EXISTENCE THEORY FOR BOUNDARY VALUE PROBLEM
OF RANDOM DIFFERENTIAL INCLUSION

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ABSTRACT

In this paper, I prove the existence of random solution for the boundary value problem of second order multi-valued differential inclusion with non-convex case using a nonlinear alternative of Leray-Schauder type and on a selection theorem for lower semi continuous maps.

Keywords: Random differential inclusion, multi-valued random operator, random solution, boundary value problem.

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1. INTRODUCTION

Consider the two point boundary value problem (BVP) of second order differential inclusions

\[ x^{\prime\prime}(t) \in F\left(t, x(t), x^{\prime}(t)\right) \quad \text{a.e. } t \in J \]  \hspace{1cm} (1.1)

satisfying the boundary conditions

\[ \begin{aligned}
    a_0 x(t_0) - a_1 x^{\prime}(t_1) &= c_0 \\
    b_0 x(t_0) + b_1 x^{\prime}(t_1) &= c_1
\end{aligned} \]  \hspace{1cm} (1.2)

where the functions involved in (1.1) and (1.2) satisfy the following properties:

(a) \( F: J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}_f\left(\mathbb{R}\right) \),
(b) \( a_0, a_1, b_0, b_1 \in \mathbb{R}^+ \) satisfying \( a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0 > 0 \) and
(c) \( c_0, c_1 \in \mathbb{R} \).

Let \( J = [t_0, t_1] \) be a closed and bounded interval in \( \mathbb{R} \) for some real numbers \( t_0, t_1 \in \mathbb{R} \) with \( t_0 < t_1 \). Let \( \mathcal{P}_f\left(\mathbb{R}\right) \) denote the class of all non-empty subsets of \( \mathbb{R} \) with a property \( f \). By a solution of the BVP (1.1) – (1.2), I mean a function \( x \in AC^1(J, \mathbb{R}) \) whose second derivative exists and is a member of \( L^1(J, \mathbb{R}) \) in \( F\left(t, x(t), x^{\prime}(t)\right) \), i.e. there exists a \( v \in L^1(J, \mathbb{R}) \) such that \( v(t) \in F\left(t, x(t), x^{\prime}(t)\right) \) for a.e. \( t \in J \) and \( x^{\prime\prime}(t) = v(t) \) for all \( t \in J \) satisfying (1.2), where \( AC^1(J, \mathbb{R}) \) is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on \( J \).

The special cases of the BVP (1.1)-(1.2) have been discussed in the literature for existence of the solutions. The special case of the form

\[ x^{\prime\prime}(t) = f\left(t, x(t), x^{\prime}(t)\right) , \quad \text{a.e. } t \in J \]  \hspace{1cm} (1.3)

satisfying the boundary conditions (1.2) where \( f: J \times \mathbb{R} \rightarrow \mathbb{R} \), \( a_0, a_1, b_0, b_1 \in \mathbb{R}^+ \), \( c_0, c_1 \in \mathbb{R} \) and \( a_0 a_1 (t_1 - t_0) + a_0 b_1 + a_1 b_0 > 0 \) has been discussed in Bernfeld and Lakshmikantham [2] for the existence of solutions. Again, when \( c_0 = c_1, a_1 = 0 = b_1, a_0 = b_0 \), and \( F \) not depending on \( x^{\prime} \), the BVP (1.1) – (1.2) reduces

\[ y^{\prime\prime} \in F\left(t, y\right) , \quad \text{a.e. } t \in J , \quad y(t_0) = y(t_1). \]  \hspace{1cm} (1.4)

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where \( y = x \). This is a BVP of second order differential inclusions considered in Benchohra and Ntouyas [3]. Finally, the special case of the BVP consisting of the equation
\[
y''(t) \in F\left(t, y(t)\right), \quad a.e. \quad t \in J
\]
(1.5)
satisfying the boundary conditions (1.2) has been studied in Halidias and Papageorgiou [5] via the method of lower and upper solutions. Thus the BVP (1.1) - (1.2) is more general and so is its importance in the theory of differential inclusions. Here in the present paper, I discuss the BVP (1.1) - (1.2) via a nonlinear alternative of Leray-Schauder type and on a selection theorem for lower semi continuous maps. I prove the main existence results for the BVP (1.1) – (1.2) when the right hand side has nonconvex values.

2. AUXILIARY RESULTS

I apply the following nonlinear alternative in the sequel.

**Theorem 2.1** (O'Regan [8]) Let \( U \) and \( \overline{U} \) be the open and closed subsets in a normed linear space \( X \) such that \( 0 \in U \) and let \( T : \overline{U} \to \mathcal{P}_{cp,cv}(X) \) be a completely continuous multi-valued operator. Then either

(i) the operator inclusion \( x \in Tx \) has a solution, or

(ii) there is an element \( u \in \partial U \) such that \( \lambda u \in Tu \) for some \( \lambda > 1 \), where \( \partial U \) is the boundary of \( U \).

**Corollary 2.1** Let \( \overline{B}_r(0) \) and \( \overline{B}_r(O) \) be the open and closed balls in a normed linear space \( X \) centered at origin 0 of radius \( r \) and let \( T : \overline{B}_r(0) \to \mathcal{P}_{cp,cv}(X) \) be a completely continuous multi-valued operator. Then either

(i) the operator inclusion \( x \in Tx \) has a solution, or

(ii) there is an element \( u \in X \) such that \( \|u\| = r \) satisfying \( \lambda u \in Tu \) for some \( \lambda > 1 \).

**Corollary 2.2** Let \( \overline{B}_r(0) \) and \( \overline{B}_r(O) \) be the open and closed balls in a normed linear space \( X \) centered at origin 0 of radius \( r \) and let \( T : \overline{B}_r(0) \to X \) be a completely continuous multi-valued map. Then either (1) the operator inclusion \( x = Tx \) has a solution, or (2) there is an element \( u \in X \) such that \( \|u\| = r \) and \( u = \lambda Tu \) for some \( \lambda < 1 \).

Now, I state a selection theorem due to Bressan and Colombo [4].

**Theorem 2.2** Let \( Y \) be a separable metric space and let \( N : Y \to \mathcal{P}_c(L^1(J, \mathbb{R})) \) be a multi-valued operator which has property \((BC)\). Then \( N \) has a continuous selection, i.e. there exists a continuous function (single-valued) \( g : Y \to L^1(J, \mathbb{R}) \) such that \( g(y) \in N(y) \) for every \( y \in Y \).

3. EXISTENCE RESULTS

I have written a useful result from the theory of boundary value problems of ordinary differential equations.

**Lemma 3.1** If \( f \in L^1(J, \mathbb{R}) \), then the BVP
\[
x''(t) = f(t) \quad a.e. \quad t \in J \quad and \quad \left\{
\begin{align*}
    a_0 x(t_0) - a_1 x'(t_1) &= c_0 \\
    b_0 x(t_0) + b_1 x'(t_1) &= c_1
\end{align*}
\right.
\]
(3.1)
has a unique solution \( x \) given by
\[
x(t) = z(t) + \int_{t_0}^{t_f} G(t,s)f(s)ds, \quad t \in J,
\]
(3.2)
where \( z \) is a unique solution of the homogeneous differential equation
\[
x''(t) = 0 \quad a.e. \quad t \in J \quad and \quad \left\{
\begin{align*}
    a_0 x(t_0) - a_1 x'(t_1) &= c_0 \\
    b_0 x(t_0) + b_1 x'(t_1) &= c_1
\end{align*}
\right.
\]
(3.3)
given by
\[ z(t) = \frac{c_0a_1(t_1-t) + c_0b_1 + c_1a_0(t-t_0) + c_1b_0}{a_0a_1(t_1-t_0) + a_0b_1 + a_1b_0}, \quad t \in J \] (3.4)
and \( G(t,s) \) is the Green’s function associated to the differential equation
\[ x''(t) = 0 \quad a.e. \quad t \in J \quad \text{and} \quad \begin{cases} a_0x(t_0) - a_1x'(t_1) = 0 \\ b_0x(t_0) + b_1x'(t_1) = 0 \end{cases} \] (3.5)
given
\[
G(t,s) = \begin{cases} \frac{(a_1(t_1-t) + b_1)(a_0(s-t_0) + b_0)}{a_0a_1(t_1-t_0) + a_0b_1 + a_1b_0}, & t_0 \leq s \leq t \leq t_1, \\ \frac{(a_1(t_1-s) + b_1)(a_0(t-t_0) + b_0)}{a_0a_1(t_1-t_0) + a_0b_1 + a_1b_0}, & t_0 \leq t \leq s \leq t_1, \end{cases}
\] (3.6)

**Remark 3.1** It is known that the function \( z \) belongs to the class \( C^1(J, \mathbb{R}) \). Therefore it is bounded on \( J \) and there is a constant \( C_1 > 0 \) with
\[
C_1 = \max \left\{ \frac{c_0a_1(t_1-t_0) + c_0b_1 + c_1a_0(t_1-t_0) + c_1b_0}{a_0a_1(t_1-t_0) + a_0b_1 + a_1b_0}, \frac{c_0b_1 - c_0a_1 + c_1a_0 + c_1b_0}{a_0a_1(t_1-t_0) + a_0b_1 + a_1b_0} \right\}
\]
such that
\[
\|z\| = \max \left\{ \sup_{t \in J} |z(t)|, \sup_{t \in J} |z'(t)| \right\} \leq C_1.
\]

**Remark 3.2** It is easy to see that the Green’s function \( G(t,s) \) of Lemma 3.1 is continuous in \( J \times J \) and \( G(t,s) \) is continuous in \( (a,b) \times (a,b) \setminus \{(t,t) \mid t \in J\} \) and satisfy the inequalities
\[
|G(t,s)| = G(t,s) \leq \frac{(a_1(t_1-t_0) + b_1)(a_0(t_1-t_0) + b_0)}{a_0a_1(t_1-t_0) + a_0b_1 + a_1b_0} = K_1,
\] (3.7)
\[
|G(t,s)| = \begin{cases} \frac{-a_1(a_0(s-t_0) + b_0)}{a_0a_1(t_1-t_0) + a_0b_1 + a_1b_0}, & t_0 < s < t < t_1, \\ \frac{(a_1(t_1-s) + b_1)a_0}{a_0a_1(t_1-t_0) + a_0b_1 + a_1b_0}, & t_0 < t < s < t_1. \end{cases}
\]

and
\[
= \max \left\{ \frac{a_1(a_0(t_1-t_0) + b_0)}{a_0a_1(t_1-t_0) + a_0b_1 + a_1b_0}, \frac{(a_1(t_1-t_0) + b_1)a_0}{a_0a_1(t_1-t_0) + a_0b_1 + a_1b_0} \right\}
\] (3.8)

Now, I study the case, when \( F \) is not necessarily convex valued. I give result, based on the Leray-Schauder Alternative for single valued maps combined with a selection theorem due to Bressan and Colombo [4] for lower semicontinuous multi-valued operators with decomposable values.

The following assumptions will be needed for proving main existence resul
\[
(H_1) \text{There exists a function } \phi \in L^1(J, \mathbb{R}) \text{ with } \phi(t) > 0 \text{ for a.e. } t \in J \text{ and there is a nondecreasing function } \psi : \mathbb{R}^+ \rightarrow (0, \infty) \text{ such that }
\]
\[
\|F(t,x,y)\|_p = \sup_{u \in F(t,x,y)} |u| \leq \phi(t) \psi \left( \max \{|x|, |y|\} \right) \text{ for a.e. } t \in J \text{ and for all } x, y \in \mathbb{R}.
\]
The multi-valued function $F(t, x, y)$ is measurable and integrably bounded for all $x, y \in \mathbb{R}$.

The multi-function $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ satisfies

$$H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq \ell_1(t)|x_1 - y_1| + \ell_2(t)|x_2 - y_2| \quad \text{a.e. } t \in J$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

The multi-function $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ satisfies

a) $(t, x, y) \mapsto F(t, x, y)$ is $(\mathcal{L} \otimes \mathcal{B} \otimes \mathcal{B})$-measurable, and

b) $(x, y) \mapsto F(t, x, y)$ is lower semi-continuous for almost every $t \in J$.

Lemma 3.2 Let $F : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be an integrably bounded multi-valued function satisfying $(H_4)$. Then $F$ is of lower semi-continuous type.

**MAIN RESULT**

**Theorem 3.3** Assume the hypotheses $(H_1)$-$(H_4)$ hold and there exists a real number $r > 0$ satisfying

$$r > C_1 + \max\{K_1, K_2\} \frac{\|\phi\|_{L^1}}{\psi(r)}.$$  \hspace{1cm} (3.9)

where $C_1, K_1$ and $K_2$ are the constants defined in Remark 3.2. Then the BVP (1.1) – (1.2) has at least one solution on $J$.

**Proof:** First, I transform the BVP (1.1) – (1.2) into a fixed-point problem in a suitable normed linear space. The problem of existence of a solution of BVP (1.1) – (1.2) reduces to finding a solution of the integral equation

$$x(t) = z(t) + \int_{t_0}^{t_1} k(t, s) f(x(s)) ds, \quad t \in J,$$  \hspace{1cm} (3.10)

where $f(x(\cdot)) \in L^1$ with $f(x(t)) \in F(t, x(t), x'(t))$ a.e. $t \in J$. I study the integral equation (6.3.13) in the space $AC^1(J, R)$. Let $X = AC^1(J, R)$ and define an open ball $\mathcal{B}_r(0)$ in $X$ centered at origin 0 of radius $r$, where the real number $r > 0$ satisfies the inequality (3.9). Define the operator $T$ on $\mathcal{B}_r(0)$ by

$$Tx(t) = z(t) + \int_{t_0}^{t_1} k(t, s) f(x(s)) ds.$$  \hspace{1cm} (3.11)

Then the integral equation (3.11) is equivalent to the operator equation

$$x(t) = Tx(t), \quad t \in J.$$  \hspace{1cm} (3.12)

I will show that the multi-valued operator $T$ satisfies all the conditions of Corollary 2.2.

First, I show that $T$ is continuous on $\overline{\mathcal{B}_r(0)}$. Since $(H_1)$ holds, then

$$\left| f(x(t)) \right| \leq \phi(t) \psi\left(\max\{|x(t)|, |x'(t)|\}\right) \quad \text{a.e. } t \in J$$

for all $x \in AC^1(J, R)$. Let $\{x_n\}$ be a sequence in $\overline{\mathcal{B}_r(0)}$ converging to a point $x \in \overline{\mathcal{B}_r(0)}$.

Then,

$$\left| f(x_n(t)) \right| \leq \phi(t) \psi(r) \quad \text{a.e. } t \in J.$$
Hence, by the dominated convergence theorem and continuity of \( f \), I have

\[
\lim_{n \to \infty} T x_n(t) = z(t) + \int_{t_0}^{t} G(t,s) f(x_n(s)) \, ds
\]

\[
= T x(t)
\]

and

\[
\lim_{n \to \infty} (T x_n)'(t) = z'(t) + \int_{t_0}^{t} G'(t,s) f(x_n(s)) \, ds
\]

\[
= (T x)'(t)
\]

for all \( t \in J \). As a result, \( T \) is continuous on \( \overline{B_r(0)} \). Next, using theorem as “Assume that (1) \( F \) is Carathéodory and (2) (\( H_3 \)) hold. Suppose that there is a real number \( r > 0 \) such that \( r > C_1 + \max \{ K_1, K_2 \} \| \phi \|_{L^1} \psi(r) \). Then the BVP (1.1)-(1.2) has at least one solution \( u \) such that \( \| u \| \leq r \). Following the arguments as in above mentioned theorem, it is shown that \( T \) is a compact operator on \( \overline{B_r(0)} \). Now an application of Corollary 2.2 yields that either (i) the operator equation \( x = T x \) has a solution \( \overline{B_r(0)} \), or (ii) there is an element \( u \in X \) such that \( \| u \| = r \) and \( u = \lambda T u \) for some \( \lambda \in (0,1) \). If the assertion (ii) holds, then we obtain a contradiction to multivalued operator definition. Hence assertion (i) holds and the BVP (1.1 – (1.2) has a solution \( u \in AC^1(J,R) \) such that \( \| u \| \leq r \). This completes the proof.

REFERENCES


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