A COINCIDENCE POINT THEOREM FOR FOUR SELF MAPPINGS IN DISLOCATED QUASI METRIC SPACES

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ABSTRACT

In this paper we prove a coincidence point theorem in dislocated metric spaces, provide a supporting example and extend the theorem to dislocated quasi metric spaces. We observe that the supporting example of a result of K.P.R. Rao and P. Ranga Swamy ([7]) , on the existence of a coincidence point for four self maps on a dislocated metric space is not valid. We also make a modification of their result.

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1. INTRODUCTION

In 2005, F.M. Zayeda, G.H. Hassan and M.A. Ahmed [10] defined dislocated quasi metric spaces and dislocated metric spaces. C.T. Aage and J.N. Salunke [1] and A. Isufati [4] proved fixed point theorems for a single self map and a pair of self mappings in dislocated metric spaces, K.P.R. Rao and P. Ranga Swamy ([7]) proved a common coincidence point theorem for four self maps in a dislocated metric space. In this paper we prove a coincidence point theorem for four self maps on a dislocated quasi metric space and observe that (Theorem 2.1, [7]) is a special case of our result.

First we recall some Definitions from [10]

Definition 1.1: Let X be a non empty set and let d : X × X → [0, ∞) be a function. The following conditions on d are referred subsequently

(i) d(x,x) = 0 ∀ x ∈ X . (1.1.1)
(ii) d(x,y) = d(y,x) = 0 ⇒ x = y ∀ x,y ∈ X . (1.1.2)
(iii) d(x,y) = d(y,x) ∀ x,y ∈ X . (1.1.3)
(iv) d(x,y) = max{d(x,z), d(z,y)} ∀ x,y,z ∈ X . (1.1.4)
(v) d(x,y) ≤ max{d(x,z), d(z,y)} ∀ x,y,z ∈ X . (1.1.5)

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We observe that every ultra metric is a metric.

**Definition 1.2** ([10]): A sequence \( \{x_n\} \) in a dq – metric space \((X, d)\) is called a Cauchy sequence if for any given \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that for all \( m, n \geq n_0, \) \( d(x_m, x_n) < \varepsilon. \)

**Definition 1.3** ([10]): A sequence \( \{x_n\} \) in a dq – metric space is said to be dislocated quasi convergent to \( x \) if \( \lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0. \)

In this case \( x \) is called a dq – limit of \( \{x_n\} \) and we write \( x_n \to x. \)

**Lemma 1.4** ([10]): Dq – limit in a dq – metric space is unique.

**Definition 1.5** ([10]): A dq – metric space \((X, d)\) is called complete if every Cauchy sequence in it is dq – convergent.

**Definition 1.6** ([10]): Let \((X, d_1)\) and \((Y, d_2)\) be dq – metric spaces and let \( f : X \to Y \) be a function. Then \( f \) is said to be continuous at \( x_0 \in X, \) if the sequence \( f\{x_n\} \) is \( d_2 \)-convergent to \( f(x_0) \in Y \) whenever the sequence \( \{x_n\} \) in \( X \) is \( d_1 \)-convergent to \( x_0. \)

**Definition 1.7** ([10]): Let \((X, d)\) be a dq – metric space. A map \( T : X \to X \) is called a contraction if there exists \( 0 \leq \lambda < 1 \) such that
\[
d(Tx, Ty) \leq \lambda d(x, y) \forall x, y \in X.
\]

**Note:** In a metric space, a contraction map is continuous. However, in a dq – metric space a contraction map need not be continuous (Rutten [9], Example [3.6]). Zayed et al. [10] proved the dq - metric version of Banach contraction principle.

**Theorem 1.8** ([10]): Let \((X, d)\) be a dq – metric space and let \( T : X \to X \) be a continuous contraction mapping. Then \( T \) has unique fixed point.

K. P. R. Rao and P. Ranga Swamy [7] proved the following coincidence theorem for four self maps on a dislocated metric space.

**Theorem 1.9:** ([7], Theorem 2.1) Let \((X, d)\) be a complete dislocated metric space and let \( F, G, S \) and \( T : X \to X \) be continuous mappings satisfying
\[
S(X) \subseteq G(X) \quad \text{and} \quad T(X) \subseteq F(X) \tag{1.9.1}
\]
\[
SF = FS \quad \text{and} \quad TG = GT \quad \text{and} \tag{1.9.2}
\]
\[
d(Sx, Ty) \leq \varphi(\max\{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty), \frac{d(Fx, Sx)d(Gy, Ty)}{d(Fx, Gy)}\}) \tag{1.9.3}
\]

for all \( x, y \in X, \) where \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is monotonically non – decreasing and \( \sum_{n=1}^{\infty} \varphi^n(t) < \infty \) for all \( t > 0. \)

Then (i) \( F \) and \( S \) (or) \( G \) and \( T \) have coincidence point (or) (ii) the pairs \((F, S) \) and \((G, T)\) have a common coincidence point.

The following Example is given in [7] in support of Theorem 1.9 ([7], Example 2.2)

**Example 1.10:** Let \( X = [0, 1] \) and \( d(x, y) = \max\{x, y\}. \) Then \((X, d)\) is a dislocated metric space. Define \( Sx = 0, \)
\[
Tx = \frac{x}{6}, Fx = x, Gx = \frac{x}{3}
\]
Clearly \( S, T, F \) and \( G \) are continuous and (1.9.1) and (1.9.2) are satisfied.

Also \( d(Sx, Ty) \leq \varphi(d(Fx, Gy)) \) for all \( x, y \in X, \) where \( \varphi(t) = \frac{t}{2}, \) clearly 0 is a common coincidence point of \((F, S) \) and \((G, T)\). However the above example does not support Theorem 1.9, since at \( x = y = 0 \) (1.9.3) is not satisfied (In fact it has no meaning). Consequently condition (1.9.3) should be assumed to hold whenever \( d(Fx, Gy) \neq 0. \) Example 1.10 cannot be considered as an Example in support of Theorem 1.9.

since \( d(Fx, Gy) = 0 \) for \( x = y = 0 \).
2. MAIN RESULTS

In this section, we prove a coincidence point theorem for four self maps on a dislocated metric space and provide a supporting example. Extend this to dislocated quasi metric spaces. We also obtain the result of [7], with modification, as a corollary.

Theorem 2.1: Let \((X, d)\) be a complete dislocated metric space and let \(F, G, S\) and \(T: X \rightarrow X\) be continuous mappings satisfying

\[
S(X) \subseteq G(X) \quad \text{and} \quad T(X) \subseteq F(X) \tag{2.1.1}
\]

\[
SF = FS \quad \text{and} \quad TG = GT \tag{2.1.2}
\]

\[
d(Sx, Ty) \leq \varphi \max \{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty)\} \tag{2.1.3}
\]

for all \(x, y \in X\), where \(\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is monotonically non-decreasing and \(\sum_{n=1}^{\infty} \varphi^n(t) < \infty\) for all \(t > 0\). Then the pairs \((F, S)\) and \((G, T)\) have a common coincidence point.

Proof: It is clear that \(\varphi^n(t) \to 0\) as \(n \to \infty\) and \(\varphi(t) < t \quad \forall \quad t > 0\). Suppose \(x_0 \in X\). Define the sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
y_{2n} = Sx_{2n} = Gx_{2n+1}
\]

\[
y_{2n+1} = Tx_{2n+1} = Fx_{2n+2}, \quad n = 0, 1, 2, \ldots
\]

Now

\[
d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}) \quad \text{(by (2.1.3))}
\]

\[
\leq \varphi \left( \max \{d(Fx_{2n}, Gy_{2n+1}), d(Fx_{2n}, Sx_{2n}), d(Gy_{2n+1}, Tx_{2n+1})\} \right)
\]

\[
= \varphi \left( \max \{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \right)
\]

\[
= \varphi \left( \max \{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \right) \tag{2.1.4}
\]

Now from (2.1.4)

\[
d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1})
\]

\[
\Rightarrow d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n}, y_{2n+1}))
\]

\[
\Rightarrow d(y_{2n}, y_{2n+1}) = 0
\]

\[
\Rightarrow d(y_{2n}, y_{2n+1}) \leq 0 \leq \varphi(d(y_{2n-1}, y_{2n})) \tag{2.1.5}
\]

Now suppose

\[
d(y_{2n-1}, y_{2n}) > d(y_{2n}, y_{2n+1}). \quad \text{Then from (2.1.4)}
\]

follows that

\[
d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n-1}, y_{2n})) \tag{2.1.6}
\]

From (2.1.5) and (2.1.6) follows that

\[
d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n-1}, y_{2n})) \quad \text{for} \quad n = 1, 2, 3, \ldots
\]

Similarly we can show that

\[
d(y_{2n+1}, y_{2n+2}) = d(Tx_{2n+1}, Sx_{2n+2}) \quad \text{(by (1.1.3))}
\]

\[
= d(x_{2n+2}, x_{2n+1})
\]

\[
\leq \varphi \left( \max \{d(Fx_{2n+1}, Gy_{2n+2}), d(Fx_{2n+2}, Sx_{2n+2}), d(Gy_{2n+1}, Tx_{2n+1})\} \right)
\]

\[
= \varphi \left( \max \{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1})\} \right)
\]

\[
= \varphi \left( \max \{d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1})\} \right)
\]

\[
\leq \varphi(d(y_{2n}, y_{2n+1}))
\]
Hence $d(y_n, y_{n+1}) \leq \varphi(d(y_{n-1}, y_n))$ for $n = 1, 2, 3, \ldots$

\[
\therefore d(y_n, y_{n+1}) \leq \varphi(d(y_{n-1}, y_n)) \leq \cdots \leq \varphi^n(d(y_0, y_1)) \leq \varphi^n(d(y_0, y_1))
\]

Since $\varphi^n(t) \to 0$ as $n \to \infty$, we may suppose without loss of generality that $d(y_0, y_1) < 1$

\[
\therefore d(y_n, y_{n+k}) \leq d(y_n, y_{n+1}) + \cdots + d(y_{n+k-1}, y_{n+k}) = \varphi^n(d(y_0, y_1)) + \cdots + \varphi^{n+k-1}(d(y_0, y_1)) = \frac{\varphi^n(d(y_0, y_1))}{\varphi(d(y_0, y_1))} \to 0 \text{ as } n \to \infty
\]

\[
\therefore \{y_n\} \text{ is a Cauchy sequence. Hence there exists } u \in X \text{ such that } y_n \to u \quad \text{whenever } d(x, y) = \max\{x, y\}
\]

We have $Su = \lim_{n \to \infty} SFX_{2n} = \lim_{n \to \infty} FFX_{2n} = Fu$, since $TG = GT$ and $T$ and $G$ are continuous,

we have $Tu = \lim_{n \to \infty} TGX_{2n+1} = \lim_{n \to \infty} GTX_{2n+1} = Gu$

Thus $u$ is a common coincidence point of the pair $(F, S)$ and $(G, T)$,

consequently $u$ is a common coincidence point of $F, S, G$ and $T$.

The following Example supports our theorem

**Example 2.2:** Let $X = [0, 1]$, $d(x, y) = \max\{x, y\}$. Then $(X, d)$ is a complete dislocated metric space. Define $Sx = 0$, $Tx = \frac{x}{2}$, $Fx = Gx = x$. Take $\varphi(t) = \frac{t}{2}$. Then clearly (2.1.1), (2.1.2) and (2.1.3) are satisfied and $0$ is a common coincidence point of $S, F, G$ and $T$.

The following is a modified version of Theorem 1.9

**Theorem 2.3:** Let $(X, d)$ be a complete dislocated metric space and let $F, G, S$ and $T : X \to X$ be continuous mappings satisfying

\[
S(X) \subseteq G(X) \quad \text{and} \quad T(X) \subseteq F(X) \quad (2.3.1)
\]

\[
SF = FS \text{ and } TG = GT \quad \text{and} \quad (2.3.2)
\]

\[
d(Tx, Sy) \leq \varphi(\max\{d(Gx, Fy), d(Fy, Sy), d(Gx,Tx), \frac{d(Fy, Sy) d(Gx, Tx)}{d(Fx, Fy)}\}) \quad (2.3.3)
\]

for all $x, y \in X$, whenever $d(Gx, Fy) \neq 0$, where $\varphi : R^+ \to R^+$ is monotonically non-decreasing and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$.

Then $F, S, G$ and $T$ have a common coincidence point.

**Proof:** It is clear that $\varphi^n(t) \to 0$ as $n \to \infty$ and $\varphi(t) < t \forall t > 0$. Suppose $x_0 \in X$.

Define the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

\[
y_{2n} = Sx_{2n} = Gx_{2n+1}
\]

\[
y_{2n+1} = Tx_{2n+1} = Fx_{2n+2}, n = 0, 1, 2, \ldots
\]
If \( y_{2n} = y_{2n+1} \) for some \( n \) then \( Gx_{2n+1} = Tx_{2n+1} \). Hence \( x_{2n+1} \) is a coincidence point of \( G \) and \( T \).

If \( y_{2n+1} = y_{2n+2} \) for some \( n \) then \( Fx_{2n+2} = Sx_{2n+2} \). Hence \( x_{2n+2} \) is a coincidence point of \( F \) and \( S \). Assume that \( y_n \neq y_{n+1} \) for all \( n \). Then as in the proof of Theorem 2.1, we can show that \( \{ y_n \} \) is a Cauchy sequence.

Hence there exists \( u \in X \) such that \( \{ y_n \} \) converges to \( u \). Since \( FS = SF \) and \( S \) and \( F \) are continuous, we have \( Su = \lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} FSx_{2n} = Fu \), since \( TG = GT \) and \( T \) and \( G \) are continuous, we have \( Tu = \lim_{n \to \infty} TGx_{2n+1} = \lim_{n \to \infty} GTx_{2n+1} = Gu \).

Thus \( u \) is a common coincidence point of the pairs \((F, S)\) and \((G, T)\), consequently \( u \) is a common coincidence point of \( F, S, G \) and \( T \).

The following Example shows that Theorem 2.3 may not hold, even in metric spaces, if (2.3.2) is dropped.

**Example 2.4:** Let \( X = [0,1] \) with the usual metric \( d \). Define \( Sx = \frac{x}{2}, Tx = 0, Fx = x, Gx = 1 - x \forall x \in X \). Then \((X, d)\) is a complete metric space, \((2.3.1)\) is clearly satisfied \((2.3.3)\) is satisfied with \( \varphi(t) = \frac{t}{2} \). But \((2.3.2)\) is not satisfied since \( TG \neq GT \). Here the pair \((F, S)\) has a coincidence point, namely 0. The pair \((G, T)\) has a coincidence point namely 1. But \( S, T, F \) and \( G \) do not have a common coincidence point.

**Now we extend Theorem 2.1 to dislocated quasi metric spaces as follows**

**Theorem 2.5:** Let \((X, d)\) be a complete dislocated quasi metric space, let \( F, G, S \) and \( T : X \to X \) be continuous mappings satisfying

\[
S(X) \subseteq G(X) \quad \text{and} \quad T(X) \subseteq F(X) \tag{2.5.1}
\]

\[
SF = FS \quad \text{and} \quad TG = GT \quad \text{and} \tag{2.5.2}
\]

\[
d(Sx, Ty) \leq \varphi(\max\{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty)\}) \tag{2.5.3}
\]

\[
d(Tx, Sy) \leq \varphi(\max\{d(Gx, Fy), d(Fy, Sy), d(Gx, Tx)\}) \tag{2.5.4}
\]

for all \( x, y \in X \), where \( \varphi : R^+ \to R^+ \) is monotonically non-decreasing and \( \sum_{n=1}^{\infty} \varphi^n(t) < \infty \) for all \( t > 0 \).

Then the pairs \((F, S)\) and \((G, T)\) have a common coincidence point which is also a common coincidence point of \( F, G, S \) and \( T \).

**Proof:** It is clear that \( \varphi^n(t) \to 0 \) as \( n \to \infty \) and \( \varphi(t) < t \forall t > 0 \). Suppose \( x_0 \in X \).

Define the sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
y_{2n} = Sx_{2n} = Gx_{2n+1}
\]

\[
y_{2n+1} = Tx_{2n+1} = Fx_{2n+2}, \quad n = 0,1,2,...
\]

Now

\[
d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}) \quad \text{(by \ref{2.5.3})}
\]

\[
\leq \varphi(\max\{d(Fx_{2n}, Gx_{2n+1}), d(Fx_{2n}, Sx_{2n}), d(Gx_{2n+1}, Tx_{2n+1})\})
\]

\[
= \varphi(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\})
\]

\[
= \varphi(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}) \quad \text{(2.5.5)}
\]

Now from \ref{2.5.5}

\[
d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1})
\]

\[
\Rightarrow d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n}, y_{2n+1}))
\]

\[
\Rightarrow d(y_{2n}, y_{2n+1}) = 0
\]
From (2.5.6) and (2.5.7) shows that

\[ \Rightarrow d(y_{2n}, y_{2n+1}) \leq 0 \leq \varphi(d(y_{2n-1}, y_{2n})) \]  

(2.5.6)

Now suppose

\[ d(y_{2n-1}, y_{2n}) > d(y_{2n}, y_{2n+1}) \]  

Then from (2.5.5)

follows that

\[ d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n-1}, y_{2n})) \]  

(2.5.7)

From (2.5.6) and (2.5.7) shows that

\[ d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n-1}, y_{2n})) \quad \text{for } n = 1, 2, 3, \ldots \]

Similarly using (2.5.4), we can show that

\[ d(y_{2n+1}, y_{2n+2}) \leq \varphi(d(y_{2n}, y_{2n+1})) \]

Now

\[ d(y_{2n}, y_{2n+1}) = d(Tx_{2n+1}, Sx_{2n}) \quad \text{(by (2.5.4))} \]

\[ \leq \varphi(\max\{d(Gx_{2n+1}, Fx_{2n}), d(Fx_{2n}, Sx_{2n}), d(Gx_{2n+1}, Tx_{2n+1})\}) \]

\[ = \varphi(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n+1})\}) \]

\[ = \varphi(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\}) \]

\[ \Rightarrow \text{If } d(y_{2n}, y_{2n+1}) \leq d(y_{2n}, y_{2n+1}) < d(y_{2n}, y_{2n+1}) \]

since \( d(y_{2n}, y_{2n+1}) > 0 \), which is contradiction

\[ \Rightarrow d(y_{2n}, y_{2n+1}) \leq d(y_{2n}, y_{2n}) \]

\[ \Rightarrow d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n-1}, y_{2n})) \quad \text{this is true for } n = 1, 2, 3, \ldots \]

\[ \Rightarrow d(y_{n}, y_{n+1}) \leq \varphi^{2}(d(y_{n-1}, y_{n})) \]

\[ \vdots \]

\[ \leq \varphi^{n}(d(y_{0}, y_{1})) \]

Here again, we may suppose without loss of generality that \( \varphi(d(y_{0}, y_{1})) < 1 \)

\[ \Rightarrow d(y_{n}, y_{n+k}) \leq d(y_{n}, y_{n+1}) + \cdots + d(y_{n+k-1}, y_{n+k}) \]

\[ = \varphi^{0}(d(y_{0}, y_{1})) + \cdots + \varphi^{n+k-1}(d(y_{0}, y_{1})) \]

\[ = \frac{\varphi^{n}(d(y_{0}, y_{1}))}{1-\varphi(d(y_{0}, y_{1}))} \to 0 \quad \text{as } n \to \infty \]

\[ \Rightarrow \{y_{n}\} \text{ is a Cauchy sequence}. \]

Since \( X \) is a complete dislocated quasi metric space there exists \( u \in X \) such that \( \{y_{n}\} \) converges to \( u \). Since \( FS = SF \) and \( S \) and \( F \) are continuous,

we have \( Su = \lim_{n \to \infty} SFx_{2n} = \lim_{n \to \infty} FSx_{2n} = Fu \),

since \( TG = GT \) and \( T \) and \( G \) are continuous,

we have \( Tu = \lim_{n \to \infty} TGx_{2n+1} = \lim_{n \to \infty} GTx_{2n+1} = Gu \)

Thus the pairs \( (F, S) \) and \( (G, T) \) have common coincidence point which is also a common coincidence point of \( F, S, G \) and \( T \).
The following is a common fixed point theorem, with the control function containing rational terms.

**Theorem 2.6**: Let \((X, d)\) be a complete dislocated quasi metric space.

Let \(F, G, S\) and \(T: X \rightarrow X\) be continuous mappings satisfying

\[
S(X) \subseteq G(X) \quad \text{and} \quad T(X) \subseteq F(X)
\]  
\[
SF = FS \quad \text{and} \quad TG = GT \quad \text{and} \quad (2.6.2)
\]

Then from (2.6.5)

\[
d(Sx, Ty) \leq \varphi(max\{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty), \frac{d(Fx, Sx)d(Gy, Ty)}{d(Fx, Gy)}\}) \quad \text{for all} \quad x, y \in X \quad \text{and} \quad (2.6.6)
\]

\[
d(Fx, Gy) \neq 0 \quad \text{and} \quad (2.6.7)
\]

\[
d(Tx, Sy) \leq \varphi(\max\{d(Gx, Fy), d(Fy, Sy), d(Gx, Tx), \frac{d(Fy, Sy)d(Gx, Tx)}{d(Gx, Fy)}\})
\]  
\[
(2.6.4)
\]

for all \(x, y \in X\) and \(d(Gx, Fy) \neq 0\), where \(\varphi: R^+ \rightarrow R^+\) is monotonically non-decreasing and

\[
\sum_{n=1}^{\infty} \varphi^n(t) < \infty \quad \text{for all} \quad t > 0.
\]

Then the pairs \((F, S)\) and \((G, T)\) have a common coincidence point.

**Proof**: It is clear that

\[
\varphi^n(t) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad \text{and} \quad \varphi(t) < t \quad \forall \quad t > 0.
\]

Suppose \(x_0 \in X\).

Define the sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
y_{2n} = Sx_{2n} = Gx_{2n+1}
\]

\[
y_{2n+1} = Tx_{2n+1} = Fx_{2n+2}, \quad n = 0, 1, 2, \ldots
\]

Now

\[
d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}) \quad \text{(by (2.6.3))}
\]

\[
\leq \varphi(\max\{d(Fx_{2n}, Gx_{2n+1}), d(Fx_{2n}, Sx_{2n}), d(Gx_{2n+1}, Tx_{2n+1}), \frac{d(Fx_{2n}, Sx_{2n})d(Gx_{2n+1}, Tx_{2n+1})}{d(Fx_{2n}, Gx_{2n+1})}\})
\]

\[
= \varphi(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})}{d(y_{2n-1}, y_{2n})}\})
\]

\[
= \varphi(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}) \quad \text{(2.6.5)}
\]

Now from (2.6.5)

\[
d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1})
\]

\[
\Rightarrow d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n}, y_{2n+1}))
\]

\[
\Rightarrow d(y_{2n}, y_{2n+1}) = 0
\]

\[
\Rightarrow d(y_{2n}, y_{2n+1}) \leq 0 \leq \varphi(d(y_{2n-1}, y_{2n})) \quad \text{(2.6.6)}
\]

Now suppose

\[
d(y_{2n-1}, y_{2n}) > d(y_{2n}, y_{2n+1}).\] Then from (2.6.5)

follows that

\[
d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n-1}, y_{2n})) \quad \text{(2.6.7)}
\]

From (2.6.6) and (2.6.7) follows that

\[
d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n-1}, y_{2n})) \quad \text{for} \quad n = 1, 2, 3, \ldots
\]
Similarly we can show that

\[ d(y_{2n+1}, y_{2n+2}) \leq \varphi(d(y_{2n}, y_{2n+1})) \quad \text{for} \quad n = 1, 2, 3 \ldots \]

Now

\[ d(y_{2n}, y_{2n+1}) = d(Tx_{2n+1}, Sx_{2n}) \quad (\text{by (2.6.4)}) \]

\[ \leq \varphi(\max\{d(Gx_{2n+1}, Fx_{2n}), d(Fx_{2n}, Sx_{2n}), d(Gx_{2n+1}, Tx_{2n+1})\}, \frac{d(Fx_{2n}, Sx_{2n}) + d(Gx_{2n+1}, Tx_{2n+1})}{d(Gx_{2n+1}, Fx_{2n})}) \]

\[ = \varphi(\max\{d(y_{2n}, y_{2n-1}), (y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}, \frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{d(y_{2n-1}, y_{2n})}) \]

\[ = \varphi(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\}) \]

\[ \therefore \text{If} \quad d(y_{2n}, y_{2n+1}) \leq d(y_{2n}, y_{2n+1}) < d(y_{2n}, y_{2n+1}) \quad \text{since} \quad d(y_{2n}, y_{2n+1}) > 0, \quad \text{which is contradiction} \]

\[ \therefore d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n}) \]

\[ \therefore d(y_{2n}, y_{2n+1}) \leq \varphi(d(y_{2n-1}, y_{2n})) \quad \text{this is true for} \quad n = 1, 2, 3 \ldots \]

Similarly

\[ \therefore d(y_n, y_{n+1}) \leq \varphi(d(y_{n-1}, y_n)) \]

\[ \vdots \]

\[ \leq \varphi^n(d(y_0, y_1)) \]

Since \( \varphi^n(t) \to 0 \) as \( n \to \infty \), we may suppose with out loss of generality that \( d(y_0, y_1) < 1 \)

\[ \therefore d(y_n, y_{n+k}) \leq d(y_n, y_{n+1}) + \cdots + d(y_{n+k-1}, y_{n+k}) \]

\[ = \varphi^n(d(y_0, y_1)) + \cdots + \varphi^{n+k-1}(d(y_0, y_1)) \]

\[ = \frac{\varphi^n(d(y_0, y_1))}{1-\varphi(d(y_0, y_1))} \to 0 \quad \text{as} \quad n \to \infty \]

\[ \therefore \{y_n\} \text{ is a Cauchy sequence.} \]

Since \( X \) is a complete dislocated quasi metric space there exists \( u \in X \) such that \( \{y_n\} \) converges to \( u \).

Since \( FS = SF \) and \( S \) and \( F \) are continuous, we have \( Su = \lim_{n \to \infty} SFx_{2n} = \lim_{n \to \infty} FSx_{2n} = Fu \), since \( TG = GT \) and \( T \) and \( G \) are continuous,

we have \( Tu = \lim_{n \to \infty} TGx_{2n+1} = \lim_{n \to \infty} GTx_{2n+1} = Gu \)

Thus the pairs \( (F, S) \) and \( (G, T) \) have common coincidence point which is also a common coincidence point of \( F \), \( S, G \) and \( T \).

\textbf{Note:} In theorem 2.6, if we assume the space to be a complete dislocated metric space, (2.6.4) can be drooped and consequently we get a modified version of theorem 1.9 as a corollary.

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