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EDIFICE OF THE REAL NUMBERS BY ALTERNATING SERIES

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ABSTRACT

In this paper, we introduce a procedure which represents a given real number as an alternating series of rational numbers. This procedure is a generalization of the alternating-Sylvester series and has the same properties as that. The methods are almost same as that of Arnold Knopfmacher and John Knofmacher [2]. However, I added my own favor in treatment of proofs, whenever necessary in the case of [1].

Key Words: *Sylvester series, real number, alternating series.*

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1. INTRODUCTION

In [1], Soichi Ikeda discussed comprehensive theory on alternating series of a real number. However, I could not find proper investigation in the treatments of proofs. I would like to add my own flavor to this published paper and discussed the same by interesting proofs and treatments. Honestly, the credit of writing this paper will go to [1] without any more doubts.

There are some methods which represent any *given* real number as a certain series of rational numbers. In this regard, Arnold Knopfmacher and John Knopfmacher constructed the real numbers by the Sylvester series and the Engel series [2].

In 1880, Sylvester showed that, for given real number A has unique representation $A = c + \sum_{i=1}^{\infty} \frac{1}{a_i}$ where c is an integer with $a_i \geq 2$, such that $a_{i+1} > a_i^2 - a_i$. Conversely, series of this kind is convergent and its sum A is irrational if and only if $a_{i+1} > a_i^2 - a_i + 1$.

For instance, $A = \sum_{i=0}^{\infty} \frac{1}{2^{2^i}}$ is irrational.

In 1883, Luroth showed that, for given real number A has unique representation $A = c + \frac{1}{a_1} + \sum_{i=1}^{\infty} \frac{1}{a_1(a_1-1)a_2(a_2-1)...a_i(a_i-1)} \cdot \frac{1}{a_{i+1}}$ where c is an integer with $a_i \geq 2$. Conversely, series of this kind is convergent and its sum A is irrational if and only if $\{a_i\}$ is not periodic.

In 1913, Engel showed that, for given real number A has unique representation $A = c + \sum_{i=1}^{\infty} \frac{1}{a_1 a_2 ... a_i}$ where c is an integer with $a_i \in [2, \infty)$. Conversely, series of this kind is convergent and its sum A is irrational if and only if $\underset{i \to \infty}{Lt}\, a_i = \infty$.

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As per [2], we see that, for *given* any real number *A*, there exist three different sequences of integers *{a_i}* such that

(i) $A = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots$ where $a_1 \geq 2, a_{i+1} \geq a_i^{\,2} - a_i + 1, \forall i \geq 1,$ **(#)**

(ii) $A = a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \ldots$ where $a_1 \geq 2, a_{i+1} \geq a_i \forall i \geq 1,$

(iii) $A = a_0 + \frac{1}{a_1} + \frac{1}{(a_1-1)a_1}.\frac{1}{a_2} + \frac{1}{(a_1-1)(a_2-1)}.\frac{1}{a_1 a_2 a_3} + \ldots$ where $a_1 \geq 2, \forall i \geq 1.$

2. DISCUSSIONS ON SYLVESTER, ENGEL, AND LUROTH SERIES

We are given in introduction, expression of A from (i), (ii) and (iii) are respectively known as Sylvester, Engel and Luroth series. After close look at [2] and [3], I realized as below and produced my own understand on those series.

The treatment is given in [2] & [3] is not very much motivating. However, I consider those observations in my best of knowledge. I suspect that *A* has a single representation, from (i)

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots$$

for integers a_i where $a_1 \geq 2, a_{i+1} \geq a_i^{\,2} - a_i + 1, \forall i \geq 1$. This is (at least for an irrational A) often described as the Egyptian fractions [4] representation for example [5] gives:

$$\pi = 3 + \frac{1}{8} + \frac{1}{61} + \frac{1}{5020} + \frac{1}{128541455} + \ldots$$

To show it's exist and is unique, consider the partial sum A_i up to the $\frac{1}{a_i}$ term, where $a_0 = \lceil A \rceil - 1$ and $a_{i+1} = \left\lfloor \frac{1}{A - A_i} \right\rfloor + 1.$ to complete the proof the following are essential.

$$(*) A_i < A \leq A_{i-1} + \frac{1}{a_i - 1}$$

$$(**) \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}$$

$$(***) b_{j+1} = b_j^{\,2} - b_j + 1 \Rightarrow \sum_{j=1}^{\infty} \frac{1}{b_j} = \frac{1}{b_k - 1}$$

However, we can handle the (ii), same as above with suitable adjustments: for instance, (***) becomes $\sum_{j=1}^{\infty} \left(\frac{1}{b_k} \right)^j = \frac{1}{b_k - 1}$.

Remark: From (i) to (iii) can be proved by experimenting with various values of A and use $a_0 = \lceil A \rceil - 1$ and $a_{i+1} = \left\lfloor \frac{1}{A - A_i} \right\rfloor + 1,$ by keeping track of $A - A_i$ and $\frac{1}{A - A_i}$.

Now, if we upsurge each term of **(#)** up to *k* times:

$$A = a_0 + \left(\frac{1}{a_1} \right)^k + \left(\frac{1}{a_2} \right)^k + \left(\frac{1}{a_3} \right)^k + \ldots \qquad \textbf{(\#\#)}$$

where $i \geq 1$ and the recurrence relation $a_{i+1} \geq a_i \geq 2$ with *k* is a fixed real and ≥ *1*.

Discussion: Since, $a_{i+1} = a_i$, if *k* is some integer, we can reduce the problem for *k > 1* to the case *k = 1* by repeating $\frac{1}{a_i^{\,k}}$ for $a_i^{\,k-1}$ times before taking a new value. By doing the same repetition, we can see the sum of $\frac{a_i^{\,k-1}}{a_i^{\,k}} = \frac{1}{a_i}$.

For instance, for *k = 1*, we see the following:

$$A = 1 + \frac{1}{2} + \frac{1}{8} + \ldots$$

For *k = 2*, the new expansion reads,

$$A = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2 + \ldots$$

Since every real number can be written as the product of an integer and a real number in the range *[1, 2)*, the same strategy can be used to reduce the problem for arbitrary non-integer *k > 1* to the problem *1 < k < 2*.

Now, we can do little modification to (##) and can show the A as follows:

$$A = a_0 + \left(\frac{1}{a_1}\right)^k + \left(\frac{1}{a_1}\right)^k \left(\frac{1}{a_2}\right)^k + \left(\frac{1}{a_1}\right)^k \left(\frac{1}{a_2}\right)^k \left(\frac{1}{a_3}\right)^k + \ldots \qquad \textbf{(\#\#\#)}$$

Where $a_1 = 2$ and $a_i \in [1, 2]$ for all $i \geq 2$ and $a_i = 2$ and $k \geq 1$ and fixed real, infinitely often.

Here, discussing the proof of the above result is ridiculous. It should be fairly easy to prove even with *k* fixed to *1*. In this setting, each of the terms in the sum (apart from a_0), is going to be some power of ½ and one can control whether the specific term will be the same as the previous one (by setting $a_i = 1$) or twice smaller (when $a_i = 2$). In order to determine which one needs to be used in which case, you can look at the binary (= base 2) expansion of A. To be frank! After close observation of **(###)**, we can realize that, *"All real numbers can be defined in terms of (Cauchy) convergent sequence of rational numbers"*. For instance, the power series [6] representation of a number is just converging sequence of a rational numbers. As we know that, the value of π is 3.1459… We can see the same as the limit of the rational sequence {3, 3.1, 3.14, 3.145, 3.1459, …}.

A classic way to define a real number is by a sequence of rational numbers whose successive differences converge to zero, which is popularly known as Cauchy convergence.

3. CONSTRUCTION OF THE REAL NUMBERS

I recommend the readers to verify the paper written by Sochi Ikeda at [1] and see the last section. In this section, one can find more propositions and interesting theorems namely **theorem 3.1** and

Theorem 3.2. Now, I would like to cover both the theorem in terms of one theorem and one proof.

Theorem 3.1. *Let M be a non-empty subset of S. If M is bounded from above (below) then there exists a supremum (an infimum). If S totally ordered field with l.u.b is unique up to isomorphism.*

Proof: If *S* is a totally ordered field with the lub-property, we will not be able to prove constructively that any nonempty subset $M \subset S$ which is bounded from above has a least upper bound. This is a property one can *assume* that *S* has, not a property to *prove* that *S* has.

It is, however, possible to prove that any nonempty subset $H \subset R$ which is bounded from above has a least upper bound. The proof will depend on how one can have defined the set of real numbers **R**. Note that there are several definitions possible, all of which are equivalent. (The most common ones are by using Dedekind cuts of rational numbers [6], or by using equivalence classes of Cauchy sequences of rational numbers.)

If *S* is a totally ordered field with the lub-property then there is a unique order-preserving field isomorphism ϕ: **R**→*S*. This means that ϕ will have the following properties:

ϕ is 1-1 and onto

$$\phi(x + y) = \phi(x) + \phi(y)$$

$$\phi(xy) = \phi(x)\, \phi(y)$$

$$\text{If } x < y \text{ then } \phi(x) < \phi(y)$$

In order to prove that there exists such a map ϕ, you follow the steps as follow. Of course, there is some work involved here, if we want to write out all details, but it is healthy work which will deepen the reader understanding of all the involved concepts.

Now, we begin by showing that *if* there is such a map ϕ, then it is necessary that $\phi(0) = 0S$ and $\phi(1) = 1S$, where $0S$ and $1S$ are the zero element and the unit element of the field S, respectively. Hence we define $\phi(0) = 0S$ and $\phi(1) = 1S$. Now use the field axioms to define $\phi(x)$ for any *rational* x. Show that this can only be done in one way if ϕ is to be a field isomorphism [7]. Show that ϕ will then be 1-1 on $\mathbf{Q}$. Also show that ϕ will preserve the ordering on rational numbers, so that if $x,y \in \mathbf{Q}$ and $x < y$ then $\phi(x) < \phi(y)$ as well.

Now, one needs to extend ϕ so that it becomes a map which is defined on all of $\mathbf{R}$, and not only on $\mathbf{Q}$. I'm sure it can be done in several (equivalent) ways, but the following seems natural: If $x \in \mathbf{R}$, let $E = \{y \in \mathbf{Q}: y < x\}$. Note that E is bounded above in $\mathbf{R}$, and that x is a least upper bound of E. Show that $\phi(E)$ is bounded above in S, and define $\phi(x)$ to be the least upper bound of $\phi(E)$. Show that this does not change $\phi(x)$ in the case when $x \in \mathbf{Q}$. Show that the extension still has the properties $\phi(x + y) = \phi(x) + \phi(y)$, $\phi(xy) = \phi(x)\phi(y)$, and that it still is order-preserving and 1-1.

Now we have an order-preserving injective map ϕ: $\mathbf{R} \to S$ which preserves the field structure. The final step is to show that it is necessarily surjective. Let $s \in S$, we need to find $x \in \mathbf{R}$ such that $\phi(x) = s$. First show that there are a, $b \in \mathbf{R}$ such that $\phi(a) < s < \phi(b)$. Now, define $E = \{y \in \mathbf{R}: \phi(y) < s\}$. Show that E is bounded above, and let x be the least upper bound of E. Show that $\phi(x) = s$.

To conclude the last part, here R and S are both complete (in the sense of lub property) ordered fields and we want to prove they are isomorphic. And so far we have constructed an isomorphism from R into S, preserving order. So one can pick any s in S, we might as well assume $s > 0$. Pick any positive $x \in R$.

If $f(x) = s$, we are done, otherwise $f(x) < s$ or $f(x) > s$. In either case, s lies between $f(x)$ and $\dfrac{1}{f(x)}$, hence we have shown that there are a and b with $a < b$ in R such that $f(a) < f(s) < f(b)$. However, one might better proceed as follows: if s is rational, we are done, otherwise let $B = \{y \in S \mid y \text{ rational}, y < s\}$ and let $E = f^{-1}(B)$. Of course, we need to use the fact that both R and S are Archimedean.

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REFERENCES

[1] http://arxiv.org/pdf/1208.1434v1.pdf

[2] http://www.fq.math.ca/Scanned/27-1/knopfmacher.pdf

[3] http://emis.matem.unam.mx/journals/HOA/IJMMS/Volume12_3/613.html

[4] Richard Guy, *"Unsolved Problems in Number Theory"*, Springer Science + Business Media, Inc , 3rd Edition, 2004.

[5] http://oeis.org/A001466

[6] Ethan D. Bloch *"The Real Numbers and Real Analysis"*, Springer Science + Business Media, LLC 2011, London.

[7] Joseph A. Gallian, *"Contemporary Abstract Algebra"*, 7th edition, Cengage Learning, 2010, USA.

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