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Certain Special classes of ideals in Boolean like Semi ring of Fractions

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ABSTRACT

In this paper we introduce the notions of Extended, Primary and Almost Primary ideals of Boolean like semi ring of fractions. Further we obtain certain properties regarding extended ideals in this class. Also we prove that the contraction of an almost primary (and hence primary and prime) ideal is also almost primary (primary and prime respectively) in a Boolean like semi ring (a special class of near ring).

Key words: *Boolean like semi ring of fractions, Extended ideal, Contracted ideal, Almost Primary ideal.*

Mathematics Subject Classification: *16Y30, 16 Y 60.*

INTRODUCTION

Foster A.L. [2] introduced the notion of Boolean like ring R as a commutative ring with unity and is of characteristic 2 in which ab (1+ a) (1+b) = 0 for all a, b $\in$ R. Later in [4] Venkateswarlu et al generalized the notion of Boolean like ring by introducing the concept of Boolean like semi ring. Boolean like semi ring is a special class of near ring. Venkateswarlu and Murthy [1, 5, 7] made an extensive study of the class of Boolean like semi rings and also have studied certain properties of contraction of ideals in [8].

Recently the present authors have introduced the notion Boolean like semi ring of fractions in [4] and it was proved that every Boolean like semi ring of fraction is a Boolean like ring. However there are many interesting facts about the class of Boolean like semi ring of fractions which do not subsume the properties of Boolean like rings.

This paper is divided into 3 sections of which the first sections is devoted for recollecting certain definitions and results concerning Boolean like semi rings and as well as Boolean like semi ring of fractions. In section 2, we introduce the concept of extended ideal in a Boolean like semi ring of fractions and prove that $S^{-1}I$ is an ideal of $S^{-1}R$ .Further, pre image of any ideal of $S^{-1}R$ is also an ideal in the Boolean like semi ring R (see definition 2.3). Also we observe in theorem 2.4 that $S^{-1}P$ is a prime ideal of $S^{-1}R$ whenever P is a prime ideal in R which is disjoint from S. In theorem 2.8 we prove $f^{-1}(J)$ is a prime ideal of R and $f^{-1}(J) \cap S = \emptyset$ provided J is a prime ideal in $S^{-1}R$ with right unity.

In section 3, we introduce the notions of primary and almost primary ideal in Boolean like semi ring of fractions and further we prove that $S^{-1}P$ is a almost primary (hence primary) ideal of $S^{-1}R$ whenever P is almost primary (respectively primary) in R (3.5 & 3.7). Finally we establish in theorems 3.6 & 3.8 that the contraction of an ideal J is almost primary (hence primary) in a Boolean like semi ring R whenever J is almost primary (primary) ideal of $S^{-1}R$.

1. PRELIMINARIES.

In this, we recall certain definitions and results on Boolean like semi rings and Boolean like semi ring of fractions from [2], [4] and [6]

Definition 1.1 A non empty set R together with two binary operations + and $\cdot$ satisfying the following conditions is called Boolean like semi ring;

1. (R, +) is an abelian group.
2. (R, $\cdot$) is a semi group.
3. a $\cdot$ (b + c) = a.b + a.c

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4. $a + a = 0$ for all $a \in R$.
5. $ab\,(a + b + ab) = ab$ for all a, b, c $\in$ R.

Lemma 1.2 Let R be a Boolean like semi ring. Then a.0 = 0 for all a in R.

Definition 1.3 A Boolean like semi ring R is said to be weak commutative if abc = acb for all a, b and c $\in$ R.

Lemma 1.4 Let R be weak commutative Boolean like semi ring and let m and n be integers. Then
(i) $a^m a^n = a^{m+n}$
(ii) $(a^m)^n = a^{mn}$
(iii) $(ab)^n = a^n b^n$ for all a, b $\in$ R

Definition 1.5. A non empty subset I of R is said to be an ideal if

1 (I, +) is a sub group of (R, +), i.e, for a, b $\in$ R $\Rightarrow$ a + b $\in$ R
2. ra $\in$ R for all a $\in$ I , r $\in$ R ,i.e RI $\subseteq$I
3. (r+a)s + rs $\in$ I for all r, s $\in$ R , a $\in$ I

Definition 1.6 A non empty subset S of a Boolean like semi ring R is called multiplicatively closed whenever a, b $\in$ S implies ab $\in$S.

Lemma 1.7 Let R be a weak commutative Boolean like semi ring and S be a multiplicatively closed subset of R. Then

1. $\frac{r}{s} = \frac{rt}{st} = \frac{tr}{st} = \frac{tr}{ts}$ for all r in R and for all s, t in S
2. $\frac{rs}{s} = \frac{rs'}{s'}$ for all r in R and for all s, s′ in S.
3. $\frac{s}{s} = \frac{s'}{s'}$ for all s, s' in S.

Theorem 1.8. Let S be a multiplicatively closed sub set in a weak commutative Boolean like semi ring R. Define binary operations + and · on S^{-1}R as follows:

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2} \text{ for } s_1, s_2 \in S \text{ and } r_1, r_2 \in R$$

Then (S^{-1}R, +, ·) is a Boolean like semi ring.

Definition 1.9. If R and R′ are Boolean like semi rings, a mapping f: R→R′ is said to be a Boolean like semi ring homomorphism (or simply homomorphism) of R into R′ if

f(a + b) = f (a) + f (b) and f (ab) = f(a)f(b) for all a, b $\in$ R.

Definition 1.10. Let R and S be two Boolean like semi rings. If I is an ideal of S and f : R→S is a homomorphism then f^{-1}(I) is an ideal of R, called the contraction of I and is denoted by I^c

2. EXTENDED IDEALS IN BOOLEAN LIKE SEMI RING OF FRACTIONS

We begin with the following

Theorem 2.1 Let R be a weak commutative Boolean like semi ring and let S be a multiplicatively closed subset in R. Define f: R →S^{-1}R by f(r) = $\frac{rs}{s}$ where s$\in$S. Then

1. f is a homomorphism.
2. If 0$\notin$S and S contain no divisors of zero then f is a monomorphism.

Proof. To prove 1, Let r, t $\in$ R. By repeated application of 1 of lemma 1.7, we have

f (r + t) = $\frac{(r+t)s}{s} = \frac{s(r+t)}{s} = \frac{rs}{s} + \frac{ts}{s}$ = f(r) f(t) and f (rt) = $\frac{(rt)s}{s} = \frac{s(rt)s}{ss} = \frac{rs}{s} \cdot \frac{ts}{s}$ = f(r) f(t).

Hence f is a homomorphism.

Now let r, t ∈R su1.1ch that f(r) = f(t). Then $\frac{rs}{s} = \frac{ts}{s}$

⇒ there exists m ∈ S such that m[s(rs) + s(ts)] = 0 and hence $ms^2 [r + t] = 0$

⇒ r + t = 0 since $ms^2 \in S$ and S doesn't contain (non zero) zero divisors. Thus r = t.

Hence f is a monomorphism.

Theorem 2.2. Let R be a weak commutative Boolean like semi ring, S be a multiplicatively closed subset in R and I be an ideal of R. Let $S^{-1}I = \{\frac{a}{s} / a \in I, s \in S\}$. Then $S^{-1}I$ is an ideal of $S^{-1}R$.

Proof. Since 0 ∈ I, it is clear that $\frac{0}{s} \in S^{-1}I$ and hence $S^{-1}I$ is non empty.

Let $t_1, t_2 \in S^{-1}I$. Then $t_1 = \frac{a_1}{s_1}$, $t_2 = \frac{a_2}{s_2}$ for some $a_1, a_2 \in I$ and $s_1, s_2 \in S$.

⇒ $t_1 + t_2 = \frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2 a_1 + s_1 a_2}{s_1 s_2} \in S^{-1}I$ since $s_2 a_1 + s_1 a_2 \in I$.

Further let $\lambda \in S^{-1}R$ and $\beta \in S^{-1}I$. Then $\lambda = \frac{r}{s}$ and $\beta = \frac{a}{s'}$ for some r ∈ R and a ∈ I and s, $s' \in S$

Thus $\lambda\beta = \frac{r}{s}\frac{a}{s'} = \frac{ra}{ss'} \in S^{-1}I$.

Finally, Let $\lambda_1, \lambda_2 \in S^{-1}R$ and $\beta \in S^{-1}I$, then $\lambda_1 = \frac{r_1}{s_1}$, $\lambda_2 = \frac{r_2}{s_2}$ and $\beta = \frac{a}{s}$ for some $r_1, r_2 \in R$, a ∈ I and s, s_1, $s_2 \in S$. By using the definitions of 1.1, 1.5 and lemma 1.7, we have

$$
\begin{aligned}
[\lambda_1 + \beta]\lambda_2 + \lambda_1\lambda_2 &= \left[\frac{r_1}{s_1} + \frac{a}{s}\right]\frac{r_2}{s_2} + \frac{r_1}{s_1}.\frac{r_2}{s_2} = \left[\frac{sr_1 + s_1 a}{s_1 s}\right]\frac{r_2}{s_2} + \frac{r_1 r_2}{s_1 s_2} \\
&= \frac{s_1 s_2[sr_1 + s_1 a]r_2 + (s_1 s)s_2(r_1 r_2)}{(s_1 s)s_2(s_1 s_2)} \\
&= \frac{[s_1 s_2][sr_1 + s_1 a]r_2 + (sr_1)r_2)]}{[s_1 s_2]s(s_1 s_2)} \\
&= \frac{[sr_1 + s_1 a]r_2 + (sr_1)r_2]}{s(s_1 s_2)} \in S^{-1}I
\end{aligned}
$$

Since by the definition 1.5, we have that $[sr_1 + s_1 a]r_2 + (sr_1)r_2] \in I$

Hence $S^{-1}I$ is an ideal of $S^{-1}R$.

In view of the above theorem , we have the following

Definition 2.3. If I is an ideal of R and f: R $\to S^{-1}R$ is a homomorphism then f (I) = $S^{-1}I$ is an ideal of $S^{-1}R$, called the Extension of I and is denoted by I^e .

Theorem 2.4. Let P be a prime ideal of a weak commutative Boolean like semi ring R such that $P \cap S = \emptyset$ then $S^{-1}P$ is a prime ideal of $S^{-1}R$.

Proof. First we show $S^{-1}P \neq S^{-1}R$. Suppose $S^{-1}P = S^{-1}R$

Then $\frac{s}{s} \in S^{-1}P$ since $\frac{s}{s} \in S^{-1}R$.

⇒ $\frac{s}{s} = \frac{a}{s'}$ for some a ∈ P and $s' \in S$.

⇒ $m[sa + s's] = 0$ for some m ∈ S.

⇒ $ms's = msa$ and msa ∈ P since a∈P.

⇒ $ms's \in P \cap S$

$\Rightarrow P \cap S \neq \emptyset$ contradiction to our supposition. Hence $S^{-1}P \neq S^{-1}R$.

It is clear from the above theorem 2.2, that $S^{-1}P$ is an ideal of $S^{-1}R$.

Next, suppose $t_1, t_2 \in S^{-1}R$ such that $t_1t_2 \in P$. Then $t_1 = \frac{r_1}{s_1}$ and $t_2 = \frac{r_2}{s_2}$.
$\Rightarrow \frac{r_1 r_2}{s_1 s_2} = \frac{a}{s}$ for some a in P and s in S.
$\Rightarrow s'(s_1 s_2 a + s r_1 r_2) = 0$ for some s' in S.
$\Rightarrow (s's)(r_1 r_2) = (s' s_1 s_2) a \in P$
$\Rightarrow s's \in P$ or $r_1 r_2 \in P$. But $s's \notin P$ for $P \cap S = \emptyset$ and $s's \in S$. Hence $r_1 r_2 \in P$.
$\Rightarrow r_1 \in P$ or $r_2 \in P$.
$\Rightarrow \frac{r_1}{s_1} \in S^{-1}P$ or $\frac{r_2}{s_2} \in S^{-1}P$.
$\Rightarrow t_1 \in S^{-1}P$ or $t_2 \in S^{-1}P$.

Hence $S^{-1}P$ is a prime ideal of $S^{-1}R$.

Theorem 2.5 Let J be an ideal of $S^{-1}R$. Let $f: R \to S^{-1}R$ be the homomorphism given by $f(r) = \frac{rs}{s}$.

Then $f^{-1}(J) = \{x \in R / f(x) \in J\}$ is an ideal of R.

Proof. Since $f(0) = 0 \in J$, we have $0 \in f^{-1}(J)$. Hence $f^{-1}(J)$ is non empty.

Let $x, y \in f^{-1}(J)$, then $f(x+y) = f(x) + f(y) \in J$. Hence $x + y \in f^{-1}(J)$. Also let $r \in R$ and $x \in f^{-1}(J)$.

Then $f(rx) = f(r) f(x) \in J$ since $f(x) \in J$. Thus $rx \in f^{-1}(J)$ .Finally let $r, s \in R$ and $x \in^{-1}(J)$. Then

$f[(r + x)s + rs] = f[(r+x)s] + f(rs) = [f(r) + f(x)]f(s) + f(r)f(s)$ lies in J since J is an ideal of $S^{-1}R$.

Hence $f^{-1}J$ is an ideal of R.

Definition 2.6. The ideal $f^{-1}J$ is called the contraction of J to R and denoted by J^c.

Definition 2.7. Let R be a Boolean like semi ring. The element 1 in R is called right unity if $a1 = a$ for all $a \in R$.

Theorem 2.8 Let R be a weak commutative Boolean like semi ring with right unity. Let J be a prime ideal of $S^{-1}R$. Then $f^{-1}(J)$ is a prime ideal of R and $f^{-1}(J) \cap S = \emptyset$

Proof. By theorem 2.5, $f^{-1}(J)$ is an ideal of R. Now claim $f^{-1}(J)$ is proper in R.

Suppose $f^{-1}(J)$ is not proper in R. Then $f^{-1}(J) = R$. Hence $f(x) \in J$ for all $x \in R$.

$\Rightarrow f(1) = \frac{1s}{s} \in J \Rightarrow \frac{s1}{s} \in J$ (by lemma 1.7)
$\Rightarrow \frac{s}{s} \in J$ which contradicts that J is prime.

Now let $t_1, t_2 \in R \ni t_1t_2 \in f^{-1}(J)$
$\Rightarrow f(t_1t_2) \in J \Rightarrow f(t_1) f(t_2) \in J$
$\Rightarrow f(t_1) \in J$ or $f(t_2) \in J$ since J is a prime ideal of $S^{-1}R$
$\Rightarrow t_1 \in f^{-1}(J)$ or $t_2 \in f^{-1}(J)$. Hence $f^{-1}(J)$ is a prime ideal of R.

Finally, for if $f^{-1}(J) \cap S \neq \emptyset$ let $s \in f^{-1}(J) \cap S$

$\Rightarrow s \in f^{-1}(J)$ and $s \in S$ and hence $f(s) \in J$ and $s \in S$

$\Rightarrow \frac{ss'}{s'} = \frac{s''}{s'} \in J$ and $s \in S$ for some s' in S and $s'' = ss'$.

$\Rightarrow \frac{s}{s} = (\frac{s''}{s'})\ \frac{s'}{s''} \in J$, for all $s \in S$ and since J is an ideal. But this contradicts the hypothesis that J is prime. Hence the theorem.

3. PRIMARY AND ALMOST PRIMARY IDEALS IN BOOLEAN LIKE SEMI RING OF FRACTIONS

We recall from [5] with the following

Definition 3.1[5]. A proper ideal p of a Boolean like semi ring R is called primary if $x \in P$ or $y^2 \in P$ whenever $xy \in P$ for every $x, y \in R$.

Definition 3.2 A proper ideal p of a Boolean like semi ring R is called almost primary if $x \in P$ or $y^2 \in P$ whenever $xy \in P - P^2$ for every $x, y \in R$.

Remark 3.3.

(a). It is clear that every primary ideal is almost primary but not conversely.

(b). In a Boolean like semi ring R, $a^n = a$ or a^2 or a^3 for any $n \geq 1$.Hence in definition 3.1 and 3.2 , it is appropriate to define primary and almost primary in the above fashion instead of defining the usual way as in the case of rings.

Theorem 3.4. Let S be a multiplicative set and I be an ideal in a Boolean like semi ring R. Then for all $r \in R, s \in S$ $\frac{r}{s} \in S^{-1}I \Leftrightarrow mr \in I$ for some $m \in S$.

Proof. Let $\frac{r}{s} \in S^{-1}I$. Then $\frac{r}{s} = \frac{a}{t}$ for some $a \in I$ and $t \in S$.Then $n[tr + sa] = 0$ for some n in S. Hence $(nt)r = (ns)a \in I$. Hence choose $m = nt$ in S.

Conversely, if mr is in I for some m in S, we have $\frac{r}{s} = \frac{mr}{ms} \in S^{-1}I$.

Theorem 3.5 Let P be a primary ideal of a Boolean like semi ring R and S be a multiplicative subset of R such that $P \cap S = \emptyset$. Then $S^{-1}P$ is a primary ideal of $S^{-1}R$.

Proof. By theorem 2.2 we have that $S^{-1}P$ is an ideal of $S^{-1}R$. And it is routine to verify that $S^{-1}P$ is a proper ideal.

Now let $\frac{r_1}{s_1}, \frac{r_2}{s_2}$ be in $S^{-1}R$ such that $\frac{r_1 r_2}{s_1 s_2} \in S^{-1}P$. Then if $\frac{r_1}{s_1} \in S^{-1}P$ we are done. Otherwise, by theorem 3.4, $mr_1 \notin P$ for all m in S. But $\frac{r_1 r_2}{s_1 s_2} \in S^{-1}P$ implies $\frac{r_1 r_2}{s_1 s_2} = \frac{a}{s}$ for some a in P and s in S.

$\Rightarrow t[sr_1r_2] = t[s_1s_2a] = (ts_1s_2)a \in P$ since P is an ideal and $a \in P$

$\Rightarrow (tsr_1)r_2 \in P$

$\Rightarrow r_2^2 \in P$ since $ts_1 \in S$, $mr_1 \notin P$ for all $m \in S$, hence $tsr_1 \notin P$ and P is primary.

$\Rightarrow \frac{r_2^2}{s_2^2} \in S^{-1}P$. Hence $S^{-1}P$ is a primary ideal in $S^{-1}R$.

Theorem 3.6 Let J be a primary ideal of $S^{-1}R$. Then $J^c = \{r \in R / f(r) \in J\}$ is also a primary ideal of R. Moreover $J^c \cap S = \emptyset$.

Proof. Clearly J^c is a proper ideal of R. Now let $r_1, r_2 \in R$ such that $r_1r_2 \in J^c$.

$\Rightarrow f(r_1r_2) \in J$

$\Rightarrow f(r_1)f(r_2) \in J$. $\Rightarrow f(r_1) \in J$ or $f(r_2^2) = [f(r_2)]^2 \in J$ since f is homomorphism and J is primary.

$\Rightarrow r_1 \in J^c$ or $r_2^2 \in J^c$. Hence J^c is primary. Now suppose that $J^c \cap S \neq \emptyset$. Then there exists $s \in J^c \cap S$. Letting $s'' = ss'$ then $\frac{s''}{s'} = \frac{ss'}{s'} = f(s) \in J$. Thus $\frac{s'}{s'} = (\frac{s''}{s'})(\frac{s'}{s''}) \in J$, which is a contradiction to the fact that J is primary and hence proper.

Theorem 3.7. Let P be an almost primary ideal of a Boolean like semi ring R and S be a multiplicative subset of R such that $P \cap S = \emptyset$. Then $S^{-1}P$ is an almost primary ideal of $S^{-1}R$.

Proof. Clearly $S^{-1}P$ is a proper ideal of $S^{-1}R$.

Let $t_1, t_2 \in S^{-1}R$ such that $t_1t_2 \in S^{-1}P - (S^{-1}P)^2$. Then $t_1 = \frac{r_1}{s_1}$, $t_2 = \frac{r_2}{s_2}$.

Then $t_1t_2 = \frac{r_1}{s_1}\frac{r_2}{s_2} \in S^{-1}P - (S^{-1}P)^2$. Now, if $t_1 = \frac{r_1}{s_1}$ lies in $S^{-1}P$ we are done.

Otherwise let $\frac{r_1}{s_1} \notin S^{-1}P$. Hence by theorem 3.4 we have $mr_1 \notin P$ for every m in S.

Finally we end this by the following

Theorem 3.8 Let J be an almost primary ideal of $S^{-1}R$. Then $J^c = \{r \in R / f(r) \in J\}$ is also an almost primary ideal of R. Moreover $J^c \cap S = \emptyset$.

Proof. in the similar lines of the above theorem and by the definition of almost primary ideal.

REFERENCES

[1] Murthy B.V.N. and Venkateswarlu. K. Contraction of ideals in Boolean like semi rings, International Journal of Mathematical Archive, 3(4) – 2012, pp1324 – 1328

[2] Foster. A. L. The theory of Boolean like rings, Trans. Amer. Math. Soc. Vol.59, 1946.

[3] Ketsela Hailu, Berhanu B, Zelelam.T and Venkateswarlu. K. Boolean like semi ring of fractions. International Journal of Mathematical Archive, 3(4) – 2012, pp 1554 – 1560

[4] Venkateswarlu. K and Murthy B.V.N and Amaranth N. Boolean like semi rings, Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 13, pp 619 – 635

[5] Venkateswarlu. K and Murthy B.V.N: Primary ideals in a Boolean like semi rings. Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 28, pp 1367 – 1377

[6] Venkateswarlu. K and Murthy B.V.N. Semi prime ideals and Annihilators in Boolean like semi rings, International Journal of Algebra, Vol. 5, 2011, no. 28, 1363 – 1370, Hikari Publishers, Bulgaria

[7] Venkateswarlu. K and Murthy B.V.N. Spectrum of Boolean like semi rings, International Journal of Mathematical Sciences and Applications, Vol. 1, No. 3, September 2011 pages 1083 – 1088.

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