International Journal of Mathematical Archive-3(9), 2012, 3315-3325
Available online through www.ijma.info ISSN 2229 – 5046

FIXED POINT THEOREMS IN COMPLETE METRIC SPACE BY INTEGRAL TYPE MAPPING

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(Received on: 24-08-12; Revised & Accepted on: 19-09-12)

ABSTRACT

In the present paper, we establish some fixed point theorem and common fixed point theorems for integral type mapping in complete metric spaces. Our results are generalization and extension of various known results.

***AMS Subject Classification**: 54H25, 47H10.*

***Key words:** Fixed point, common fixed point, complete metric space, Lebesgue-integrable map.*

1. INTRODUCTION AND PRELIMINARIES

Impact of fixed point theory in different branches of mathematics and its applications is immense. The most important result on fixed points for contractive type mapping was Banach's contraction principle by S. Banach [1] in 1922. In the general setting of complete metric space, this theorem runs as follows ([4] see Theorem 2.1 or [10] Theorem 1.2.2).

Theorem 1.1. (Banach's contraction principle) [1] Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f: X \to X$ be a mapping such that for each $x, y \in X$,

$$d(fx, fy) \leq c\, d(x, y) \tag{1.1}$$

then f has a unique fixed point $a \in X$, such that for each $x \in X$, $lim_{n\to\infty} f^n x = a$

After this classical result, a number of mathematicians have been working on fixed point theory dealing with mappings satisfying various type of contractive conditions (see [3], [5] [7], [8], [9] and [11] for details).

In 2002, A. Branciari [2] analyzed the existence of fixed point for mapping f defined on a complete metric space (X, d) satisfying a general contractive condition of integral type

Theorem 1.2 (Branciari) [2] Let (X, d) be a complete metric space, $c \in (0,1)$ and let $f : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \varphi(t)\, dt \leq \mathrm{c} \int_0^{d(x,y)} \varphi(t)\, dt \tag{1.2}$$

Where $\varphi: [0, +\infty) \to [0, +\infty)$ is a Lesbesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t) > 0$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $lim_{n\to\infty} f^n x = a$

After the paper of Branciari, a lot of research works have been carried out on generalising contractive conditions of integral type for different contractive mappings satisfying various known properties.

The aim of this paper is to generalise some mixed type of contractive conditions to the mapping and then a pair of mappings satisfying a general contractive condition of integral type.

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In this paper, we obtain an extension of theorem 1.2 through rational expression.

2. MAIN RESULTS

Theorem 2.1. Let f be a self mapping of a complete metric space (X, d) satisfying the following condition:

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le \alpha \int_0^{\left[\frac{d^3(x,fx)+d^3(y,fy)}{1+d^2(x,fx)+d^2(y,fy)}\right]} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x,fy)+d^2(y,fx)}{1+d(x,fy)+d(y,fx)}} \varphi(t)\,dt + \gamma \int_0^{d(x,y)} \varphi(t)\,dt \tag{2.1}$$

for each $x, y \in X$ with nonnegative reals α, β, γ such that $2\alpha + 2\beta + \gamma < 1$, where $\varphi: R^+ \to R^+$ is a Lesbesgue-integrable mapping which is summable on each compact subset of R^+, such that

for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)\,dt > 0$ (2.2)

then f has a unique fixed point $z \in X$ and for each $x \in X$, $lim_{n\to\infty} f^n x = z$

Proof. For any arbitrary $x_0 \in X$, there is x_1 in X such that $x_1 = f x_0$. Proceeding the same way, *we* construct a sequence $\{x_n\}$ of element of X, such that $x_{n+1} = f x_n$.

For each integer $n = 0, 1, 2\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

from (2.1) we get

$$\int_0^{d(x_{n+1}, x_{n+2})} \varphi(t)\,dt = \int_0^{d(fx_n, f x_{n+1})} \varphi(t)\,dt$$

$$\le \alpha \int_0^{\frac{d^3(x_n, fx_n)+d^3(x_{n+1}, f x_{n+1})}{1+d^2(x_n, fx_n)+d^2(x_{n+1}, f x_{n+1})}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_n, f x_{n+1})+d^2(x_{n+1}, f x_n)}{1+d(x_n, f x_{n+1})+d(x_{n+1}, f x_n)}} \varphi(t)\,dt + \gamma \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\le \alpha \int_0^{\frac{d^3(x_n, x_{n+1})+d^3(x_{n+1}, x_{n+2})}{1+d^2(x_n, x_{n+1})+d^2(x_{n+1}, x_{n+2})}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_n, x_{n+2})+d^2(x_{n+1}, x_{n+1})}{1+d(x_n, x_{n+2})+d(x_{n+1}, x_{n+1})}} \varphi(t)\,dt + \gamma \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\le \alpha \int_0^{\frac{(d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2}))(d^2(x_n, x_{n+1})+d^2(x_{n+1}, x_{n+2}))}{1+d^2(x_n, x_{n+1})+d^2(x_{n+1}, x_{n+2})}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_n, x_{n+2})}{1+d(x_n, x_{n+2})}} \varphi(t)\,dt + \gamma \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\le \alpha \int_0^{d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2})} \varphi(t)\,dt + \beta \int_0^{d(x_n, x_{n+2})} \varphi(t)\,dt + \gamma \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\le (\alpha + \gamma) \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt + \alpha \int_0^{d(x_{n+1}, x_{n+2})} \varphi(t)\,dt + \beta \int_0^{d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2})} \varphi(t)\,dt$$

$$\int_0^{d(x_{n+1}, x_{n+2})} \varphi(t)\,dt \le \frac{(\alpha+\beta+\gamma)}{(1-\alpha-\beta)} \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\int_0^{d(x_{n+1}, x_{n+2})} \varphi(t)\,dt \le r \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt \tag{2.3}$$

where $\frac{(\alpha+\beta+\gamma)}{(1-\alpha-\beta)} = r$ (say) < 1 (Since $2\alpha + 2\beta + \gamma < 1$)

Thus by routine calculation

$$\int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt \le r^n \int_0^{d(x_0, x_1)} \varphi(t)\,dt \tag{2.4}$$

Taking limit of (2.4) as n→∞, we get

$\lim_{n\to\infty} \int_0^{d(x_n, x_{n+1})} \varphi(t)\, dt = 0$

Which, from (2.2) implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{2.5}$$

We show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ such that for each p $\in N$ there are subsequence {m(p)} and {n(p)} in N such that {m(p)} < {n(p)}< {m(p+1)} with

$$d\left(x_{m(p)}, x_{n(p)}\right) \geq \epsilon\ ,\ d\left(x_{m(p)}, x_{n(p)-1}\right) < \epsilon \tag{2.6}$$

Now

$$d\left(x_{m(p)-1}, x_{n(p)-1}\right) \leq d\left(x_{m(p)-1}, x_{m(p)}\right) + d\left(x_{m(p)}, x_{n(p)-1}\right)$$

$$< d\left(x_{m(p)-1}, x_{m(p)}\right) + \epsilon \tag{2.7}$$

Hence

$$\lim_{p\to\infty} \int_0^{d\left(x_{m(p)-1}, x_{n(p)-1}\right)} \varphi(t)\, dt \leq \int_0^{\epsilon} \varphi(t)\, dt \tag{2.8}$$

Using (2.3), (2.6), and (2.8), we get

$$\int_0^{\epsilon} \varphi(t)\, dt \leq \int_0^{d\left(x_{m(p)}, x_{n(p)}\right)} \varphi(t)\, dt \leq r \int_0^{d\left(x_{m(p)-1}, x_{n(p)-1}\right)} \varphi(t)\, dt \leq r \int_0^{\epsilon} \varphi(t)\, dt$$

Which is a contradiction, since $r \in (0, 1)$. Therefore, $\{x_n\}$ is Cauchy sequence converges to z$\in X$.

From (2.1) we get

$$\int_0^{d(z,fz)} \varphi(t)\, dt \leq \int_0^{d(z,f x_n)+d(f x_n,fz)} \varphi(t)\, dt = \int_0^{d(z,f x_n)} \varphi(t)\, dt + \int_0^{d(f x_n,fz)} \varphi(t)\, dt$$

$$\leq \int_0^{d(z,f x_n)} \varphi(t)\, dt + \alpha \int_0^{\frac{d^3(x_n, f x_n)+d^3(z, f z)}{1+d^2(x_n, f x_n)+d^2(z, f z)}} \varphi(t)\, dt + \beta \int_0^{\frac{d^2(x_n, fz)+d^2(z, f x_n)}{1+d(x_n, f z)+d(z, f x_n)}} \varphi(t)\, dt + \gamma \int_0^{d(x_n,z)} \varphi(t)\, dt$$

$$\leq \int_0^{d(z,x_{n+1})} \varphi(t)\, dt + \alpha \int_0^{\frac{d^3(x_n, x_{n+1})+d^3(z, f z)}{1+d^2(x_n, x_{n+1})+d^2(z, f z)}} \varphi(t)\, dt + \beta \int_0^{\frac{d^2(x_n, fz)+d^2(z, x_{n+1})}{1+d(x_n, f z)+d(z, x_{n+1})}} \varphi(t)\, dt + \gamma \int_0^{d(x_n,z)} \varphi(t)\, dt$$

$$\leq \int_0^{d(z,x_{n+1})} \varphi(t)\, dt + \alpha \int_0^{d(x_n, x_{n+1})+d(z, fz)} \varphi(t)\, dt + \beta \int_0^{d(x_n, f z)+d(z, x_{n+1})} \varphi(t)\, dt + \gamma \int_0^{d(x_n,z)} \varphi(t)\, dt$$

$$\leq (1+\beta) \int_0^{d(z,x_{n+1})} \varphi(t)\, dt + \alpha \int_0^{d(x_n, x_{n+1})+d(z, fz)} \varphi(t)\, dt + \beta \int_0^{d(x_n, f z)} \varphi(t)\, dt + \gamma \int_0^{d(x_n,z)} \varphi(t)\, dt$$

$$\leq (1+\beta) \int_0^{d(z,x_{n+1})} \varphi(t)\, dt + \alpha \int_0^{d(x_n, x_{n+1})} \varphi(t)\, dt + (\alpha+\beta) \int_0^{d(z, fz)} \varphi(t)\, dt + (\gamma+\beta) \int_0^{d(x_n,z)} \varphi(t)\, dt$$

Taking limit on both sides as n→∞, we get

$$\lim_{n\to\infty} \int_0^{d(z, fz)} \varphi(t)\, dt = 0$$

Which from (2.2), implies that $\lim_{n\to\infty} d(z, f z) = 0$ or $f z = z$

⇨ z is a fixed point of f

FOR UNIQUENESS

Suppose that *w (≠z)* be another fixed point of *f*, different from z i.e. *f(w) = w*. Then from (2.1) we have

$$\int_0^{d(z,w)} \varphi(t)\,dt = \int_0^{d(fz,fw)} \varphi(t)\,dt$$

$$\leq \alpha \int_0^{\left[\frac{d^3(z,fz)+d^3(w,fw)}{1+d^2(z,fz)+d^2(w,fw)}\right]} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(z,fw)+d^2(w,fz)}{1+d(z,fw)+d(w,fz)}} \varphi(t)\,dt + \gamma \int_0^{d(z,w)} \varphi(t)\,dt$$

$$\leq \beta \int_0^{\frac{d^2(z,w)+d^2(w,z)}{1+d(z,w)+d(w,z)}} \varphi(t)\,dt + \gamma \int_0^{d(z,w)} \varphi(t)\,dt$$

$$\leq \beta \int_0^{2\,d(z,w)} \varphi(t)\,dt + \gamma \int_0^{d(z,w)} \varphi(t)\,dt$$

$$\leq (2\beta+\gamma) \int_0^{d(z,w)} \varphi(t)\,dt$$

Since, $(2\beta+\gamma) < 1$, this implies that

$$\int_0^{d(z,w)} \varphi(t)\,dt = 0$$

Which, from (2.2), implies that *d(z,w) = 0, or z = w* and so *f* has unique fixed point in *X*.

Theorem 2.2. Let *f* and *g* be self mappings of a complete metric space (X, d) satisfying the following conditions:

(i) *f* and *g* are commutative

(ii) $\int_0^{d(fx,gy)} \varphi(t)\,dt \leq \alpha \int_0^{\left[\frac{d^3(x,fx)+d^3(y,gy)}{1+d^2(x,fx)+d^2(y,gy)}\right]} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x,gy)+d^2(y,fx)}{1+d(x,gy)+d(y,fx)}} \varphi(t)\,dt + \gamma \int_0^{d(x,y)} \varphi(t)\,dt$ (2.9)

for each $x, y \in X$ with nonnegative reals α, β, γ such that $2\alpha + 2\beta + \gamma < 1$, where $\varphi: R^+ \to R^+$ is a Lesbesgue-integrable mapping which is summable on each compact subset of *R*, and such that

for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)\,dt > 0$ (2.10)
then *f* and *g* have a unique common fixed point $z \in X$.

Proof. For any arbitrary $x_0 \in X$ there is x_1 in X such that $x_1 = f x_0$ *and* $x_2 = g x_1$. Proceeding the same way, we construct a sequence $\{x_n\}$ of element of *X*, Such that $x_{n+1} = f x_n$. and $x_{n+2} = g\, x_{n+1}$.

For each integer *n =0, 1, 2…..*

From (2.9) we get

$$\int_0^{d(x_{n+1},x_{n+2})} \varphi(t)\,dt = \int_0^{d(fx_n,gx_{n+1})} \varphi(t)\,dt$$

$$\leq \alpha \int_0^{\frac{d^3(x_n,fx_n)+d^3(x_{n+1},gx_{n+1})}{1+d^2(x_n,fx_n)+d^2(x_{n+1},gx_{n+1})}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_n,gx_{n+1})+d^2(x_{n+1},fx_n)}{1+d(x_n,gx_{n+1})+d(x_{n+1},fx_n)}} \varphi(t)\,dt + \gamma \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt$$

$$\leq \alpha \int_0^{\frac{d^3(x_n,x_{n+1})+d^3(x_{n+1},x_{n+2})}{1+d^2(x_n,x_{n+1})+d^2(x_{n+1},x_{n+2})}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_n,x_{n+2})+d^2(x_{n+1},x_{n+1})}{1+d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})}} \varphi(t)\,dt + \gamma \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt$$

$$\leq \alpha \int_0^{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})} \varphi(t)\,dt + \beta \int_0^{d(x_n,x_{n+2})} \varphi(t)\,dt + \gamma \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt$$

$$\leq \alpha \int_0^{d(x_{n+1},x_{n+2})} \varphi(t)\,dt + \beta \int_0^{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})} \varphi(t)\,dt + (\alpha+\gamma) \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt$$

$$\leq (\alpha+\beta) \int_0^{d(x_{n+1},x_{n+2})} \varphi(t)\,dt + (\alpha+\beta+\gamma) \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt$$

$\int_0^{d(x_{n+1},x_{n+2})} \varphi(t)\,dt \le \frac{(\alpha+\beta+\gamma)}{(1-\alpha-\beta)} \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt$

$\int_0^{d(x_{n+1},x_{n+2})} \varphi(t)\,dt \le r \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt$ (2.11)

where $\frac{(\alpha+\beta+\gamma)}{(1-\alpha-\beta)} = r$ (say) < 1 (Since $2\alpha+2\beta+\gamma < 1$)

Similarly

$\int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt \le r \int_0^{d(x_{n-1},x_n)} \varphi(t)\,dt$ (2.12)

Thus by routine calculation

$\int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt \le r^n \int_0^{d(x_0,x_1)} \varphi(t)\,dt$ (2.13)

Taking limit of as $n\to\infty$, we get

$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt = 0$

Which from (2.10) implies that

$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ (2.14)

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequence $\{m(p)\}$ and $\{n(p)\}$ such that m (p) < $\{n(p)\}$ < $\{m(p+1)\}$ with

$d(x_{m(p)}, x_{n(p)}) \ge ,\ d(x_{m(p)}, x_{n(p)-1}) < \epsilon$ (2.15)

Now

$d(x_{m(p)-1}, x_{n(p)-1}) < d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)-1})$

$< d(x_{m(p)-1}, x_{m(p)}) + \epsilon$ (2.16)

Hence

$\lim_{p\to\infty} \int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)\,dt \le \int_0^{\epsilon} \varphi(t)\,dt$ (2.17)

Using (2.15), (2.12), and (2.17), we get

$\int_0^{\epsilon} \varphi(t)\,dt \le \int_0^{d(x_{m(p)},x_{n(p)})} \varphi(t)\,dt \le r \int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)\,dt \le r\int_0^{\epsilon} \varphi(t)\,dt$

Which is a contradiction, since $r \in (0, 1)$.Therefore, $\{x_n\}$ is Cauchy sequence converges to z ∈ X.

From (2.9) we get

$$\int_0^{d(z,fz)} \varphi(t)\,dt = \int_0^{d(z,gx_{n+1})+d(gx_{n+1},fz)} \varphi(t)\,dt = \int_0^{d(z,x_{n+2})+d(gx_{n+1},fz)} \varphi(t)\,dt$$

$$= \int_0^{d(z,x_{n+2})} \varphi(t)\,dt + \int_0^{d(gx_{n+1},fz)} \varphi(t)\,dt$$

$$\le \int_0^{d(z,x_{n+2})} \varphi(t)\,dt + \alpha \int_0^{\frac{d^3(x_{n+1},gx_{n+1})+d^3(z,fz)}{1+d^2(x_{n+1},gx_{n+1})+d^2(z,fz)}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_{n+1},fz)+d^2(z,gx_{n+1})}{1+d(x_{n+1},fz)+d(z,gx_{n+1})}} \varphi(t)\,dt + \gamma \int_0^{d(x_{n+1},z)} \varphi(t)\,dt$$

$$\le \int_0^{d(z,x_{n+2})} \varphi(t)\,dt + \alpha \int_0^{\frac{d^3(x_{n+1},x_{n+2})+d^3(z,fz)}{1+d^2(x_{n+1},x_{n+2})+d^2(z,fz)}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_{n+1},fz)+d^2(z,x_{n+2})}{1+d(x_{n+1},fz)+d(z,x_{n+2})}} \varphi(t)\,dt + \gamma \int_0^{d(x_{n+1},z)} \varphi(t)\,dt$$

$$\leq \int_0^{d(z,x_{n+2})} \varphi(t)\,dt + \alpha \int_0^{d(x_{n+1},x_{n+2})+d(z,fz)} \varphi(t)\,dt + \beta \int_0^{d(x_{n+1},fz)+d(z,x_{n+2})} \varphi(t)\,dt + \gamma \int_0^{d(x_{n+1},z)} \varphi(t)\,dt$$

Taking limit on both sides as n→∞, we get

$$\lim_{n\to\infty} \int_0^{d(z,fz)} \varphi(t)\,dt = 0$$

Which from (2.10), implies that $\lim_{n\to\infty} d(z, fz) = 0$ *or* $fz = z$ .Similarly it can be shown that $g z = z$.

So f and g have a common fixed point $z \in X$.

FOR UNIQUENESS:

We now show that z is the unique common fixed point of f and g. If not, then let w be another common fixed point of f and g. Then from (2.9) we have

$$\int_0^{d(z,w)} \varphi(t)\,dt = \int_0^{d(fz,gw)} \varphi(t)\,dt$$

$$\leq \alpha \int_0^{\left[\frac{d^3(z,fz)+d^3(w,gw)}{1+d^2(z,fz)+d^2(w,gw)}\right]} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(z,gw)+d^2(w,fz)}{1+d(z,gw)+d(w,fz)}} \varphi(t)\,dt + \gamma \int_0^{d(z,w)} \varphi(t)\,dt$$

$$\leq \beta \int_0^{2\,d(z,w)} \varphi(t)\,dt + \gamma \int_0^{d(z,w)} \varphi(t)\,dt$$

$$\leq (2\beta + \gamma) \int_0^{d(z,w)} \varphi(t)\,dt$$

Since, $(2\beta + \gamma) < 1$, this implies that

$$\int_0^{d(z,w)} \varphi(t)\,dt = 0$$

which, from (2.10), implies that $d(z, w) = 0$ or, $z = w$ and so the fixed point is unique.

Theorem 2.3 Let f ,g and h be self mappings of a complete metric space (X, d) satisfying the following condition:

(i) $fg = gf$ *and* $hg = gh$ (2.18)

(ii) $\int_0^{d(fgx,hgy)} \varphi(t)\,dt \leq \alpha \int_0^{\left[\frac{d^3(x,fgx)+d^3(y,hgy)}{1+d^2(x,fgx)+d^2(y,hgy)}\right]} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x,hgy)+d^2(y,fgx)}{1+d(x,hgy)+d(y,fgx)}} \varphi(t)\,dt + \gamma \int_0^{d(x,y)} \varphi(t)\,dt$ (2.19)

for each $x, y \in X$ with nonnegative reals α, β, γ such that $2\alpha + 2\beta + \gamma < 1$, where $\varphi: R^+ \to R^+$ is a Lesbesgue-integrable mapping which is summable on each compact subset of R, and such that

for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)\,dt > 0$ (2.20)

then f , g and h have a unique common fixed point $z \in X$.

Proof. For any arbitrary $x_0 \in X$, there is x_1 in X such that $x_1 = fgx_0$ *and* $x_2 = hg\,x_1$. Proceeding the same way, we construct a sequence $\{x_n\}$ of element of X, Such that $x_{n+1} = fg\,x_n$. and $x_{n+2} = hg\,x_{n+1}$.

For each integer $n = 0, 1, 2\ldots$

From (2.19)

$$\int_0^{d(x_{n+1},x_{n+2})} \varphi(t)\,dt = \int_0^{d(fgx_n,hgx_{n+1})} \varphi(t)\,dt$$

$$\leq \alpha \int_0^{\frac{d^3(x_n, fg\,x_n)+d^3(x_{n+1}, hg\,x_{n+1})}{1+d^2(x_n, f\,x_n)+d^2(x_{n+1}, g\,x_{n+1})}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_n,\ hg\,x_{n+1})+d^2(x_{n+1}, fg\,x_n)}{1+d\left(x_n,\ hg\,x_{n+1}\right)+d(x_{n+1}, f\,x_n)}} \varphi(t)\,dt + \gamma \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\leq \alpha \int_0^{\frac{d^3(x_n, x_{n+1})+d^3(x_{n+1}, x_{n+2})}{1+d^2(x_n, x_{n+1})+d^2(x_{n+1}, x_{n+2})}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_n, x_{n+2})+d^2(x_{n+1}, x_{n+1})}{1+d(x_n, x_{n+2})+d(x_{n+1}, x_{n+1})}} \varphi(t)\,dt + \gamma \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\leq \alpha \int_0^{d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2})} \varphi(t)\,dt + \beta \int_0^{d(x_n, x_{n+2})} \varphi(t)\,dt + \gamma \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\leq \alpha \int_0^{d(x_{n+1}, x_{n+2})} \varphi(t)\,dt + \beta \int_0^{d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2})} \varphi(t)\,dt + (\alpha+\gamma) \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\leq (\alpha+\beta) \int_0^{d(x_{n+1}, x_{n+2})} \varphi(t)\,dt + (\alpha+\beta+\gamma) \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\int_0^{d(x_{n+1}, x_{n+2})} \varphi(t)\,dt \leq \frac{(\alpha+\beta+\gamma)}{(1-\alpha-\beta)} \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt$$

$$\int_0^{d(x_{n+1}, x_{n+2})} \varphi(t)\,dt \leq r \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt \tag{2.21}$$

where $\frac{(\alpha+\beta+\gamma)}{(1-\alpha-\beta)} = r$ (say) < 1 (Since $2\alpha + 2\beta + \gamma < 1$)

Similarly

$$\int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt \leq r \int_0^{d(x_{n-1}, x_n)} \varphi(t)\,dt \tag{2.22}$$

Thus by routine calculation

$$\int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt \leq r^n \int_0^{d(x_0, x_1)} \varphi(t)\,dt \tag{2.23}$$

Taking limit of (2.23)as $n \to \infty$, we get

$$\lim_{n\to\infty} \int_0^{d(x_n, x_{n+1})} \varphi(t)\,dt = 0$$

Which from (2.20) implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{2.24}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequence *{m (p)}* and *{n (p)}* such that m(*p) < {n(p)} < {m(p+1)}* with

$$d\left(x_{m(p)}, x_{n(p)}\right) \geq \epsilon,\ d\left(x_{m(p)}, x_{n(p)-1}\right) < \epsilon \tag{2.25}$$

Now

$$d\left(x_{m(p)-1}, x_{n(p)-1}\right) < d\left(x_{m(p)-1}, x_{m(p)}\right) + d\left(x_{m(p)}, x_{n(p)-1}\right)$$

$$< d\left(x_{m(p)-1}, x_{m(p)}\right) + \epsilon \tag{2.26}$$

Hence

$$\lim_{p\to\infty} \int_0^{d\left(x_{m(p)-1}, x_{n(p)-1}\right)} \varphi(t)\,dt \leq \int_0^{\epsilon} \varphi(t)\,dt \tag{2.27}$$

Using (2.25), (2.22), and (2.27), we get

$$\int_0^{\epsilon} \varphi(t)\,dt \leq \int_0^{d\left(x_{m(p)}, x_{n(p)}\right)} \varphi(t)\,dt \leq r \int_0^{d\left(x_{m(p)-1}, x_{n(p)-1}\right)} \varphi(t)\,dt \leq r\int_0^{\epsilon} \varphi(t)\,dt$$

Which is a contradiction, since $r \in (0, 1)$.Therefore,$\{x_n\}$ is Cauchy sequence converges to $z \in X$.

From (2.19) we get

$$\int_0^{d(z,fgz)} \varphi(t)\,dt \le \int_0^{d(z,hgx_{n+1})+d(hgx_{n+1},fgz)} \varphi(t)\,dt = \int_0^{d(z,x_{n+2})+d(hgx_{n+1},fgz)} \varphi(t)\,dt$$

$$= \int_0^{d(z,x_{n+2})} \varphi(t)\,dt + \int_0^{d(hgx_{n+1},fgz)} \varphi(t)\,dt$$

$$\le \int_0^{d(z,x_{n+2})} \varphi(t)\,dt + \alpha \int_0^{\frac{d^3(x_{n+1},ghx_{n+1})+d^3(z,fgz)}{1+d^2(x_{n+1},ghx_{n+1})+d^2(z,fgz)}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_{n+1},fgz)+d^2(z,hgx_{n+1})}{1+d(x_{n+1},fgz)+d(z,hgx_{n+1})}} \varphi(t)\,dt + \gamma \int_0^{d(x_{n+1},z)} \varphi(t)\,dt$$

$$\le \int_0^{d(z,x_{n+2})} \varphi(t)\,dt + \alpha \int_0^{\frac{d^3(x_{n+1},x_{n+2})+d^3(z,fgz)}{1+d^2(x_{n+1},x_{n+2})+d^2(z,fgz)}} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(x_{n+1},fgz)+d^2(z,x_{n+2})}{1+d(x_{n+1},fgz)+d(z,x_{n+2})}} \varphi(t)\,dt + \gamma \int_0^{d(x_{n+1},z)} \varphi(t)\,dt$$

$$\le \int_0^{d(z,x_{n+2})} \varphi(t)\,dt + \alpha \int_0^{d(x_{n+1},x_{n+2})+d(z,fgz)} \varphi(t)\,dt + \beta \int_0^{d(x_{n+1},fgz)+d(z,x_{n+2})} \varphi(t)\,dt + \gamma \int_0^{d(x_{n+1},z)} \varphi(t)\,dt$$

$$\le (1+\beta) \int_0^{d(z,x_{n+2})} \varphi(t)\,dt + \alpha \int_0^{d(x_{n+1},x_{n+2})} \varphi(t)\,dt + (\alpha+\beta) \int_0^{d(z,fgz)} \varphi(t)\,dt + (\beta+\gamma) \int_0^{d(x_{n+1},z)} \varphi(t)\,dt$$

Taking limit as $n\to\infty$, we get

$$\lim_{n\to\infty} \int_0^{d(z,fgz)} \varphi(t)\,dt = 0$$

Which from (2.20), implies that $\lim_{n\to\infty} d(z, fg\,z)= 0$ *or* $fg\,z = z$. Similarly it can be shown that $hg\,z = z$. So *f*, g *and h* have a common fixed point $z \in X$.

FOR UNIQUENESS

We now show that *z* is the unique common fixed point of *f* , *g and h*. If not, then let *w* be another common fixed point of *f, and g* . Then from (2.19) we have

$$\int_0^{d(z,w)} \varphi(t)\,dt = \int_0^{d(fz,gw)} \varphi(t)\,dt$$

$$\le \alpha \int_0^{\left[\frac{d^3(z,fz)+d^3(w,gw)}{1+d^2(z,fz)+d^2(w,gw)}\right]} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(z,gw)+d^2(w,fz)}{1+d(z,gw)+d(w,fz)}} \varphi(t)\,dt + \gamma \int_0^{d(z,w)} \varphi(t)\,dt$$

$$\le \beta \int_0^{2\,d(z,w)} \varphi(t)\,dt + \gamma \int_0^{d(z,w)} \varphi(t)\,dt$$

$$\le (2\beta+\gamma) \int_0^{d(z,w)} \varphi(t)\,dt$$

Since, $(2\beta + \gamma) < 1$, this implies that

$$\int_0^{d(z,w)} \varphi(t)\,dt = 0$$

which, from (2.20), implies that $d(z, w) = 0$ or, $z = w$ and so the fixed point is unique.

Now we prove another fixed point theorem which is generalization of theorem 2.3

Theorem 2.4 Let *f, g* and *h* be self mappings of a complete metric space (X, d) satisfying the following conditions:

(i) $f(X) \subseteq h(X)$ and $g(X) \subseteq h(X)$ (2.28)

(ii) $$\int_0^{d(fx,gy)} \varphi(t)\,dt \le \alpha \int_0^{\left[\frac{d^3(hx,fx)+d^3(hy,gy)}{1+d^2(hx,fx)+d^2(hy,gy)}\right]} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(hx,gy)+d^2(hy,fx)}{1+d(hx,gy)+d(hy,fx)}} \varphi(t)\,dt + \gamma \int_0^{d(hx,hy)} \varphi(t)\,dt \quad (2.29)$$

(iii) $fg = gf\ , fh = hf\ and\ gh = hg$

for each $x, y \in X$ with nonnegative reals α, β, γ such that $2\alpha + 2\beta + \gamma < 1$, where $\varphi: R^+ \to R^+$ is a Lesbesgue-integrable mapping which is summable on each compact subset of R, and such that

for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)\, dt > 0$ (2.30)

then f, g and h have a unique common fixed point $z \in X$.

Proof. For any arbitrary $x_0 \in X$, there is x_1 in X such that $fx_0 = h\,x_1 = y_0$ and $gx_1 = hx_1 = y_1$. Proceeding the same way we construct a sequence $\{x_n\}$ of element of X, Such that
$fx_n = hx_{n+1} = y_n$. and gx_{n+1}= h $x_{n+2} = y_{n+1}$.

For each integer $n = 0, 1, 2\ldots\ldots$

From (2.29)

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)\, dt = \int_0^{d(fx_n,gx_{n+1})} \varphi(t)\, dt$$

$$\leq \alpha \int_0^{\frac{d^3(hx_n,fx_n)+d^3(h\,x_{n+1},gx_{n+1})}{1+d^2(hx_n,fx_n)+d^2(h\,x_{n+1},gx_{n+1})}} \varphi(t)\, dt + \beta \int_0^{\frac{d^2(h\,x_n,gx_{n+1})+d^2(h\,x_{n+1},fx_n)}{1+d(h\,x_n,gx_{n+1})+d(h\,x_{n+1},fx_n)}} \varphi(t)\, dt + \gamma \int_0^{d(hx_n,hx_{n+1})} \varphi(t)\, dt$$

$$\leq \alpha \int_0^{\frac{d^3(y_{n-1},y_n)+d^3(y_n,\ y_{n+1})}{1+d^2(y_{n-1},y_n)+d^2(y_n,\ y_{n+1})}} \varphi(t)\, dt + \beta \int_0^{\frac{d^2(y_{n-1},y_{n+1})+d^2(y_n,\ y_n)}{1+d(y_{n-1},y_{n+1})+d(y_n,\ y_n)}} \varphi(t)\, dt + \gamma \int_0^{d(y_{n-1},y_n)} \varphi(t)\, dt$$

$$\leq \alpha \int_0^{d(y_{n-1},y_n)+d(y_n,y_{n+1})} \varphi(t)\, dt + \beta \int_0^{d(y_{n-1},y_{n+1})} \varphi(t)\, dt + \gamma \int_0^{d(y_{n-1},y_n)} \varphi(t)\, dt$$

$$(1-\alpha-\beta) \int_0^{d(y_n,y_{n+1})} \varphi(t)\, dt \leq (\alpha+\beta+\gamma) \int_0^{d(y_{n-1},y_n)} \varphi(t)\, dt$$

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)\, dt \leq r \int_0^{d(y_{n-1},y_n)} \varphi(t)\, dt \tag{2.31}$$

where $\frac{(\alpha+\beta+\gamma)}{(1-\alpha-\beta)} = r$ (say) < 1 (Since $2\alpha + 2\beta + \gamma < 1$)

Thus by routine calculation

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)\, dt \leq r^n \int_0^{d(y_0,y_1)} \varphi(t)\, dt \tag{2.32}$$

Taking limit of (2.32) as $n \to \infty$, we get

$$\lim_{n\to\infty} \int_0^{d(y_n,y_{n+1})} \varphi(t)\, dt = 0$$

Which from (2.30) implies that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \tag{2.33}$$

We now show that $\{y_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequence $\{m(p)\}$ and $\{n(p)\}$ such that m(p) < $\{n(p)\}$ < $\{m(p+1)\}$ with

$$d\left(y_{m(p)}, y_{n(p)}\right) \geq \epsilon,\ d\left(y_{m(p)}, y_{n(p)-1}\right) < \epsilon \tag{2.34}$$

Now

$$d\left(y_{m(p)-1}, y_{n(p)-1}\right) < d\left(y_{m(p)-1}, y_{m(p)}\right) + d\left(y_{m(p)}, y_{n(p)-1}\right)$$

$$< d\left(y_{m(p)-1}, y_{m(p)}\right) + \epsilon \tag{2.35}$$

Hence

$$\lim_{p\to\infty} \int_0^{d\,(y_{m(p)-1}\,, y_{n(p)-1})} \varphi(t)\,dt \le \int_0^{\epsilon} \varphi(t)\,dt \tag{2.36}$$

Using (2.34), (2.31), and (2.36), we get

$$\int_0^{\epsilon} \varphi(t)\,dt \le \int_0^{d\,(y_{m(p)}\,, y_{n(p)})} \varphi(t)\,dt \le r \int_0^{d\,(y_{m(p)-1}\,, y_{n(p)-1})} \varphi(t)\,dt \le r\int_0^{\epsilon} \varphi(t)\,dt$$

Which is a contradiction, since $r \in (0, 1)$.Therefore, $\{y_n\}$ is Cauchy sequence converges to z ∈ X.

From (2.29) we get

$$\int_0^{d(fz,z)} \varphi(t)\,dt = \int_0^{d(fz,y_{n+1})+d(y_{n+1},z)} \varphi(t)\,dt = \int_0^{d(fz,gx_{n+1})+d(y_{n+1},z)} \varphi(t)\,dt$$

$$\le \alpha \int_0^{\left[\frac{d^3(hz,fz)+d^3(hx_{n+1},gx_{n+1})}{1+d^2(hz,fz)+d^2(hx_{n+1},gx_{n+1})}\right]} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(hz,gx_{n+1})+d^2(hx_{n+1},fz)}{1+d(hz,gx_{n+1})+d(hx_{n+1},fz)}} \varphi(t)\,dt$$

$$+ \gamma \int_0^{d(hz,hx_{n+1})} \varphi(t)\,dt + \int_0^{d(hx_{n+2},z)} \varphi(t)\,dt$$

$$\le \alpha \int_0^{[d(hz,fz)+d(hx_{n+1},gx_{n+1})]} \varphi(t)\,dt + \beta \int_0^{d(hz,gx_{n+1})+d(hx_{n+1},fz)} \varphi(t)\,dt$$

$$+ \gamma \int_0^{d(hz,hx_{n+1})} \varphi(t)\,dt + \int_0^{d(hx_{n+2},z)} \varphi(t)\,dt$$

$$\le \alpha \int_0^{[d(z,fz)+d(y_n,y_{n+1})]} \varphi(t)\,dt + \beta \int_0^{d(z,y_{n+1})+d(y_n,fz)} \varphi(t)\,dt + \gamma \int_0^{d(z,y_n)} \varphi(t)\,dt + \int_0^{d(y_{n+1},z)} \varphi(t)\,dt$$

$$(1-\alpha-\beta) \int_0^{d(fz,z)} \varphi(t)\,dt \le \alpha \int_0^{[d(y_n,y_{n+1})]} \varphi(t)\,dt + (1+\beta) \int_0^{d(z,y_{n+1})} \varphi(t)\,dt + (\beta+\gamma) \int_0^{d(z,y_n)} \varphi(t)\,dt$$

Taking limit on both sides as n→∞, we get

$$\lim_{n\to\infty} \int_0^{d\,(z,fz)} \varphi(t)\,dt = 0$$

Which from (2.30), implies that $\lim_{n\to\infty} d(z, fz) = 0$ or $fz = z$ .Similarly it can be shown that $gz = z$. So f ,g *and* h have a common fixed point $z \in X$.

FOR UNIQUENESS

We now show that z is the unique common fixed point of f, g *and* h. If not, then let w be another common fixed point of f *and* g . Then from (2.29) we have

$$\int_0^{d(z,w)} \varphi(t)\,dt = \int_0^{d(fz,gw)} \varphi(t)\,dt$$

$$\le \alpha \int_0^{\left[\frac{d^3(z,fz)+d^3(w,gw)}{1+d^2(z,fz)+d^2(w,gw)}\right]} \varphi(t)\,dt + \beta \int_0^{\frac{d^2(z,gw)+d^2(w,fz)}{1+d(z,gw)+d(w,fz)}} \varphi(t)\,dt + \gamma \int_0^{d(z,w)} \varphi(t)\,dt$$

$$\le \beta \int_0^{2\,d(z,w)} \varphi(t)\,dt + \gamma \int_0^{d(z,w)} \varphi(t)\,dt$$

$$\le (2\beta+\gamma) \int_0^{d(z,w)} \varphi(t)\,dt$$

Since, $(2\beta + \gamma) < 1$, this implies that

$$\int_0^{d(z,w)} \varphi(t)\,dt = 0$$

which, from (2.30), implies that $d(z, w) = 0$ or, $z = w$ and so the fixed point is unique.

REMARKS

(a) In theorem 2.1,
(i) if we take $\alpha = \beta = 0$ and $\gamma \in (0,1)$ gives Branciari mapping of integral type.
(ii) by taking $\varphi(t) = 1$ over R^+, the contractive condition of integral type transforms into a Banach contractive condition.

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Source of support: Nil, Conflict of interest: None Declared

