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COMMON FIXED POINT THEOREMS OF COMPATIBLE MAPPINGS OF TYPE (A-1) AND TYPE (A-2)

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ABSTRACT

In this paper we prove some common fixed point theorems for compatible mappings of type (A-1) and type (A-2) in fuzzy metric spaces. Our results modify and extend some results in the literature.

Mathematics subject classification: *54H25, 54E50.*

Key words: *Compatible mappings, Compatible mappings of type (A), Compatible mappings of type (A-1), Compatible mappings of type (A-2), Common fixed point, Fuzzy metric space.*

1. INTRODUCTION

The first important result in the theory of fixed point of compatible mappings was obtained by Gerald Jungck in 1986 [8] as a generalization of commuting mappings. In 1993 Jungck, Murthy and Cho [9] introduced the concept of compatible mappings of type (A) by generalizing the definition of weakly uniformly contraction maps. Pathak and Khan [14] introduced the concept of A-compatible and S-compatible by splitting the definition of compatible mappings of type (A). Pathak *et. al.* [10] renamed A-compatible and S-compatible as compatible mappings of type (A-1) and compatible mappings of type (A-2) respectively and introduced it in fuzzy metric space.

Zadeh [20] introduced the concept of fuzzy sets. The idea of fuzzy metric space was introduced by Kramosil and Michalek [13] which was modified by George and Veeramani [3, 4]. Bijendra Singh and M. S. Chauhan [18] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems in fuzzy metric spaces in the sence of George and Veeramani with continuous t-norm * defined by $a*b=\min\{a, b\}$ for all $a, b\in [0,1]$.

The aim of the paper is to prove some common fixed point theorems of compatible mappings of type (A-1) and type (A-2). These results modify and extend the results in [10, 14, 15, 16, 19].

2. PRELIMINARIES

Definition 2.1. [17] A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfies the following conditions

(i) * is associative and commutative.
(ii) * is continuous.
(iii) $a*1 = a$ for all $a \in [0, 1]$.
(iv) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d\in [0, 1]$.

Definition 2.2. [3] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, * is continuous t-norm, and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions:
(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,

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(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(5) $M(x, y, .) : (0, \infty) \to [0,1]$ is continuous

for all $x, y, z \in X$ and $t, s > 0$.

Let (X, d) be a metric space, and let $a*b = ab$ or $a*b = \min\{a, b\}$. Let $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, M,*)$ is a fuzzy metric M induced by d is called the standard fuzzy metric space [3].

Definition 2.3. [5] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$), if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x, t) > 1-\varepsilon$ for all $n \geq n_0$.

The completeness and non completeness of fuzzy metric space was discussed in George and Veeramani [3] and M. Grabiec[5].

Definition 2.4. [3] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called Cauchy sequence if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1-\varepsilon$ for all $n, m \geq n_0$.

Definition 2.5. [10] Self mappings A and S of a fuzzy metric space $(X, M, *)$ is said to be compatible if $\lim_{n\to\infty} M(ASx_n, SAx_n\ t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 2.6 [10] Self mappings A and S of a fuzzy metric space $(X, M, *)$ is said to be compatible of type (A) if $\lim_{n\to\infty} M(ASx_n, SSx_n, t) = \lim_{n\to\infty} M(SAx_n, AAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 2.7. [10] Self mappings A and S of a fuzzy metric space $(X, M, *)$ is said to be compatible of type (A-1) if $\lim_{n\to\infty} M(SAx_n, AAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 2.8. [10] Self mappings A and S of a fuzzy metric space $(X, M, *)$ is said to be compatible of type (A-2) if $\lim_{n\to\infty} M(ASx_n, SSx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Lemma 2.9. [5] Let $(X, M, *)$ be a fuzzy metric space. Then for all x, y in X, $M(x, y, .)$ is non-decreasing.

Lemma 2.10. [19] Let $(X, M, *)$ be a fuzzy metric space. If there exists $q \in (0, 1)$ such that $M(x, y, qt) \geq M(x, y, t/q^n)$ for positive integer n. Taking limit as $n \to \infty$, $M(x, y, t) \geq 1$ and hence $x = y$.

Lemma 2.11. [12] The only t-norm $*$ satisfying $r*r \geq r$ for all $r \in [0,1]$ is the minimum t-norm, that is, $a*b = \min\{a, b\}$ for all $a,b \in [0,1]$.

Proposition 2.12. [10] Let $(X, M, *)$ be a fuzzy metric space and let A and S be continuous mappings of X then A and S are compatible if and only if they are compatible of type (A).

Proposition 2.13. [10] Let $(X, M, *)$ be a fuzzy metric space and let A and S be compatible mappings of type (A-1) and $Az=Sz$ for some $z \in X$, then $SAz=AAz$.

Proposition 2.14. [10] Let $(X, M, *)$ be a fuzzy metric space and let A and S be compatible mappings of type (A-1) and $Az=Sz$ for some $z \in X$, then $ASz=SSz$.

Proposition 2.15. [10] Let $(X, M, *)$ be a fuzzy metric space and let A and S be compatible mappings of type (A-1) and let $Ax_n, Sx_n \to z$ as $n \to \infty$ for some $z \in X$ then $AAx_n \to Sz$ if S is continuous at z.

Proposition 2.16. [10] Let $(X, M, *)$ be a fuzzy metric space and let A and S be compatible mappings of type (A-2) and let $Ax_n, Sx_n \to z$ as $n \to \infty$ for some $z \in X$ then $SSx_n \to Az$ if A is continuous at z.

3. MAIN RESULTS

We prove the following theorem.

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be a self mappings of X satisfying the following conditions:*

(*i*) $A(X) \subset T(X)$, $B(X) \subset S(X)$,
(*ii*) *S and T are continuous*
(*iii*) *the pairs{A ,S} and {B, T} are compatible mapping of type (A-1) on X.*
(*iv*) *there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$,*

$M(Ax, By, qt) \geq M(Sx, Ty ,t)*M(Ax, Sx , t *M(By, Ty, t)*M(Ax, Ty ,t)$

Then A, B, S and T have a unique common point in X.

Proof: Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, for any $x_0 \in X$, there exists $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this $x_1 \in X$, there exists $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we define a sequence $\{y_n\}$ in X such that

$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$, for all $n = 0, 1, 2, \ldots\ldots$

From (iv),
$M(y_{2n+1}, y_{2n+2}, qt) = M(Ax_{2n}, Bx_{2n+1}, qt)$.

$$\geq M(Sx_{2n}, Tx_{2n+1}, t)*M(Ax_{2n}, Sx_{2n}, t)*M(Bx_{2n+1}, Tx_{2n+1}, t)*M(Ax_{2n},Tx_{2n+1}, t)$$

$$= M(y_{2n}, y_{2n+1}, t)*M(y_{2n+1}, y_{2n}, t)*M(y_{2n+2}, y_{2n+1}, t)*M(y_{2n+1}, y_{2n+1}, t)$$

$$\geq M(y_{2n}, y_{2n+1}, t)*M(y_{2n+1}, y_{2n+2}, t)$$

From lemma 2.9 and 2.10, we have

$$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t) \quad (3.1.1)$$

Similarly, we have

$$M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t) \quad (3.1.2)$$

From (3.1.1) and (3.1.2), we have

$$M(y_{n+1}, y_{n+2}, qt) \geq M(y_n, y_{n+1}, t) \quad (3.1.3)$$

From (3.1.3), we have

$$\begin{aligned} M(y_n, y_{n+1}, t) &\geq M(y_n, y_{n-1}, \tfrac{t}{q}) \\ &\geq M(y_{n-2}, y_{n-1}, \tfrac{t}{q^2}) \\ &\geq \ldots\ldots\ldots.. \geq M(y_1, y_2, \tfrac{t}{q^n}) \to 1 \text{ as } n \to \infty. \end{aligned}$$

So, $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$ for any $t > 0$. For each $\varepsilon > 0$ and each $t > 0$, we can choose $n_0 \in \mathbf{N}$ such that

$M(y_n, y_{n+1}, t) > 1 - \varepsilon$ for all $n > n_0$.

For $m, n \in \mathbf{N}$, we suppose $m \geq n$. Then we have that

$$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, \tfrac{t}{m-n})*M(y_{n+1}, y_{n+2}, \tfrac{t}{m-n})* \ldots\ldots\ldots\ldots..*M(y_{m-1}, y_m, \tfrac{t}{m-n})$$

$\geq (1-\varepsilon)*(1-\varepsilon)* \ldots\ldots\ldots\ldots..(m–n)$ times.

$\geq (1-\varepsilon)$

and hence $\{y_n\}$ is a Cauchy sequence in X.

Since $(X, M, *)$ is complete, $\{y_n\}$ converges to some point $z \in X$, and so

$\{Ax_{2n-2}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z.

From proposition 2.15 and (iii), we have

$$AAx_{2n-2} \to Sz \tag{3.1.4}$$

and $BBx_{2n-1} \to Tz$ (3.1.5)

From (iv), we get

$M(AAx_{2n-2}, BBx_{2n-1}, qt) \geq M(SAx_{2n-2}, TBx_{2n-1}, t)*M(AAx_{2n-2}, SAx_{2n-2}, t)*M(BBx_{2n-1}, TBx_{2n-1}, t)* M(AAx_{2n-2}, TBx_{2n-1}, t)$

Taking limit as $n \to \infty$ and using (3.1.4) and (3.1.5), we have

$M(Sz, Tz, qt) \geq M(Sz, Tz, t)*M(Sz, Sz, t)*M(Tz, Tz, t)*M(Sz, Tz, t)$

$\geq M(Sz, Tz, t)* 1* 1*M(Sz, Tz, t)$

$\geq M(Sz, Tz, t)$.

It follows that $Sz = Tz$. (3.1.6)

Now, from (iv),

$M(Az, BBx_{2n-1}, qt) \geq M(Sz, TBx_{2n-1}, t)*M(Az, Sz, t)*M(BBx_{2n-1}, TBx_{2n-1}, t)*M(Az, TBx_{2n-1}, t)$

Again, taking limit as $n \to \infty$ and using (3.1.5) and (3.1.6), we have

$M(Az, Tz, qt) \geq M(Sz, Sz, t)*M(Az, Tz, t)*M(Az, Tz, t)*M(Az, Tz, t)$

$\geq M(Az, Tz, t)$.

and hence $Az = Tz$. (3.1.7)

From (iv), (3.1.6) and (3.1.7),

$M(Az, Bz, qt) \geq M(Sz, Tz, t)*M(Az, Sz, t)*M(Bz, Tz, t)*M(Az, Tz, t)$

$= M(Az, Az, t)*M(Az, Az, t)*M(Bz, Az, t)*M(Az, Az, t)$

$\geq M(Az, Bz, t)$.

and hence $Az = Bz$. (3.1.8)

From (3.1.6), (3.1.7) and (3.1.8), we have

$$Az = Bz = Tz = Sz. \tag{3.1.9}$$

Now, we show that $Bz = z$.

From (iv),

$M(Ax_{2n}, Bz, qt) \geq M(Sx_{2n}, Tz, t)*M(Ax_{2n}, Sx_{2n}, t)*M(Bz, Tz, t)*M(Ax_{2n}, Tz, t)$

And, taking limit as $n \to \infty$ and using (3.1.6) and (3.1.7), we have

$$M(z, Bz, qt) \geq M(z, Tz, t)*M(z, z, t)*M(Bz, Tz, t)*M(z, Tz, t)$$

$$= M(z, Bz, t)*1*M(Az, Az, t)*M(z, Bz, t)$$

$$\geq M(z, Bz, t).$$

And hence $Bz = z$. Thus from (3.1.9), $z = Az = Bz = Tz = Sz$ and z is a common fixed point of A, B, S and T.

In order to prove the uniqueness of fixed point, let w be another common fixed point of A, B, S and T. Then

$$M(z, w, qt) = M(Az, Bw, qt)$$

$$\geq M(Sz, Tw, t)*M(Az, Sz, t)*M(Bw, Tw, t)*M(Az, Tw, t)$$

$$\geq M(z, w, t).$$

From lemma 2.10, $z = w$. This completes the proof of theorem.

Corollary 3.2. *Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be a self mappings of X satisfying (i) - (iii) of theorem 3.1 and there exists $q \in (0,1)$ such that*

$$M(Ax, By, qt) \geq M(Sx, Ty, t)*M(Ax, Sx, t)*M(By, Ty, t)*M(By, Sx, 2t)*M(Ax, Ty, t)$$

for every $x, y \in X$ and $t > 0$. Then A, B, S and T have a unique common point in X.

Corollary 3.3. *Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be a self mappings of X satisfying (i) - (iii) of theorem3.1 and there exists $q \in (0,1)$ such that*

$$M(Ax, By, qt) \geq M(Sx, Ty, t)$$

for every $x, y \in X$ and $t > 0$. Then A, B, S and T have a unique common point in X.

Corollary 3.4. *Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be a self mappings of X satisfying (i) - (iii) of theorem3.1 and there exists $q \in (0,1)$ such that*

$$M(Ax, By, qt) \geq M(Sx, Ty, t)*M(Sx, Ax, t)*M(Ax, Ty, t)$$

for every $x, y \in X$ and $t > 0$. Then A, B, S and T have a unique common point in X.

Theorem 3.5. *Let $(X, M, *)$ be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping A of X such that the following condition are satisfied:*

(*i*) $A(X) \subset T(X) \cap S(X)$,
(*ii*) *the pairs $\{A, S\}$ and $\{A, T\}$ are compatible mapping of type $(A\text{-}1)$ on X,*
(*iii*) *there exists $q \in (0,1)$ such that for every $x, y \in X$ and $t > 0$,*

$$M(Ax, Ay, qt) \geq M(Sx, Ty, t)*M(Ax, Sx, t)*M(Ay, Ty, t)*M(Ax, Ty, t)$$

In fact A, S and T have a unique common point in X.

Proof: We shown that the necessity of the conditions (i)-(iii). Suppose that S and T have a common fixed point in X, say z. Then $Sz = z = Tz$.

Let $Ax = z$ for all $x \in X$. Then we have $A(X) \subset T(X) \cap S(X)$ and we know that $[A, S]$ and $[A, T]$ are compatible mapping of type (A-1), in fact $A \circ S = S \circ A$ and $A \circ T = T \circ A$, and hence the conditions (i) and (ii) are satisfied.

For some $q \in (0, 1)$, we get

$$M(Ax, Ay, qt) = 1 \geq M(Sx, Ty, t)*M(Ax, Sx, t)*M(Ay, Ty, t)*M(Ax, Ty, t)$$

for every $x, y \in X$ and $t > 0$ and hence Tthe condition (iii) is satisfied.

Now, for the sufficiency of the conditions, let $A = B$ in theorem 3.1. Then A, S and T have a unique common fixed point in X.

Corollary 3.6. *Let $(X, M, *)$ be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping A of X satisfying (i) – (ii) of theorem 3.5 and there exists $q \in (0,1)$ such that for every $x, y \in X$ and $t > 0$*

$$M(Ax, Ay, qt) \geq M(Sx, Ty, t)*M(Ax, Sx, t)*M(Ay, Ty, t)*M(Ax,Sx,2t)*M(Ax, Ty, t)$$

In fact A, S and T have a unique common point in X.

Corollary 3.7. *Let $(X, M, *)$ be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping A of X satisfying (i) – (ii) of theorem 3.5 and there exists $q \in (0,1)$ such that for every $x, y \in X$ and $t > 0$*

$$M(Ax, Ay, qt) \geq M(Sx, Ty, t)$$

In fact A, S and T have a unique common point in X.

Corollary 3.8. *Let $(X, M, *)$ be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping A of X satisfying (i) – (ii) of theorem 3.5 and there exists $q \in (0,1)$ such that for every $x, y \in X$ and $t > 0$*

$$M(Ax, Ay, qt) \geq M(Sx, Ty, t)*M(Sx, Ax, t)*M(Ax, Ty, t)$$

In fact A, S and T have a unique common point in X.

Remark: Corresponding results for compatible mappings of type (A-2) can also be obtained.

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