ON THE STABILITY OF A FOUR SPECIES SYN ECO-SYSTEM WITH COMMENSAL PREY-PREDATOR PAIR WITH PREY-PREDATOR PAIR OF HOSTS-II
(1st Level Prey-Predator Washed Out States)

B. Hari Prasad1* & N. Ch. Pattabhi Ramacharyulu2

1Dept. of Mathematics, Chaitanya Degree College (Autonomous), Hanamkonda, India
2Former Faculty, Dept. of Mathematics, NIT Warangal, India

(Received on: 07-07-12; Accepted on: 30-07-12)

ABSTRACT

The present paper is devoted to an investigation on a Four Species (S 1, S 2, S 3, S 4) Syn Eco-System with Commensal Prey-Predator pair with Prey-Predator pair of Hosts [Prey (S 1) and Predator (S 2) Washed Out States]. The System comprises of a Prey (S 1), a Predator (S 2) that survives upon S 1, two Hosts S 3 and S 4 for which S 1, S 2 are Commensal respectively i.e., S 3 and S 4 benefit S 1 and S 2 respectively, without getting effected either positively or adversely. Further S 3 is Prey for S 4 and S 4 is Predator for S 3. The pair (S 3, S 4) may be referred as 1st level Prey-Predator and the pair (S 1, S 2) the 2nd level Prey-Predator. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of three of these sixteen equilibrium points: 1st Level Prey-Predator Washed Out States are established. The system would be stable if all the characteristic roots are negative, in case they are real, and have negative real parts, in case they are complex. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories are illustrated.

Keywords: Commensal, Eco-System, Equilibrium point, Host, Prey, Predator, Stable, Trajectories.

AMS Classification: 92D25, 92D40.

1. INTRODUCTION

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [1] and by Volterra [2]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Meyer [3], Kushing [4], Paul colinvaux [5], Kapur [6] etc. The ecological interactions can be broadly classified as Prey-predation, Competition, Commensalism, Ammensalism, Neutralism and so on. N.C. Srinivas [7] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later, Lakshminarayan [8], Lakshminarayan and Pattabhi Ramacharyulu [9] studied Prey-predator ecological models with a partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Krishna Gandhi [10] and by Bhaskara Rama Sarma and Pattabhi Ramacharyulu [11], while Ravindra Reddy [12] investigated mutualism between two species. Recently Phani Kumar [13] studied some mathematical models of ecological commensalism. More recently the criteria for a four species syn eco-system was discussed at length by the present authors [14-25].

A Schematic Sketch of the system under investigation is shown here under Fig.1.

![Schematic Sketch of the Syn Eco-System](image-url)

Fig. 1: Schematic Sketch of the Syn Eco-System
2. BASIC EQUATIONS:

The model equations for a four species syn eco-system is given by the following system of first order non-linear ordinary differential equations employing the following notation:

Notation:

- **S<sub>i</sub>** : Prey for S<sub>1</sub> and commensal for S<sub>3</sub>.
- **S<sub>2</sub>** : Predator surviving upon S<sub>1</sub> and commensal for S<sub>4</sub>.
- **S<sub>3</sub>** : Host for the commensal (S<sub>1</sub>) and Prey for S<sub>4</sub>.
- **S<sub>4</sub>** : Host of the commensal (S<sub>2</sub>) and Predator surviving upon S<sub>4</sub>.
- **Ni(t)** : The Population strength of S<sub>i</sub> at time t, i = 1, 2, 3, 4
- **t** : Time instant
- **a<sub>i</sub>** : Natural growth rate of S<sub>i</sub>, i = 1, 2, 3, 4
- **aii** : Self inhibition coefficient of S<sub>i</sub>, i = 1, 2, 3, 4
- **a<sub>12</sub>, a<sub>21</sub>** : Interaction (Prey-Predator) coefficients of S<sub>1</sub> due to S<sub>2</sub> and S<sub>2</sub> due to S<sub>1</sub>
- **a<sub>34</sub>, a<sub>43</sub>** : Interaction (Prey-Predator) coefficients of S<sub>3</sub> due to S<sub>4</sub> and S<sub>4</sub> due to S<sub>3</sub>
- **a<sub>13</sub>, a<sub>24</sub>** : Coefficients for commensal for S<sub>1</sub> due to the Host S<sub>3</sub> and S<sub>2</sub> due to the Host S<sub>4</sub>

Further the variables N<sub>i</sub>, i = 1, 2, 3, 4 are non-negative and the model parameters a<sub>i</sub>, i = 1, 2, 3, 4; a<sub>ii</sub>, i = 1, 2, 3, 4; a<sub>12</sub>, a<sub>21</sub>, a<sub>13</sub>, a<sub>24</sub>, a<sub>34</sub>, a<sub>43</sub> are assumed to be non-negative constants.

The model equations for the growth rates of S<sub>1</sub>, S<sub>2</sub>, S<sub>3</sub>, S<sub>4</sub> are

\[
\frac{dN_i}{dt} = a_i N_i - a_{11} N_1^2 - a_{12} N_1 N_2 + a_{13} N_1 N_3, \quad i = 1, 2, 3, 4
\]

(2.1)

\[
\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{23} N_1 N_2 + a_2 N_2 N_4
\]

(2.2)

\[
\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 - a_{34} N_3 N_4
\]

(2.3)

\[
\frac{dN_4}{dt} = a_4 N_4 - a_{44} N_4^2 + a_{43} N_3 N_4
\]

(2.4)

3. EQUILIBRIUM STATES:

The system under investigation has sixteen equilibrium states defined by

\[
\frac{dN_i}{dt} = 0, \quad i = 1, 2, 3, 4
\]

(3.1)

as given in the following Table-1.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Equilibrium State</th>
<th>Equilibrium Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Fully Washed out state</td>
<td>( N'_1 = 0, N'_2 = 0, N'_3 = 0, N'_4 = 0 )</td>
</tr>
<tr>
<td>2*</td>
<td>Only the Host (S&lt;sub&gt;4&lt;/sub&gt;) of S&lt;sub&gt;2&lt;/sub&gt; survives</td>
<td>( N'_1 = 0, N'_2 = 0, N'_3 = 0, N'<em>4 = \frac{a_4}{a</em>{44}} )</td>
</tr>
<tr>
<td>3*</td>
<td>Only the Host (S&lt;sub&gt;3&lt;/sub&gt;) of S&lt;sub&gt;1&lt;/sub&gt; survives</td>
<td>( N'_1 = 0, N'<em>2 = 0, N'<em>3 = \frac{a</em>{13}}{a</em>{33}}, N'_4 = 0 )</td>
</tr>
<tr>
<td>4</td>
<td>Only the Predator (S&lt;sub&gt;2&lt;/sub&gt;) survives</td>
<td>( N'<em>1 = 0, N'<em>2 = \frac{a</em>{22}}{a</em>{22}}, N'_3 = 0, N'_4 = 0 )</td>
</tr>
<tr>
<td>No.</td>
<td>Scenario Description</td>
<td>Stability Analysis</td>
</tr>
<tr>
<td>-----</td>
<td>-------------------------------------------------------------------------------------</td>
<td>-------------------------------------------------------------------------------------</td>
</tr>
</tbody>
</table>
| 5   | Only the Prey (S₁) survives                                                        | \[
N₁ = \frac{a₁}{a_{i₁}}, \quad N₂ = 0, \quad N₃ = 0, \quad N₄ = 0
\] |
| 6   | Prey (S₁) and Predator (S₂) washed out                                              | \[
N₁ = 0, \quad N₂ = 0, \quad N₃ = \frac{α}{β}, \quad N₄ = \frac{γ}{β}
\]
where \[
α = a₂a_{4₄} - a₄a₂, \quad β = a₃₃a_{4₄} + a₃₄a₄ > 0
γ = a₄a₂ + a₄a₃₄ > 0
\]
| 7   | Prey (S₁) and Host (S₃) of S₁ washed out                                            | \[
N₁ = 0, \quad N₂ = \frac{δ₁}{a₄₃₄}, \quad N₃ = 0, \quad N₄ = \frac{a₄}{a₄₄}
\]
where \[
δ₁ = a₂a₄₄ + a₄a₂₄ > 0
\]
| 8   | Prey (S₁) and Host (S₄) of S₂ washed out                                            | \[
N₁ = 0, \quad N₂ = \frac{a₂}{a_{2₂}}, \quad N₃ = \frac{a₂}{a₃₃}, \quad N₄ = 0
\] |
| 9   | Predator (S₂) and Host (S₃) of S₁ washed out                                         | \[
N₁ = \frac{a₃}{a_{i₁}}, \quad N₂ = 0, \quad N₃ = \frac{α}{β}, \quad N₄ = \frac{γ}{β}
\] |
| 10  | Predator (S₂) and Host (S₄) of S₂ washed out                                         | \[
N₁ = \frac{δ₂}{a₃₃}, \quad N₂ = 0, \quad N₃ = \frac{α}{β}, \quad N₄ = 0
\]
where \[
δ₂ = a₃₃a₃₃ + a₃₄a₃₄ > 0
\]
| 11  | Prey (S₁) and Predator (S₂) survives                                                | \[
N₁ = \frac{α}{β₃}, \quad N₂ = \frac{λ₁}{β₃}, \quad N₃ = 0, \quad N₄ = 0
\]
where \[
α₁ = a₃₃a₃₃ - a₃₄a₂, \quad β₁ = a₃₃a₂₄ + a₃₄a₃₄ > 0
γ₁ = a₃₃a₂₃ + a₃₄a₃₄ > 0
\]
| 12  | Only the Prey (S₁) washed out                                                        | \[
N₁ = 0, \quad N₂ = \frac{αβ}{a_{i₁}}, \quad N₃ = \frac{α}{β}, \quad N₄ = \frac{γ}{β}
\] |
| 13  | Only the predator (S₂) washed out                                                    | \[
N₁ = \frac{a₁β + a₂α}{a_{i₁}β}, \quad N₂ = 0, \quad N₃ = \frac{α}{β}, \quad N₄ = \frac{γ}{β}
\] |
| 14  | Only the Host (S₃) of S₁ washed out                                                  | \[
N₁ = \frac{a₂₃₃a₄₄ + a₃₃δ₁}{a₄₄β₃}, \quad N₂ = \frac{a₂₃₃a₄₄ + a₃₃δ₁}{a₄₄β₃}
\]
\[
N₃ = 0, \quad N₄ = \frac{a₄}{a₄₄}
\] |
| 15  | Only the Host (S₄) of S₂ washed out                                                  | \[
N₁ = \frac{a₂₃₃δ₂ - a₃₃a₂₄}{a₃₃β₃}, \quad N₂ = \frac{a₂₃₃δ₂ + a₂₄a₃₃}{a₃₃β₃}
\]
\[
N₃ = \frac{a₃₃}{a₄₃₃}, \quad N₄ = 0
\] |
| 16  | The co-existent state (or) Normal steady state                                       | \[
N₁ = \frac{a₂₃α₂ - a₁₂γ₂}{β₁}, \quad N₂ = \frac{a₁₂γ₂ + a₂₃α₂}{β₁}
\]
\[
N₃ = \frac{α}{β}, \quad N₄ = \frac{γ}{β}
\]
where \[
α₂ = a₁₃ + a₃₃α, \quad γ₂ = a₂₄ + a₄₃₄γ > 0
\]
The present paper deals with the 1st level Prey-Predator washed out states only (Sr. Nos. 2, 3, 6 marked * in the above Table -1). The stability of the other equilibrium states were published several National and International Journals.

4. STABILITY OF THE EQUILIBRIUM STATES:

Let \( N = (N_1, N_2, N_3, N_4) = \bar{N} + U \) (4.1)

where \( U = (u_1, u_2, u_3, u_4) \) is a perturbation over the equilibrium state \( \bar{N} = (\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4) \).

The basic equations (2.1), (2.2), (2.3), (2.4) are quasi-linearized to obtain the equations for the perturbed state.

\[
\frac{dU}{dt} = AU
\]

(4.2)

where

\[
A = \begin{bmatrix}
-2a_1\bar{N}_1 & -a_{12}\bar{N}_2 & a_{13}\bar{N}_3 & 0 \\
a_2 & -2a_2\bar{N}_2 & a_{23}\bar{N}_3 & a_{24}\bar{N}_4 \\
0 & a_3 & -2a_3\bar{N}_3 & a_{34}\bar{N}_4 \\
0 & 0 & a_4 & -2a_4\bar{N}_4 + a_{43}\bar{N}_3 \\
\end{bmatrix}
\]

(4.3)

The characteristic equation for the system is

\[
\det(A - \lambda I) = 0
\]

(4.4)

The equilibrium state is stable, if all the four roots of the equation (4.4) are negative, in case they are real or have negative real parts, in case they are complex.

5. STABILITY OF THE 1ST LEVEL PREY-PREDATOR WASHED OUT EQUILIBRIUM STATES: (SL. NOS. 2,3,6 MARKED * IN TABLE .1)

5.1 Equilibrium point \( \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0, \bar{N}_4 = \frac{a_4}{a_{44}} \):

The corresponding linearized equations for the perturbations \( u_1, u_2, u_3, u_4 \) are

\[
\frac{du_1}{dt} = a_1u_1, \quad \frac{du_2}{dt} = a_2u_2 \\
\frac{du_3}{dt} = a_3u_3, \quad \frac{du_4}{dt} = a_4u_4
\]

(5.1.1)

(5.1.2)

Here \( p_2 = a_2 + k_4a_{24} > 0 \), \( p_3 = a_3 - k_4a_{34} \)

(5.1.3)

The characteristic equation for which is

\[
(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda + a_4) = 0
\]

(5.1.4)

Two of the four roots \( a_1, a_2 \) are positive and \( a_4 \) is negative. Hence the state is unstable.

Case (A): If \( p_3 > 0 \) (ie, \( a_3 > k_4a_{34} \))

The solutions of the equations (5.1.1), (5.1.2) are

\[
u_1 = u_{10}e^{a_1t}, \quad u_2 = u_{20}e^{a_2t}, \quad u_3 = u_{30}e^{a_3t}, \quad u_4 = (u_{40} - P)e^{-a_4t} + Pe^{a_4t}
\]

(5.1.5)

(5.1.6)

where \( P = \frac{k_4a_{44}u_{30}}{p_3 + a_4} \)

(5.1.7)

and \( u_{10}, u_{20}, u_{30}, u_{40} \) are the initial values of \( u_1, u_2, u_3, u_4 \) respectively.

In the three equilibrium states, there would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates \( a_1, a_2, a_3, a_4 \) and the initial values of the perturbations \( u_{10}(t), u_{20}(t), u_{30}(t), u_{40}(t) \) of the
species $S_1, S_2, S_3, S_4$. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations. The solution curves are illustrated in Figures (2) to (4) and the conclusions are presented here.

Case (i): If $u_{10} < u_{30} < u_{40} < u_{20}$ and $a_1 < a_4 < p_3 < p_2$

In this case the natural birth rates of the Prey $(S_1)$, Host $(S_4)$ of $S_2$, Host $(S_3)$ of $S_1$ and the Predator $(S_2)$ are in ascending order. Initially the Host $(S_4)$ of $S_2$ dominates over the Host $(S_3)$ of $S_1$ till the time instant $t^{*}_{34}$ and thereafter the dominance is reversed. The time $t^{*}_{34}$ may be called the dominance time of $S_4$ over $S_3$.

Here $t^{*}_{34} = \frac{1}{p_3 + a_4} \log \left( \frac{u_{40} - P}{u_{30} - P} \right)$  \hspace{1cm} (5.1.8)

Case (ii): If $u_{30} < u_{10} < u_{20} < u_{40}$ and $a_1 < p_2 < a_4 < p_3$

In this case the natural birth rates of the Prey $(S_1)$, Predator $(S_2)$, Host $(S_4)$ of $S_2$ and the Host $(S_3)$ of $S_1$ are in ascending order. Initially the Host $(S_4)$ of $S_2$, Predator $(S_2)$, Prey $(S_1)$ dominates over the Host $(S_3)$ of $S_1$ till the time instant $t^{*}_{34}, t^{*}_{32}, t^{*}_{31}$ respectively and thereafter the dominance is reversed.

Here $t^{*}_{32} = \frac{1}{p_2 - p_3} \log \left( \frac{u_{30}}{u_{20}} \right)$, $t^{*}_{31} = \frac{1}{a_1 - p_3} \log \left( \frac{u_{40}}{u_{10}} \right)$ \hspace{1cm} (5.1.9)

Case (iii): If $u_{40} < u_{10} < u_{20} < u_{30}$ and $p_2 < a_1 < p_3 < a_4$

In this case the natural birth rates of the Predator $(S_2)$, Prey $(S_1)$, Host $(S_3)$ of $S_1$ and the Host $(S_4)$ of $S_2$ are in ascending order. Initially the Host $(S_3)$ of $S_1$, Predator $(S_2)$, Prey $(S_1)$ dominates over the Host $(S_4)$ of $S_2$ till the time instant $t^{*}_{34}, t^{*}_{32}, t^{*}_{41}$ respectively and thereafter the dominance is reversed. Also the Predator $(S_2)$ dominates over the Prey $(S_1)$ till the time instant $t^{*}_{12}$ and the dominance gets reversed thereafter.

Here $t^{*}_{12} = \frac{1}{a_1 - p_2} \log \left( \frac{u_{20}}{u_{10}} \right)$  \hspace{1cm} (5.1.10)

Case (B): If $p_3 < 0$ (ie, $a_3 < k_4 a_{34}$)

The solutions in this case are some as in case (A) and the solution curves are illustrated in Figures (5) to (8).

Case (i): If $u_{10} < u_{30} < u_{20} < u_{40}$ and $a_1 < p_2 < p_3 < a_4$

In this case the natural birth rates of the Host $(S_1)$ of $S_1$, Host $(S_4)$ of $S_2$, Prey $(S_1)$ and the Predator $(S_2)$ are in ascending order. Initially the Host $(S_4)$ of $S_2$ dominates over the Predator $(S_2)$, Prey $(S_1)$ till the time instant $t^{*}_{24}, t^{*}_{14}$ respectively and thereafter the dominance is reversed. Also the Host $(S_3)$ of $S_1$ dominates over the Prey $(S_1)$ till the time instant $t^{*}_{13}$ and the dominance gets reversed thereafter.

Case (ii): If $u_{20} < u_{40} < u_{10} < u_{30}$ and $p_2 < p_3 < a_4 < a_1$

In this case the natural birth rates of the Host $(S_1)$ of $S_1$, Host $(S_4)$ of $S_2$, Predator $(S_2)$ and the Prey $(S_1)$ are in ascending order. Initially the Host $(S_3)$ of $S_1$, Host $(S_4)$ of $S_2$ dominates over the Predator $(S_2)$ till the time
instant $t_{23}^*$, $t_{24}^*$ respectively and thereafter the dominance is reversed. Also the Host $(S_3)$ of $S_1$ dominates over the Host $(S_4)$ of $S_2$ till the time instant $t_{43}^*$ and the dominance gets reversed thereafter.

Case (iii): If $u_{30} < u_{20} < u_{40} < u_{10}$ and $p_3 < a_1 < p_2 < a_4$

In this case the natural birth rates are same as in case (i). Initially the Prey $(S_1)$, Host $(S_4)$ of $S_2$ dominates over the Predator $(S_2)$ till the time instant $t_{21}^*$, $t_{24}^*$ respectively and thereafter the dominance is reversed.

Case (iv): If $u_{40} < u_{30} < u_{20} < u_{10}$ and $p_2 < a_1 < a_4 < p_3$

In this case the Host $(S_4)$ of $S_2$ has the least natural birth rate and the Prey $(S_1)$ dominates the Predator $(S_2)$, Host $(S_3)$ of $S_1$, Host $(S_4)$ of $S_2$ in natural growth rate as well as in its population strength.

5.1. Trajectories of Perturbations:

The trajectories in the $u_1 - u_2$ plane given by $x^{p_2} = y_1^{u_1}$ and are shown in Fig.9 and the trajectories in the other planes are

$$x^{p_3} = y_1^{p_2}, y_1^{p_3} = y_2^{p_2}, y_3 = (1-A)x^{u_1} + A x^{u_2},$$

$$y_3 = (1-A)y_1^{p_2} + Ay_1^{p_2}, y_3 = (1-A)y_2^{p_3} + Ay_2^{p_3},$$

where $x = \frac{u_1}{u_{10}}, y_1 = \frac{u_2}{u_{20}}, y_2 = \frac{u_3}{u_{30}}, y_3 = \frac{u_4}{u_{40}}, A = \frac{P}{u_{40}}$.

5.2. Equilibrium point $N_1 = 0, N_2 = 0, N_3 = \frac{a_3}{a_{33}}, N_4 = 0$:

The corresponding linearized equations for the perturbations $u_1, u_2, u_3, u_4$ are

$$\frac{du_1}{dt} = p_1u_1, \quad \frac{du_2}{dt} = a_2u_2,$$

$$\frac{du_3}{dt} = -a_3u_3 - k_3a_3 u_4, \quad \frac{du_4}{dt} = a_4u_4$$

Here $p_1 = a_1 + k_3a_3 > 0$

The characteristic equation for which is

$$(\lambda - p_1)(\lambda - a_2)(\lambda + a_3)(\lambda - a_4) = 0$$

The roots $p_1, a_2, a_4$ are positive and $-a_3$ is negative. Hence the state is unstable and the solutions of the equations (5.2.1), (5.2.2) are

$$u_1 = u_{10} e^{p_1 t}, \quad u_2 = u_{20} e^{a_2 t},$$

$$u_3 = (u_{30} + Q)e^{-a_3 t} - Q e^{a_3 t}, \quad u_4 = u_{40} e^{a_4 t}$$

where $Q = \frac{k_3a_3 u_{40}}{a_3 + a_4}$.

The solution curves are illustrated in Figures (10) to (12) and the conclusions are presented here.

Case (i): If $u_{10} < u_{20} < u_{40} < u_{30}$ and $a_2 < p_1 < a_3 < a_4$.

In this case the natural birth rates of the Predator $(S_2)$, Prey $(S_1)$, Host $(S_3)$ of $S_1$ and the Host $(S_4)$ of $S_2$ are in ascending order. Initially the Predator $(S_2)$ dominates over the Prey $(S_1)$ till the time instant $t_{12}^*$ and thereafter the
dominance is reversed. Also the Host \( S_3 \) of \( S_1 \) dominates over the Host \( S_4 \) of \( S_2 \) till the time instant \( t_{43}^* \) and the dominance gets reversed thereafter.

Here \( t_{12}^* = \frac{1}{p_1 - a_2} \log \left( \frac{u_{20}}{u_{10}} \right) \), \( t_{43}^* = \frac{1}{a_3 + a_4} \log \left( \frac{u_{30} + Q}{u_{40} + Q} \right) \) \hspace{1cm} (5.2.8)

**Case (ii):** If \( u_{30} < u_{10} < u_{40} < u_{20} \) and \( p_1 < a_2 < a_3 < a_4 \)

In this case the natural birth rates of the Prey \( S_1 \), Predator \( S_2 \), Host \( S_3 \) of \( S_1 \) and the Host \( S_4 \) of \( S_2 \) are in ascending order. Initially the Predator \( S_2 \) dominates over the Host \( S_3 \) of \( S_1 \), Host \( S_4 \) of \( S_2 \) till the time instant \( t_{32}^* \), \( t_{42}^* \) respectively and thereafter the dominance is reversed. Also the Prey \( S_1 \) dominates its Host \( S_3 \) till the time instant \( t_{31}^* \) and the dominance gets reversed thereafter.

Here \( t_{42}^* = \frac{1}{a_2 - a_4} \log \left( \frac{u_{40}}{u_{20}} \right) \) \hspace{1cm} (5.2.9)

**Case (iii):** If \( u_{40} < u_{20} < u_{30} < u_{10} \) and \( a_3 < p_1 < a_4 < a_2 \)

In this case the natural birth rates of the Host \( S_3 \) of \( S_1 \), Prey \( S_1 \), Host \( S_4 \) of \( S_2 \) and the Predator \( S_2 \) are in ascending order. Initially the Prey \( S_1 \) dominates over the Predator \( S_2 \), Host \( S_4 \) of \( S_2 \) till the time instant \( t_{21}^* \), \( t_{41}^* \) respectively and thereafter the dominance is reversed. Also the Host \( S_3 \) of \( S_1 \) dominates over the Predator \( S_2 \), Host \( S_4 \) of \( S_2 \) till the time instant \( t_{23}^* \), \( t_{43}^* \) respectively and the dominance gets reversed thereafter.

Here \( t_{41}^* = \frac{1}{p_1 - a_4} \log \left( \frac{u_{40}}{u_{10}} \right) \) \hspace{1cm} (5.2.10)

5.2.A. **Trajectories of Perturbations:**

The trajectories in the \( u_1 - u_2 \) plane gives by \( x^{a_2} = y_1^{n_1} \) \hspace{1cm} (5.2.11)

and are shown in Fig.13 and the trajectories in the other planes are

\[
x^{a_4} = y_3^B, \quad y_1^{a_1} = y_3^{a_2}, \quad y_2 = (1 + B) x^{\frac{a_2}{n_1}} - B x^{\frac{a_2}{n_1}} \]

\[
y_2 = (1 + B) y_1^{a_2} - B y_1^{a_2}, \quad y_2 = (1 + B) y_3^{a_4} - B y_3 \]  \hspace{1cm} (5.2.12)

where \( B = \frac{P}{u_{30}} \) \hspace{1cm} (5.2.14)

5.3 **Equilibrium point** \( \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = \frac{\alpha}{\beta}, \bar{N}_4 = \frac{\gamma}{\beta} \)

Here \( \alpha = a_3 a_{44} - a_4 a_{34}, \quad \beta = a_3 a_{44} + a_4 a_{34} > 0 \) \hspace{1cm} (5.3.1)

and \( \gamma = a_4 a_{33} + a_3 a_{33} > 0 \) \hspace{1cm} (5.3.2)

The corresponding linearized equations for the perturbations \( u_1, u_2, u_3, u_4 \) are

\[
\frac{du_1}{dt} = q_1 u_1, \quad \frac{du_2}{dt} = q_2 u_2 \]

\[
\frac{du_3}{dt} = q_3 u_3 - a_{34} \frac{\alpha}{\beta} u_4, \quad \frac{du_4}{dt} = a_{43} \frac{\gamma}{\beta} u_3 + q_4 u_4 \]  \hspace{1cm} (5.3.3)

\[
\frac{du_1}{dt} = q_1 u_1, \quad \frac{du_2}{dt} = q_2 u_2 \]

\[
\frac{du_3}{dt} = q_3 u_3 - a_{34} \frac{\alpha}{\beta} u_4, \quad \frac{du_4}{dt} = a_{43} \frac{\gamma}{\beta} u_3 + q_4 u_4 \]  \hspace{1cm} (5.3.4)
Here 

\[ q_1 = a_1 + a_{13} \frac{\alpha}{\beta}, \quad q_2 = a_2 + a_{24} \frac{\gamma}{\beta} > 0 \]  (5.3.5)

\[ q_3 = a_3 - 2a_{33} \frac{\alpha}{\beta} - a_{34} \frac{\gamma}{\beta}, \quad q_4 = a_4 - 2a_{44} \frac{\gamma}{\beta} + a_{43} \frac{\alpha}{\beta} \]  (5.3.6)

The characteristic equation for which is

\[ (\lambda - q_1)(\lambda - q_2)(\lambda^2 - (q_3 + q_4)\lambda + \left(q_3q_4 + a_{43}a_{44} \frac{\alpha\gamma}{\beta^2}\right)) = 0 \]  (5.3.7)

One of the four roots \( q_2 \) is positive. Hence the state is \textit{unstable}. Let \( \lambda_1, \lambda_2 \) be the zeros of the quadratic polynomial on the R.H.S of the equation (5.3.7)

Case (A): When \( \alpha > 0 \) and \( (q_3 - q_4)^2 > 4a_{34}a_{43} \frac{\alpha\gamma}{\beta^2} \)

The roots \( q_1, \lambda_1 \) are positive and \( \lambda_2 \) is negative and the solutions of the equations (5.3.3), (5.3.4) are

\[ u_1 = u_{10} e^{\eta_1 t}, \quad u_2 = u_{20} e^{\eta_2 t} \]  (5.3.8)

\[ u_3 = \left[ a_{43}au_{40} + \beta(q_3 - q_4)u_{10}\right] e^{\eta_3 t} + \left[ a_{43}au_{40} + \beta(q_3 - \lambda_1)u_{10}\right] e^{\lambda_3 t} \]  (5.3.9)

\[ u_4 = \left[ a_{43}au_{40} + \beta(q_3 - q_4)u_{10}\right] e^{\eta_4 t} + \left[ a_{43}au_{40} + \beta(q_3 - \lambda_1)u_{10}\right] e^{\lambda_4 t} \]  (5.3.10)

The solution curves are illustrated in Figures (14), (15) and conclusions are presented here.

Case (i): If \( u_{20} < u_{30} < u_{40} < u_{10} \) and \( a_4 < q_2 < a_3 < q_1 \)

In this case the natural birth rates of the Host \( S_4 \) of \( S_2 \), Predator \( S_2 \), Host \( S_3 \) of \( S_1 \) and the Prey \( S_1 \) are in ascending order. Initially the Host \( S_4 \) of \( S_2 \) dominates over the Host \( S_3 \) of \( S_1 \), Predator \( S_2 \) till the time instant \( t^*_{43} \), \( t^*_{24} \) respectively and thereafter the dominance is reversed.

Case (ii): If \( u_{30} < u_{20} < u_{40} < u_{10} \) and \( a_3 < q_2 < a_4 < q_1 \)

In this case the Predator \( S_2 \) has the least natural birth rate and the Prey \( S_1 \) dominates the Host \( S_4 \) of \( S_2 \), Host \( S_3 \) of \( S_1 \), Predator \( S_2 \) in natural growth rate as well as in its population strength.

Case (B): when \( \alpha > 0 \) and \( (q_3 - q_4)^2 < 4a_{34}a_{43} \frac{\alpha\gamma}{\beta^2} \)

The roots \( q_1, q_2 \) are positive and \( \lambda_1, \lambda_2 \) are complex. The solutions in this case are same as in case (A) and this is illustrated in Fig. 16.

Case (C): When \( \alpha < 0 \), \( (q_3 - q_4)^2 - 4a_{34}a_{43} \frac{\alpha\gamma}{\beta^2} \) > 0

The roots \( q_1, \lambda_2 \) are negative and \( \lambda_1 \) is positive. The solutions in this case are same as in case (A) and the solution curves are illustrated in figures (17) to (20) and the conclusions are presented here.

Case (i): If \( u_{10} < u_{20} < u_{40} < u_{30} \) and \( q_1 < q_2 < a_3 < a_4 \)

In this case the natural birth rates of the Prey \( S_1 \), Predator \( S_2 \), Host \( S_3 \) of \( S_1 \) and the \( S_4 \) of \( S_2 \) are in ascending order. Initially the Host \( S_3 \) of \( S_1 \) dominates over the Host \( S_4 \) of \( S_2 \) till the time instant \( t^*_{43} \) and thereafter the dominance is reversed.
Here \( t_{d1}^{*} = \frac{1}{\lambda_1 - \lambda_2} \log \left[ \frac{(\beta b_7 - b_1 b_3) u_{30} + (\beta b_6 - b_2^2) u_{40}}{(b_1 b_2 - \beta b_5) u_{30} + (b_2^2 - \beta b_4) u_{40}} \right] \)  

(5.3.11)

where \( b_1 = a_4 \alpha \), \( b_2 = \beta (q_3 - \lambda_2) \), \( b_3 = \beta (q_3 - \lambda_1) \)  

(5.3.12)

\( b_4 = b_1 (q_3 - \lambda_1), b_5 = b_2 (q_3 - \lambda_1), b_6 = b_1 (q_3 - \lambda_2), b_7 = b_3 (q_3 - \lambda_2) \)  

(5.3.13)

**Case (ii):** If \( u_{20} < u_{40} < u_{10} < u_{30} \) and \( q_2 < q_1 < a_4 < a_3 \)

In this case the natural birth rates of the Prey \((S_1)\), Predator \((S_2)\), Host \((S_4)\) of \(S_2\) and the Host \((S_3)\) of \(S_1\) are in ascending order. Initially the Prey \((S_1)\) dominates over the Predator \((S_2)\) and its Host \((S_4)\) till the time instant \(t_{d1}^{*}\) and \(t_{d1}^{*}\) respectively and thereafter the dominance is reversed.

Here \( t_{d1}^{*} = \frac{1}{q_1 - q_2} \log \left( \frac{u_{20}}{u_{10}} \right) \)  

(5.3.14)

**Case (iii):** If \( u_{30} < u_{10} < u_{40} < u_{20} \) and \( q_2 < q_1 < a_3 < a_4 \)

In this case the natural birth rates are same as in case (i). Initially the Predator \((S_2)\) dominates over the Host \((S_3)\) of \(S_1\), Host \((S_4)\) of \(S_2\) till the time instant \(t_{d1}^{*}\), \(t_{d1}^{*}\) respectively and thereafter the dominance is reversed.

**Case (iv):** If \( u_{40} < u_{30} < u_{20} < u_{10} \) and \( a_3 < a_4 < q_1 < q_2 \)

In this case the natural birth rates of the Prey \((S_1)\), Host \((S_3)\) of \(S_1\), Host \((S_4)\) of \(S_2\) and the Predator \((S_2)\) are in ascending order. Initially the Prey \((S_1)\) dominates over the Predator \((S_2)\), Host \((S_3)\) of \(S_1\), Host \((S_4)\) of \(S_2\) till the time instant \(t_{d1}^{*}, t_{d1}^{*}, t_{d1}^{*}\) respectively and thereafter the dominance is reversed. Also the Host \((S_3)\) of \(S_1\) dominates over the Host \((S_4)\) of \(S_2\) till the time instant \(t_{d1}^{*}\) and the dominance gets reversed thereafter.

5.3. A. Trajectories of Perturbations:

The trajectories in the \(u_1 - u_2\) plane given by \( x^{\text{eq}} = y_1^{\text{eq}} \)  

(5.3.15)

and are shown in Fig.21 and the trajectories in the other planes are

\[
\begin{align*}
y_2 &= A_1 x^{\text{eq}} + B_1 x^{\text{eq}}, \quad y_3 = A_2 x^{\text{eq}} + B_2 x^{\text{eq}} \\
y_2' &= A_1 y_1^{\text{eq}} + B_1 y_1^{\text{eq}}, \quad y_3' = A_2 y_1^{\text{eq}} + B_2 y_1^{\text{eq}}
\end{align*}
\]

(5.3.16)

(5.3.17)

where

\[
\begin{align*}
A_1 &= \frac{a_3 \alpha u_{40} + \beta (q_3 - \lambda_2)}{\beta (\lambda_1 - \lambda_2)} u_{30}, \quad B_1 = \frac{a_3 \alpha u_{40} + \beta (q_3 - \lambda_1)}{\beta (\lambda_2 - \lambda_1)} u_{30} \\
A_2 &= \frac{a_3 \alpha u_{40} + \beta (q_3 - \lambda_2)}{a_3 \alpha (\lambda_1 - \lambda_2) u_{30}} (q_3 - \lambda_1) \\
B_2 &= \frac{a_3 \alpha u_{40} + \beta (q_3 - \lambda_1)}{a_3 \alpha (\lambda_2 - \lambda_1) u_{30}} (q_3 - \lambda_2)
\end{align*}
\]

(5.3.18)

(5.3.19)

(5.3.20)
6. PERTURBATION GRAPHS.

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14

Fig. 15

Fig. 16
7. NUMERICAL APPROACH OF THE GROWTH RATE EQUATIONS

The numerical solutions of the growth rate equations (2.1), (2.2), (2.3) and (2.4) computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. For this MatLab has been used and the results are illustrated in Figures (22) to (25).

Consider the model parameter values

\[ a_1=0.7, \quad a_2=1.2, \quad a_3=2.8, \quad a_4=0.48, \quad a_{12}=0.87, a_{13}=0.43, a_{21}=0.32, \quad a_{24}=0.18, \quad a_{34}=1.18, \quad a_{43}=0.18, \quad K_1=3.5, \quad K_2=4.8, \quad K_3=5.6, \quad K_4=1.2 \]

**Case (a):** If \( N_{i0} < \frac{K_i}{2} \), \( i = 1, 2, 3, 4 \).

![Graph](image)

**Figure 22:** Variation of \( N_1, N_2, N_3 \) and \( N_4 \) against time\((t)\) for \( N_{10}=1.5, N_{20}=2, N_{30}=2.5, N_{40}=0.4 \)
Case (b): If \( N_{i0} > K_i, i = 1, 2, 3, 4. \)

![Figure 23: Variation of N1, N2, N3 and N4 against time(t) for N10=4, N20=6.5, N30=6, N40=1.5](image)

Case (c): If \( \frac{K_i}{2} < N_{i0} < K_i, i = 1, 2, 3, 4. \)

![Figure 24: Variation of N1, N2, N3 and N4 against time(t) for N10=2, N20=4, N30=3.5, N40=1](image)
Case (d): If $N_{i0} = K_i$, $i = 1, 2, 3, 4$.

Figure 25: Variation of $N_1, N_2, N_3$ and $N_4$ against time(t) for $N_{10} = 3.5, N_{20} = 4.8, N_{30} = 5.6, N_{40} = 1.2$

8. OPEN PROBLEM:

Investigate some relation-chains between the species such as Prey-Predation, Neutralism, Commensalism, Mutualism, Competition and Ammensalism between four species ($S_1, S_2, S_3, S_4$) with the population relations.

$S_1$ a Prey to $S_2$ and Commensal to $S_3$, $S_2$ is a Predator living on $S_1$ and Commensal to $S_4$, $S_3$ a Host to $S_1$, $S_4$ a Host to $S_2$ and $S_3$ a Prey to $S_4$, $S_4$ a Predator to $S_3$.

The present paper deals with the study on stability of 1st level Prey-Predator washed out states only of the above problem. The stability of the other equilibrium states were published several National and International Journals.

REFERENCES


B. Hari Prasad: He works as an Assistant Prof., Department of Mathematics, Chaitanya Degree & PG College (Autonomous), Hanamkonda. He has obtained M. Phil in Mathematics. He has presented papers in various seminars and his articles are published in popular International and National journals to his credit. He has zeal to find out new vistas in Mathematics.

N. Ch. Pattabhi Ramacharyulu: He is a retired Professor in Department of Mathematics & Humanities, National Institute of Technology, Warangal. He is a stallwart in Mathematics. His yeoman services as a lecturer, professor, professor Emeritus and Deputy Director enriched the knowledge of thousands of students. He has nearly 46 Ph. Ds and plenty number of M. Phils to his credit. His research papers in areas of Applied Mathematics are more than 195 were published in various esteemed National and International Journals. He is a member of Various Professional Bodies. He published four books on Mathematics. He received several prestigious awards and rewards. He is the Chief Promoter of AP Society for Mathematical Sciences.

Corresponding author: B. Hari Prasad

Dept. of Mathematics, Chaitanya Degree College (Autonomous), Hanamkonda, India

Source of support: Nil, Conflict of interest: None Declared