Hence $x \in (a; G)$. Therefore $(a; F) \subseteq (a;G)$.

5). It is trivial that $(a; F) \cap G \subseteq (a; F) \cap (a; G)$. Conversely let $x \in (a; F) \cap (a;G)$. Then we get $a \to x \in F \cap G$.

Therefore $x \in (a; F \cap G)$.

6). Clearly $(a \vee b; F) \subseteq (a; F) \cap (b;F)$. Conversely, let $x \in (a; F) \cap (b; F)$. Then $a \to x;\ b \to x \in F$. Since F is an associative filter, we get $(a \vee b) \to x = (a \to x) \wedge (b \to x) \in F$. Hence $x \in (a \vee b; F)$. Therefore $(a; F) \cap (b; F) = (a \vee b; F)$.

7). From 6), we get $(a \vee c; F) = (a; F) \cap (c; F) = (b; F) \cap (c; F) = (b \vee c; F)$.

In [2], Liu and Xu proved the prime filter theorem in lattice implication algebras. In the following, it is generalized to the case of associative filters and proved with the help of relative annihilators.

Theorem 2.9. Let F be an associative filter and S a $\vee$-closed subset of a lattice implication algebra L such that $F \cap S = \emptyset$. Then there exists a prime associative filter P of L such that $F \subseteq P$ and $S \cap P = \emptyset$.

Proof. Suppose F is an associative filter and *S* a $\vee$-closed subset of L such that

$F \cap S = \emptyset$. Consider $\pi = \{ G \mid G$ is an associative filter, $F \subseteq G$ and $G \cap S = \emptyset \}$. Then clearly $F \in \pi$. By the Zorn's lemma π has a maximal element, say P. We now show that P is prime. Let $a, b \in L$ be such that a not in P and b not in P. Then we get $P \subset (a; P)$ and $P \subset (b; P)$. By the maximality of P, we can get that $(a, P) \cap S \neq \emptyset$ and $(b, P) \cap S \neq \emptyset$. Choose $x \in (a; P) \cap S$ and $y \in (b; P) \cap S$. Since $x \in (a; P)$, we get $a \to x \in P$. Since $x \leq x \vee y$, we get $a \to x \leq a \to (x \vee y)$ and hence $a \to (x \vee y) \in P$. By using the similar argument, it can be yielded that $b \to (x \vee y) \in P$. Since *P* is an associative filter, we get $(a \vee b) \to (x \vee y) = [a \to (x \vee y)] \wedge [b \to (x \vee y)] \in P$. Hence $x \vee y \in a \vee b; P$. If $a \vee b \in P$, then $x \vee y \in P$ and hence $x \vee y \in P \cap S$, which is a contradiction. Therefore P is a prime associative filter.

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Source of support: Nil, Conflict of interest: None Declared

