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GROWTH ANALYSIS OF DIFFERENTIAL MONOMIALS AND DIFFERENTIAL POLYNOMIALS GENERATED BY ENTIRE OR MEROMORPHIC FUNCTIONS

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ABSTRACT

In this paper we investigate the comparative growth of composite entire or meromorphic functions and differential monomials , differential polynomials generated by one of the factors which improves some earlier results .

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1 INTRODUCTION, DEFINITIONS AND NOTATIONS

For any two transcendental entire functions f and g defined in the open complex plane $\mathbb{C}$, Clunie [3] proved that

$$\lim_{r\to\infty}\frac{T(r,fog)}{T(r,f)}=\infty \text{ and } \lim_{r\to\infty}\frac{T(r,fog)}{T(r,g)}=\infty\cdot$$

Singh [14] proved some comparative growth properties of $\log T(r,fog)$ and $T(r,f)$. He also raised the problem of investigating the comparative growth of $\log T(r,fog)$ and $T(r,g)$ which he was unable to solve. However, some results on the comparative growth of $\log T(r,fog)$ and $T(r,g)$ are proved in [8].

Let f be a non-constant mereomorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_{0j}, n_{1j}, \ldots. n_{kj}$ $(k\geq 1)$ be non-negetive integers such that for each j , $\sum_{i=0}^{k} n_{ij} \geq 1$. We call $M_j[f]=A_j\,(f)^{n_{0j}}\left(f^{(1)}\right)^{n_{1j}}\ldots.\left(f^{(k)}\right)^{n_{kj}}$ where $T(r,A_j)=S(r,f)$ to be a differential monomial generated by f. The numbers $\gamma_{M_j}=\sum_{i=0}^{k} n_{ij}$ and $\Gamma_{M_j}=\sum_{i=0}^{k}(i+1)\,n_{ij}$ are called respectively the degree and weight of $M_j[f]$ {[5],[13]} . The expression $P[f]=\sum_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f The numbers $\gamma_P=\max_{1\leq j\leq s}\gamma_{M_j}$ and $\Gamma_P=\max_{1\leq j\leq s}\Gamma_{M_j}$ are called respectively the degree and weight of P[f] {[5] , [13]} . Also we call the numbers $\underline{\gamma_P}=\min_{1\leq j\leq s}\gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of P[f] respectively. If $\underline{\gamma_P}=\gamma_P$, P[f] is called a homogeneous differential polynomial. In the paper we further investigate the question of Singh [14] mentioned earlier and prove some new results relating to the comparative growths of composite entire or meromorphic functions and differential monomials , differential polynomials generated by one of

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the factors . We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [17] and [6]. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e., for which $n_{0j} = 0$ for $j = 1, 2, \ldots$ s . We consider only those P[f] , $P_0[f]$ singularities of whose individual terms do not cancel each other .

We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function f .

The following definitions are well known.

Definition 1. *The order ρ_f and lower order λ_f of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r} \cdot$$

If f is entire , one can easily verify that

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r},$$

where $\log^{[k]} \boldsymbol{x} = \log(\log^{[\boldsymbol{k}-\mathbf{1}]} \boldsymbol{x})$ for $\boldsymbol{k} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots$. and $\log^{[\mathbf{0}]} \boldsymbol{x} = \boldsymbol{x}$.

If $\rho_f < \infty$ then f is of finite order . Also $\rho_f = 0$ means that f is of order zero . In this connection Datta and Biswas [4] gave the following definition:

Definition 2. *[4] Let f be a meromorphic function of order zero. Then the quantities ρ_f^{**} and λ_f^{**} of f are defined by*

$$\rho_f^{**} = \limsup_{r \to \infty} \frac{T(r,f)}{\log r} \quad and \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{T(r,f)}{\log r} \cdot$$

If f is an entire function then clearly

$$\rho_f^{**} = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r} \quad and \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{\log M(r,f)}{\log r} .$$

Definition 3. *Let 'a' be a complex number, finite or infinite. The Nevanlinna's deficiency and the Valiron deficiency of 'a' with respect to a meromorphic function f are defined as*

$$\delta(a\,; f) = 1 - \limsup_{r \to \infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r \to \infty} \frac{m(r,a;f)}{T(r,f)}$$

and

$$\Delta(a\,; f) = 1 - \liminf_{r \to \infty} \frac{N(r,a;f)}{T(r,f)} = \limsup_{r \to \infty} \frac{m(r,a;f)}{T(r,f)} \cdot$$

Definition 4. *The quantity $\Theta(a\,; f)$ of a meromorphic function f is defined as follows*

$$\Theta(a\,; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r,a;f)}{T(r,f)} \cdot$$

Definition 5. *[16] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $n(r,a;f \mid = 1)$, the number of simple zeros of $f - a$ in $|z| \le r$. $N(r,a;f \mid = 1)$ is defined in terms of $n(r,a;f \mid = 1)$ in the usual way . We put*

$$\delta_1(a\,; f) = 1 - \limsup_{r \to \infty} \frac{N(r,a\,; f \mid = 1)}{T(r,f)},$$

the deficiency of 'a' corresponding to the simple a-points of f i.e. , simple zeros of $f - a$.

Yang [15] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a\,; f) > 0$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,; f) \le 4$.

Definition 6. *[9] For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r,a;f)$ denotes the number of zeros of $f - a$ in $|z| \le r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\ge p$ is counted exactly p times ; and $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way. We define*

$$\delta_p(a\,; f) = 1 - \limsup_{r \to \infty} \frac{N_p(r,a;f)}{T(r,f)} .$$

Definition 7. [2] $P[f]$ *is said to be admissible if*
(i) $P[f]$ is homogeneous , or
(ii) $P[f]$ is non homogeneous and $m(r,f) = S(r,f)$.

Definition 8. *A function* $\rho_f(r)$ *is called a proximate order of* f *relative to* $T(r,f)$ *if*
(i) $\rho_f(r)$ *is non-negative and continuous for* $r \geq r_0$ *, say ,*
(ii) $\rho_f(r)$ *is differentiable for* $r \geq r_0$ *except possibly at isolated points at which* $\rho_f'(r-0)$ *and* $\rho_f'(r+0)$ *exist ,*
(iii) $\lim_{r\to\infty} \rho_f(r) = \rho_f < \infty$
(iv) $\lim_{r\to\infty} r\,\rho_f'(r)\log r = 0$ *and*
(v) $\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f(r)}} = 1$.

Definition 9. *A function* $\lambda_f(r)$ *is called a lower proximate order of* f *relative to* $T(r,f)$ *if*
(i) $\lambda_f(r)$ *is non-negative and continuous for* $r \geq r_0$ *, say,*
(ii) $\lambda_f(r)$ *is differentiable for* $r \geq r_0$ *except possibly at isolated points at which* $\lambda_f'(r-0)$ *and* $\lambda_f'(r+0)$ exists
(iii) $\lim_{r\to\infty} \lambda_f(r) = \lambda_f < \infty$,
(iv) $\lim_{r\to\infty} r\,\lambda_f'(r)\log r = 0$ *and*
(v) $\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1.$

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[1] If* f *is meromorphic and* g *is entire then for all sufficiently large values of* r *,*

$$T(r, fog) \leq \{1+o(1)\}\frac{T(r,g)}{\log M(r,g)}T(M(r,g),f)\ .$$

Lemma 2. *[8] If* f *be an entire function then for* $\delta(>0)$ *the function* $r^{\rho_f+\delta-\rho_f(r)}$ *is ultimately an increasing function of* r *.*

Lemma 3. *[11] Let* f *be an entire function. Then for* $\delta(>0)$ *the function* $r^{\lambda_f+\delta-\lambda_f(r)}$ *is ultimately an increasing function of* r *.*

Lemma 4. *[2] Let* $P_0[f]$ *be admissible . If* f *is of finite order or of non-zero lower order and*

$$\sum_{a\neq\infty}\Theta(a\,;f) = 2 \quad \text{then} \quad \lim_{r\to\infty}\frac{T(r\,,P_0[f])}{T(r,f)} = \Gamma_{P_0[f]}\,.$$

Lemma 5. *[2] Let* f *be either of finite order or of non-zero lower order such that*

$\Theta(\infty\,;f) = \sum_{a\neq\infty}\delta_p(a\,;f) = 1$ *or* $\delta(\infty\,;f) = \sum_{a\neq\infty}\delta(a\,;f) = 1$ *. Then for homogeneous* $P_0[f]$ *,*

$$\lim_{r\to\infty}\frac{T(r\,,P_0[f])}{T(r,f)} = \gamma_{P_0[f]}\,.$$

Lemma 6. *Let* f *be a meromorphic function of finite order or of non zero lower order . If* $\sum_{a\neq\infty}\Theta(a\,;f) = 2$ *, then the order (lower order) of homogeneous* $P_0[f]$ *is same as that of* f *if* f *is of positive finite order .*

Proof. By Lemma 5 , $\lim_{r\to\infty}\frac{\log T(r\,,P_0[f])}{\log T(r,f)}$ exists and is equal to 1.

$$\begin{aligned}\rho_{P_0[f]} &= \limsup_{r\to\infty}\frac{\log T(r\,,P_0[f])}{\log r}\\ &= \limsup_{r\to\infty}\frac{\log T(r,f))}{\log r}\cdot\lim_{r\to\infty}\frac{\log T(r\,,P_0[f])}{\log T(r,f)}\\ &= \rho_f.1 = \rho_f.\end{aligned}$$

In a similar manner, $\lambda_{P_0[f]} = \lambda_f$.

This proves the lemma.

Lemma 7. *Let f be a meromorphic function of finite order or of non zero lower order such that* $\sum a \neq \infty$ $\delta pa; f = 1$. *Then the order (lower order) of homogeneous P0[f] and f are same when f is of finite positive order.*

We omit the proof of the lemma because it can be carried out in the line of Lemma 7 and with the help of Lemma 6. In a similar manner we can state the following lemma without proof.

Lemma 8. *Let f be a meromorphic function of finite order or of non- zero lower order such that* $\delta(\infty ; f) \sum a \neq \infty$ *δa; f* = 1. *Then for every homogeneous P0[f] , the order (lower order) of P0[f] is same as that of f. when f is of finite positive order .*

Lemma 9. *[10] Let f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a ; f) = 4$ *. Then*

$$\lim_{r \to \infty} \frac{T(r, M[f])}{T(r,f)} = \Gamma_M - (\Gamma_M - \gamma_M)\, \Theta(\infty ; f)\,,$$

where

$$\Theta(\infty ; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r,f)}{T(r,f)}\,\cdot$$

Lemma 10. *If f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a ; f) = 4$ *, then the order and lower order of M[f] are same as those of f .*

We omit the proof of the lemma because it can be carried out in the line of Lemma 6 and with the help of Lemma 9 .

Lemma 11. *[7]Let g be an entire function with* $\lambda_g < \infty$ *and assume that* a_i $(i = 1, 2, \ldots . n ; n \leq \infty)$ *are entire functions satisfying* $T(r, a_i) = \mathrm{o}\{T(r, g)\}$ *. If* $\sum_{i=1}^{n} \delta(a_i ; g) = 1$ *, then* $\lim_{r \to \infty} \frac{T(r, g)}{\log M(r,g)} = \frac{1}{\pi}\,\cdot$

3. THEOREMS

In this section we present the main results of the paper .

Theorem 1. *Let f be a meromorphic function of order zero and g be entire such that* ρ_g *is finite . Also let* $\sum_{a \neq \infty} \Theta(a ; g) = 2$ *. Then for any* $\alpha > 1$

$$\liminf_{r \to \infty} \frac{T(r, f o g)}{T(r, P_0[g])} \leq (1 + \mathrm{o}(1))\,.\left(\frac{\alpha+1}{\alpha-1}\right) \cdot \frac{\rho_f^{**}.\, \alpha^{\rho_g}}{\Gamma_{P_0[g]}}\,\cdot$$

Proof. If $\rho_f^{**} = \infty$, then the result is obvious . So we suppose that $\rho_f^{**} < \infty$. Since $\boldsymbol{T(r, g)} \leq \log^{+} \boldsymbol{M(r, g)}$, we obtain by Lemma 1 for $\varepsilon\ (> 0)$ and for all large values of r ,

$$T(r, f o g) \leq (1+ \mathrm{o}(1))\,(\rho_f^{**} + \varepsilon)\log M(r, g)$$

Since $\varepsilon\ (> 0)$ is arbitrary , it follows from above that

$$\liminf_{r \to \infty} \frac{T(r, f o g)}{T(r, P_0[g])} \leq \big(1 + \mathrm{o}(1)\big)\, \rho_f^{**} \liminf_{r \to \infty} \frac{\log M(r,g)}{T(r, P_0[g])}\,\cdot \tag{1}$$

Since $\limsup_{r \to \infty} \frac{T(r,g)}{r^{\rho_g(r)}} = 1$, for given $\varepsilon\ (0 < \varepsilon < 1)$ we get for all sufficiently large values of r

$$T(r, g) < (1 + \varepsilon)\, r^{\rho_g(r)} \tag{2}$$

and for a sequence of values of r tending to infinity

$$T(r, g) > (1 - \varepsilon)\, r^{\rho_g(r)}\,. \tag{3}$$

Since $\log M(r,g) \leq \left(\frac{\alpha+1}{\alpha-1}\right) T(\alpha r\,,g)$, {cf. [6]} for a sequence of values of r tending to infinity we get that for any $\delta\,(>0)$

$$\frac{\log M(r,g)}{T(r\,,g)} \leq \left(\frac{\alpha+1}{\alpha-1}\right)\cdot\frac{(1+\varepsilon)}{(1-\varepsilon)}\cdot\frac{(\alpha r)^{\rho_g+\delta}}{(\alpha r)^{\rho_g+\delta-\rho_g(\alpha r)}}\cdot\frac{1}{r^{\rho_g(r)}}$$
$$\leq \left(\frac{\alpha+1}{\alpha-1}\right)\cdot\frac{(1+\varepsilon)}{(1-\varepsilon)}\cdot\,\alpha^{\rho_g+\delta}$$

because by Lemma 2 , $r^{\rho_g+\delta-\rho_g(r)}$ is ultimately an increasing function of r . Since $\varepsilon(>0)$ and $\delta(>0)$ are arbitrary , we obtain that

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r\,,g)} \leq \left(\frac{\alpha+1}{\alpha-1}\right)\cdot\alpha^{\rho_g}\,. \tag{4}$$

Now in view of (4) and Lemma 4 we get that

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r\,,P_0[g])} = \liminf_{r\to\infty}\frac{\log M(r,g)}{T(r\,,g)}\cdot\lim_{r\to\infty}\frac{T(r\,,g)}{T(r\,,P_0[g])}$$
$$\leq \left(\frac{\alpha+1}{\alpha-1}\right)\frac{\alpha^{\rho_g}}{\Gamma_{P_0[g]}}\,. \tag{5}$$

Thus from (1) and (5) it follows that

$$\liminf_{r\to\infty}\frac{T(r,fog)}{T(r\,,P_0[g])} \leq (1+o(1))\cdot\left(\frac{\alpha+1}{\alpha-1}\right)\cdot\frac{\rho_f^{**}.\,\alpha^{\rho_g}}{\Gamma_{P_0[g]}}\,.$$

This proves the theorem .

Remark 1. *If we take* " $\Theta(\infty\,;g)=\sum_{a\neq\infty}\delta_p(a\,;g)=1$ *or* $\delta(\infty\,;g)=\sum_{a\neq\infty}\delta(a\,;g)=1$ " *instead of* " $\sum_{a\neq\infty}\Theta(a\,;g)=2$" *in Theorem* 1 *and the other conditions remain the same then one can easily prove that*

$$\liminf_{r\to\infty}\frac{T(r,fog)}{T(r\,,P_0[g])} \leq (1+o(1))\cdot\left(\frac{\alpha+1}{\alpha-1}\right)\cdot\frac{\rho_f^{**}.\,\alpha^{\rho_g}}{\gamma_{P_0[g]}}\,.$$

In the line of Theorem 1 and with the help of Lemma 9 we may state the following theorem without proof .

Theorem 2. *Let f be a meromorphic function of order zero and g be entire such that* ρ_g is finite . *Also let* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a\,;g)=4$ *. Then for any* $\alpha>1$

$$\liminf_{r\to\infty}\frac{T(r,fog)}{T(r\,,M[g])} \leq (1+o(1))\cdot\left(\frac{\alpha+1}{\alpha-1}\right)\cdot\frac{\rho_f^{**}.\,\alpha^{\rho_g}}{\Gamma_M-(\Gamma_M-\gamma_M)\,\Theta(\infty\,;g)}\,.$$

In the line of Theorem 1 one can easily prove the following theorem using the definition of lower proximate order.

Theorem 3. *Let f be a meromorphic function of order zero and g be entire with* $\lambda_g<\infty$ *. Also let* $\Theta(\infty\,;g)=\sum_{a\neq\infty}\delta_p(a\,;g)=1$ *or* $\delta(\infty\,;g)=\sum_{a\neq\infty}\delta(a\,;g)=1$ *. Then for any* $\alpha>1$

$$\liminf_{r\to\infty}\frac{T(r,fog)}{T(r\,,P_0[g])} \leq (1+o(1))\cdot\left(\frac{\alpha+1}{\alpha-1}\right)\cdot\frac{\rho_f^{**}.\,\alpha^{\lambda_g}}{\gamma_{P_0[g]}}\,.$$

Remark 2. *If we take* $\sum_{a\neq\infty}\Theta(a\,;g)=2$ " *instead of If we.* " $\Theta(\infty\,;g)=\sum_{a\neq\infty}\delta_p(a\,;g)=1$ *or* $\delta(\infty\,;g)=$ $\sum a\neq\infty\;\delta a\,;g\;=\;1$ *in Theorem* 3 *and the other conditions remain the same then one can easily prove that*

$$\liminf_{r\to\infty}\frac{T(r,fog)}{T(r\,,P_0[g])} \leq (1+o(1))\cdot\left(\frac{\alpha+1}{\alpha-1}\right)\cdot\frac{\rho_f^{**}.\,\alpha^{\lambda_g}}{\Gamma_{P_0[g]}}\,.$$

Theorem 4 *Let f be a meromorphic function of order zero and g be entire with* $\lambda_g<\infty$ *. Also let* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a\,;g)=4$ *. Then for any* $\alpha>1$

$$\liminf_{r\to\infty}\frac{T(r,fog)}{T(r\,,M[g])} \leq (1+o(1))\cdot\left(\frac{\alpha+1}{\alpha-1}\right)\cdot\frac{\rho_f^{**}.\,\alpha^{\lambda_g}}{\Gamma_M-(\Gamma_M-\gamma_M)\,\Theta(\infty\,;g)}\,.$$

The proof of the theorem can be established in the line of Theorem 3 and with the help of Lemma 9 and therefore it is omitted.

Theorem 5. *Let f and g be two non constant entire functions such that f is of lower order zero and λ_f^{**} and λ_g are finite . Also let* $\Theta(\infty\,;g) = \sum_{a\neq\infty} \delta_p(a\,;g) = 1$ *or* $\delta(\infty\,;g) = \sum_{a\neq\infty} \delta(a\,;g) = 1$. *Then*

$$\limsup_{r\to\infty}\frac{T(r,fog)}{T(r\,,P_0[\,g\,])} \geq (1+\,o(1))\cdot\frac{1}{3.\gamma_{P_0[\,g\,]}}\cdot\frac{\lambda_f^{**}}{4^{\lambda_g}}\cdot$$

Proof. If $\lambda_f^{**} = 0$ then the result is obvious. So we suppose that $\lambda_f^{**} > 0$.

For all sufficiently large values of r we know that

$$T(r,fog) \geq \frac{1}{3}\log M\,\{\,\frac{1}{8}\,M\,(\,\frac{r}{4}\,,g) + o(1)\;,\;f\,\}\;\{\,cf.\;[12]\,\}$$

For ε $(0 < \varepsilon < \min\{\,\lambda_f^{**}\,,1\})$ we get for all sufficiently large values of r ,

$$T(r,fog) \geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\;\log\{\,\frac{1}{8}M\,(\,\frac{r}{4}\,,g) + o(1)\,\}$$

$$i.e.,T(r,fog) \geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\log\{\,\frac{1}{9}\,M\,(\,\frac{r}{4}\,,g)\,\}$$

$$i.e.,T(r,fog) \geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\log M(\,\frac{r}{4}\,,g) + \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\log\frac{1}{9}$$

$$i.e.,T(r,fog) \geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)T(\,\frac{r}{4}\,,g) + O(1)\,. \tag{6}$$

Since $\liminf_{r\to\infty}\frac{T(r,g)}{r^{\lambda_g(r)}} = 1$, for given $\varepsilon(> 0)$ we get for all sufficiently large values of r

$$T(r,g) > (1-\varepsilon)\,r^{\lambda_g(r)} \tag{7}$$

and for a sequence of values of r tending to infinity

$$T(r,g) < (1+\varepsilon)\,r^{\lambda_g(r)}\,. \tag{8}$$

From (6) and (7) we get for $\delta(> 0)$ and for all sufficiently large values of r

$$T(r,fog) \geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)(1-\varepsilon)\,(1+o(1))\,\frac{\left(\frac{r}{4}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{r}{4}\right)}}\cdot$$

Since $r^{\lambda_g+\delta-\lambda_g(r)}$ is ultimately an increasing function of r it follows for all sufficiently large values of r that

$$T(r,fog) \geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)(1-\varepsilon)\,(1+o(1))\;\frac{r^{\lambda_g(r)}}{4^{\lambda_g+\delta}}\cdot \tag{9}$$

So by (8) and (9) we get for a sequence of values of r tending to infinity

$$T(r,fog) \geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\frac{(1-\varepsilon)}{(1+\varepsilon)}\,(1+o(1))\,\frac{T(r,g)}{4^{\lambda_g+\delta}}$$

$$i.e.,\;\frac{T(r,fog)}{T(r\,,P_0[\,g\,])} \geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\frac{(1-\varepsilon)}{(1+\varepsilon)}\,\frac{(1+o(1))}{4^{\lambda_g+\delta}}\,\frac{T(r,g)}{T(r\,,P_0[\,g\,])}$$

Since $\varepsilon(> 0)$ and $\delta(> 0)$ are arbitrary , in view of Lemma 5 it follows from above that

$$\limsup_{r\to\infty}\frac{T(r,fog)}{T(r\,,P_0[\,g\,])} \geq (1+\,o(1))\cdot\frac{1}{3.\gamma_{P_0[\,g\,]}}\cdot\frac{\lambda_f^{**}}{4^{\lambda_g}}\cdot$$

Thus the theorem is proved .

Remark 3. *If we take* " $\sum_{a\neq\infty}\Theta(a\,;g) = 2$" *instead of* " $\Theta(\infty\,;g) = \sum_{a\neq\infty}\delta_p(a\,;g) = 1$ *or* $\delta(\infty\,;g) =$ $\sum_{a\neq\infty}\delta(a\,;g) = 1$" *in Theorem* 5 *and the other conditions remain the same then one can easily prove that*

$$\limsup_{r\to\infty}\frac{T(r,fog)}{T(r\,,P_0[\,g\,])} \geq (1+\,o(1))\cdot\frac{1}{3.\Gamma_{P_0[\,g\,]}}\cdot\frac{\lambda_f^{**}}{4^{\lambda_g}}\cdot$$

In the line of Theorem 5 and with the help of Lemma 9 we may state the following theorem without proof.

Theorem 6. *Let f and g be two non constant entire functions such that f is of lower order zero and λ_f^{**} and λ_g are finite . Also let $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,;g) = 4$. Then*

$$\limsup_{r \to \infty} \frac{T(r, f o g)}{T(r, M[g])} \geq (1 + o(1)) \cdot \frac{1}{3.\Gamma_M - (\Gamma_M - \gamma_M)\Theta(\infty\,;g)} \cdot \frac{\lambda_f^{**}}{4^{\lambda_g}} .$$

Theorem 7. *Let f and g be two non constant entire functions such that ρ_f^{**} and λ_g are finite . Also suppose that there exist entire functions a_i $(i = 1, 2, \ldots . n\,; n \leq \infty)$ satisfying*

(i) $T(r, a_i) = o\{T(r, g)\}$ *as* $r \to \infty$ *for* $i = 1, 2, \ldots ., n$,

(ii) $\sum_{i=1}^{n} \delta(a_i\,;g) = 1$ *and*

(iii) $\sum_{a \neq \infty} \Theta(a\,;g) = 2$.

Then

$$\frac{\pi\,\lambda_f^{**}}{3.4^{\lambda_g}.\Gamma_{P_0[g]}} \leq \limsup_{r \to \infty} \frac{T(r, f o g)}{T(r, P_0[g])} \leq \frac{\pi\,\rho_f^{**}}{\Gamma_{P_0[g]}} .$$

Proof. For any two entire functions f and g , the following two inequalities are well known :

$$\boldsymbol{T(r,f)} \leq \log^+ \boldsymbol{M(r,f)} \leq 3\,\boldsymbol{T(2r,f)}. \; \{ cf. [6] \} \tag{10}$$

and

$$\log M(r, f o g) \leq \log M(M(r,g), f) \; . \; \{ cf. [3] \} \tag{11}$$

For $\boldsymbol{\varepsilon}(> 0)$ we get from (10) and (11) for all sufficiently large values of *r*,

$$T(r, f o g) \leq \log M(M(r,g), f)$$

$$i.e., T(r, f o g) \leq \left(\rho_f^{**} + \varepsilon\right) \log M(r, g)$$

$$i.e., \frac{T(r, f o g)}{T(r, P_0[g])} \leq \left(\rho_f^{**} + \varepsilon\right) \frac{\log M(r,g)}{T(r, P_0[g])} .$$

Hence we get from above that

$$\limsup_{r \to \infty} \frac{T(r, f o g)}{T(r, P_0[g])} \leq \left(\rho_f^{**} + \varepsilon\right) \limsup_{r \to \infty} \frac{\log M(r,g)}{T(r, P_0[g])}$$

$$i.e., \limsup_{r \to \infty} \frac{T(r, f o g)}{T(r, P_0[g])} \leq \left(\rho_f^{**} + \varepsilon\right) \limsup_{r \to \infty} \frac{\log M(r,g)}{T(r,g)} \cdot \limsup_{r \to \infty} \frac{T(r,g)}{T(r, P_0[g])}$$

Since $\boldsymbol{\varepsilon}(> 0)$ is arbitrary , it follows from Lemma 11 and Lemma 4 that

$$\limsup_{r \to \infty} \frac{T(r, f o g)}{T(r, P_0[g])} \leq \frac{\pi\,\rho_f^{**}}{\Gamma_{P_0[g]}} . \tag{12}$$

Now suppose that $0 < \varepsilon < \min\,\{\lambda_f^{**}, 1\}$ we get from (6) for all sufficiently large values of *r* that

$$\frac{T(r, f o g)}{T(r, P_0[g])} \geq \frac{1}{3}\left(\lambda_f^{**} - \varepsilon\right) \frac{\log M\left(\frac{r}{4}, g\right)}{T\left(\frac{r}{4}, g\right)} \cdot \frac{T\left(\frac{r}{4}, g\right)}{T(r,g)} \cdot \frac{T(r,g)}{T(r, P_0[g])} + O(1) . \tag{13}$$

From (7) and (8) and in the line of Lemma 3 we get for a sequence of values of r tending to infinity and for $\delta(> 0)$

$$\frac{T\left(\frac{r}{4}, g\right)}{T(r,g)} \geq \frac{1-\varepsilon}{1+\varepsilon} \cdot \frac{\left(\frac{r}{4}\right)^{\lambda_g + \delta}}{\left(\frac{r}{4}\right)^{\lambda_g + \delta - \lambda_g\left(\frac{r}{4}\right)}} \cdot \frac{1}{r^{\lambda_g(r)}}$$

$$\geq \frac{1-\varepsilon}{1+\varepsilon} \cdot \frac{1}{4^{\lambda_g + \delta}} .$$

Since $\varepsilon(> 0)$ and $\delta(> 0)$ are arbitrary we get from (13), Lemma 4, Lemma 11 and above that

$$\limsup_{r \to \infty} \frac{T(r, f o g)}{T(r, P_0[g])} \geq \frac{\pi\,\lambda_f^{**}}{3.4^{\lambda_g}.\Gamma_{P_0[g]}} . \tag{14}$$

Thus the theorem follows from (12) and (14).

Remark 4. *If we take* " $\Theta(\infty\,;g)=\sum_{a\neq\infty}\delta_p(a\,;g)=1$ *or* $\delta(\infty\,;g)=\sum_{a\neq\infty}\delta(a\,;g)=1$ " *instead of* " $\sum_{a\neq\infty}\Theta(a\,;g)=2$ " *in Theorem* 7 *and the other conditions remain the same then one can easily prove that*

$$\frac{\pi\,\lambda_f^{**}}{3.4^{\lambda_g}.\gamma_{P_0[\,g\,]}}\leq \limsup_{r\to\infty}\frac{T(r,fog)}{T(r\,,P_0[\,g\,])}\leq\frac{\pi\,\rho_f^{**}}{\gamma_{P_0[\,g\,]}}\,.$$

In the line of Theorem 7 and with the help of Lemma 9 we may state the following theorem without proof.

Theorem 8. *Let f and g be two non constant entire functions such that* ρ_f^{**} *and* λ_g *are finite. Also suppose that there exist entire functions* a_i $(i=1,2,\ldots.n\,;n\leq\infty)$ *satisfying*

(i) $T\,(r,a_i)=\mathrm{o}\{T\,(r,g)\}$ *as* $r\to\infty$ *for* $i=1,2,\ldots.,n$,

(ii) $\sum_{i=1}^{n}\delta(a_i\,;g)=1$ *and*

(iii) $\sum_{a\,\epsilon\,\mathbb{C}\,\cup\{\infty\}}\delta_1(a\,;g)=4$.

Then

$$\frac{\pi\,\lambda_f^{**}}{3.4^{\lambda_g}.\Gamma_M-(\Gamma_M-\gamma_M)\Theta(\infty\,;g)}\leq \limsup_{r\to\infty}\frac{T(r,fog)}{T(r\,,M[\,g\,])}\leq\frac{\pi\,\rho_f^{**}}{\Gamma_M-(\Gamma_M-\gamma_M)\Theta(\infty\,;g)}\,.$$

Theorem 9. *Let f and g be any two entire functions such that* $0<\lambda_f\leq\rho_f<\infty$ *and* $\rho_g^{**}>0$. *Also let* $\sum_{a\neq\infty}\Theta(a\,;f)=2$. *Then*

$$\limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^A\,,P_0[\,f\,])}\geq\frac{\lambda_f\rho_g^{**}}{A\rho_f}\,,$$

where A is any positive real number .

Proof. We know that for $r>0$ [12]

$$T(r,fog)\geq\frac{1}{3}\log M\,\{\frac{1}{8}M\,(\frac{r}{4},g)+o(1)\,,\,f\}\,. \tag{15}$$

Let us suppose that $0<\varepsilon<\min\{\lambda_f\,,\,\rho_g^{**}\}$.

Now from (15) we have for a sequence of values of r tending to infinity that

$$\log T(r,fog)\geq(\lambda_f-\varepsilon)\log M(\frac{r}{4},g)+O(1)$$

$$i.e., \log T(r,fog)\geq(\lambda_f-\varepsilon)(\rho_g^{**}-\varepsilon)\log r+O(1)\,. \tag{16}$$

Again from the definition of $\rho_{P_0[\,f\,]}$ we have for arbitrary positive ε and for all sufficiently large values of r ,

$$\log T(r^A\,,\,P_0[\,f\,])\leq A\left(\rho_{P_0[\,f\,]}+\varepsilon\right)\log r$$

$$i.e., \log T(r^A\,,\,P_0[\,f\,])\leq A\left(\rho_f+\varepsilon\right)\log r\,. \tag{17}$$

So combining (16) and (17) we get for a sequence of values of r tending to infinity that

$$\frac{\log T(r,fog)}{\log T(r^A\,,P_0[\,f\,])}\geq\frac{(\lambda_f-\varepsilon)(\rho_f^{**}-\varepsilon)\log r+O(1)}{A\,(\rho_f+\varepsilon)\log r}$$

$$i.e., \limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^A\,,P_0[\,f\,])}\geq\frac{\lambda_f\,\rho_g^{**}}{A\rho_f}\,.$$

This completes the proof.

Remark 5. *Under the same conditions of Theorem* 9 , *if f is of regular growth then*

$$\limsup_{r\to\infty}\frac{\log T(r,fog)}{\log T(r^A\,,P_0[\,f\,])}\geq\frac{\rho_g^{**}}{A}\,.$$

Remark 6 . *In Theorem* 9 *if we take* $\lambda_g^{**} > 0$ *instead of* $\rho_g^{**} > 0$ *and the other conditions remain the same then it can be shown that*

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, P_0[f])} \geq \frac{\lambda_f \lambda_g^{**}}{A\rho_f}.$$

In addition if f *is of regular growth then*

$$\liminf_{r \to \infty} \frac{T(r, fog)}{T(r^A, P_0[f])} \geq \frac{\lambda_g^{**}}{A}.$$

Remark 7. *Also if we consider* $0 < \lambda_g < \infty$ *or* $0 < \rho_g < \infty$ *instead of* $0 < \lambda_g \leq \rho_g < \infty$ *in Theorem* 9 *and the other conditions remain the same , then one can easily verify that*

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, P_0[f])} \geq \frac{\lambda_g^{**}}{A}.$$

Remark 8. *The conclusions of Theorem* 9 *, Remark* 5 *, Remark* 6 *and Remark* 7 *can also be drawn under the hypothesis* $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ *instead of* $\sum_{a \neq \infty} \Theta(a; f) = 2$.

Theorem 10. *Let* f *and* g *be two entire functions such that* $0 < \lambda_f \leq \rho_f < \infty$ *and* $\rho_g^{**} > 0$ *. Also let* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. *Then*

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, M[f])} \geq \frac{\lambda_f \rho_g^{**}}{A\rho_f},$$

where A is any positive real number .

The proof is omitted because it can be carried out in the line of Theorem 10 and with the help of Lemma 10 .

Remark 9. *Under the same conditions of Theorem* 10 *if* f *is of regular growth then*

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, M[f])} \geq \frac{\rho_g^{**}}{A}.$$

Remark 10. *In Theorem* 10 *if we take* $\lambda_g^{**} > 0$ *instead of* $\rho_g^{**} > 0$ *and the other conditions remain the same then it can be shown that*

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, M[f])} \geq \frac{\lambda_f \lambda_g^{**}}{A\rho_f}.$$

In addition if f *is of regular growth then*

$$\liminf_{r \to \infty} \frac{T(r, fog)}{T(r^A, M[f])} \geq \frac{\lambda_g^{**}}{A}.$$

Remark 11. *Further if we consider* $0 < \lambda_g < \infty$ *or* $0 < \rho_g < \infty$ *instead of* $0 < \lambda_g \leq \rho_g < \infty$ *in Theorem* 10 *and the other conditions remain the same , then one can easily verify that*

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, M[f])} \geq \frac{\lambda_g^{**}}{A}.$$

Theorem 11. *Let* f *be a transcendental meromorphic function with the maximum deficiency sum and g be an entire function such that* $0 < \lambda_f \leq \rho_f < \infty$ *and* $\rho_g^{**} < \infty$. *Also let* $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. *Then*

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, P_0[f])} \leq \frac{\rho_f \rho_g^{**}}{A\lambda_f}.$$

for any positive real number A .

Proof. In view of Lemma 1 and the inequality $\boldsymbol{T(r,g)} \leq \log^+ \boldsymbol{M(r,g)}$ we get for all sufficiently large values of r ,

$$T(r, fog) \leq \{1 + o(1)\}\, T(M(r,g), f)$$
$$i.e., \log T(r, fog) \leq (\rho_f + \varepsilon) \log M(r,g) + O(1)$$
$$i.e., \log T(r, fog) \leq (\rho_f + \varepsilon)(\rho_g^{**} + \varepsilon) + O(1). \quad (18)$$

From the definition of $\lambda_{P_0[f]}$ we have for arbitrary positive ε and for all sufficiently large values of *r*,

$$\log T(r^A, P_0[f]) \geq A(\lambda_{P_0[f]} - \varepsilon)\log r$$

$$i.e., \log T(r^A, P_0[f]) \geq A(\lambda_f - \varepsilon)\log r. \tag{19}$$

Now combining (18) and (19) we get for all sufficiently large values of *r*,

$$\frac{\log T(r, f o g)}{\log T(r^A, P_0[f])} \leq \frac{(\rho_f + \varepsilon)(\rho_g^{**} + \varepsilon) + O(1)}{A(\lambda_f - \varepsilon)\log r}$$

$$i.e., \limsup_{r\to\infty} \frac{\log T(r, f o g)}{\log T(r^A, P_0[f])} \leq \frac{\rho_f \rho_g^{**}}{A\lambda_f}.$$

This completes the proof .

Remark 12. *Under the same conditions of Theorem* 11 *if* f *is of regular growth then*

$$\limsup_{r\to\infty} \frac{\log T(r, f o g)}{\log T(r^A, P_0[f])} \leq \frac{\rho_g^{**}}{A}.$$

Remark 13. *In Theorem* 11 *if we take* $\lambda_g^{**} < \infty$ *instead of* $\rho_g^{**} < \infty$ *and the other conditions remain the same then it can be shown that*

$$\liminf_{r\to\infty} \frac{\log T(r, f o g)}{\log T(r^A, P_0[f])} \leq \frac{\rho_f \lambda_g^{**}}{A\rho_f}.$$

In addition if f *is of regular growth then*

$$\liminf_{r\to\infty} \frac{\log T(r, f o g)}{\log T(r^A, P_0[f])} \leq \frac{\rho_f \lambda_g^{**}}{A\rho_f}.$$

Remark 14. *If we take* $0 < \rho_f < \infty$ *or* $0 < \lambda_f < \infty$ *instead of* $0 < \lambda_f \leq \rho_f < \infty$ *in Theorem* 11 *and the other conditions remain the same, then one can easily verify that*

$$\liminf_{r\to\infty} \frac{\log T(r, f o g)}{\log T(r^A, P_0[f])} \leq \frac{\rho_g^{**}}{A}.$$

Remark 15. *The conclusions of Theorem* 11 , *Remark* 12 , *Remark* 13 *and Remark* 14 *can also be deduced if we replace* $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ *by* $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ *or* $\sum_{a\neq\infty} \Theta(a; f) = 2$ *respectively* .

In the line of Theorem 11 and with the help of Lemma 10 we may state the following theorem without proof.

Theorem 12. *Let* f *be a transcendental meromorphic function with the maximum deficiency sum and g be an entire function such that* $0 < \lambda_f \leq \rho_f < \infty$ *and* $\rho_g^{**} < \infty$. *Also let* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$. *Then*

$$\limsup_{r\to\infty} \frac{\log T(r, f o g)}{\log T(r^A, M[f])} \leq \frac{\rho_f \rho_g^{**}}{A\lambda_f},$$

where A is any positive real number .

Remark 16. *Under the same conditions of Theorem* 12 *if* f *is of regular growth then*

$$\limsup_{r\to\infty} \frac{\log T(r, f o g)}{\log T(r^A, M[f])} \leq \frac{\rho_g^{**}}{A}.$$

Remark 17. *In Theorem* 12 *if we take* $\lambda_g^{**} < \infty$ *instead of* $\rho_g^{**} < \infty$ *and the other conditions remain the same then it can be shown that*

$$\liminf_{r\to\infty} \frac{\log T(r, f o g)}{\log T(r^A, M[f])} \leq \frac{\rho_f \lambda_g^{**}}{A\lambda_f}.$$

In addition if f *is of regular growth then*

$$\liminf_{r\to\infty} \frac{\log T(r, f o g)}{\log T(r^A, M[f])} \leq \frac{\lambda_g^{**}}{A}.$$

Remark 18. *If we take* $0 < \rho_f < \infty$ *or* $0 < \lambda_f < \infty$ *instead of* $0 < \lambda_f \leq \rho_f < \infty$ *in Theorem* 12 *and the other conditions remain the same , then one can easily verify that*

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, M[f])} \leq \frac{\rho_g^{**}}{A}.$$

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