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# Bilateral Sequence Spaces $c_\circ(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ and $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ defined by Orlicz Function

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***ABSTRACT***

*In this paper, we construct new sequence spaces $c_\circ(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ and $\ell(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ by using OrliczfunctionM. We also examine some of the properties like containment, linearity and completeness etc of these newly constructed sequence spaces.*

***Keywords:*** *Bilateral Sequence, Sequence Space, Paranormed space, Orlicz function.*

***2010 AMS Subject Classification:*** *Primary- 46A45, Secondary- 46B45.*

## 1. INTRODUCTION

By a bilateral sequence, we mean a function whose domain is the set $\mathbb{Z}$ of all integers with natural ordering. The utility of bilateral sequences can be found in [7] and [8]. We will denote a bilateral sequence by the symbol$(a_k)_{-\infty}^{\infty}$or $\bar{a} = (a_k)_{-\infty}^{\infty}$. As usual, by the convergence of the bilateral series $\sum_{-\infty}^{\infty} a_k$ to $s$ written as $\sum_{-\infty}^{\infty} a_k = s$, we shall mean the convergence of the sequence $(S_n)_{n=1}^{\infty}$to $s$ where $S_n = \sum_{-n}^{n} a_k$ is called n-th partial sum of the bilateral series $\sum_{-\infty}^{\infty} a_k$

Again, let $M$ be the Orlicz function. The definition of Orlicz function and Orlicz sequence spaces are as follows:

**Definition 1.1:** An Orlicz function $M: [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function defined for $t \geq 0$ such that
**(i)**$M(x) > 0$for $x > 0$;
**(ii)**$M(0) = 0$and
**(iii)**$\lim_{t\to\infty} M(t) = \infty$.

An Orlicz function $M$ can always be represented in the following integral form (see [1])

$$M(x) = \int_0^x p(t)dt$$

where $p$ is known as the kernal of $M$, is right differentiable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, $p$ is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

**Definition 1.2:**Lindenstrauss and Tzafriri (see [1], [3], [4] and [5]) used the ideas of Orlicz function to construct the sequence space,

$$l_M = \left\{x \in \omega: \sum_1^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \; for\ some\ \rho > 0\right\}$$

The space $l_M$ with the norm

$$||x|| = \inf\left\{\rho > 0: \sum_1^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

becomes a Banach space which is called an Orlicz sequence space.

Now let $\bar{p} = (p_k)_{-\infty}^{\infty}$ and$\bar{q} = (q_k)_{-\infty}^{\infty}$ be bilateral sequences of strictly positive real numbers and $\bar{\lambda} = (\lambda_k)_{-\infty}^{\infty}$ and $\bar{\mu} = (\mu_k)_{-\infty}^{\infty}$be bilateral sequences of non-zero complex numbers and $M$ be an Orlicz function. Now we introduce the following classes of Banach space $X$ −valued bilateral sequences:

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$$c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) = \{\bar{x} = (x_k)_{-\infty}^{\infty}: x_k \in X,\ k \in \mathbb{Z},\ and\ \left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k} \to 0\ as\ k \to -\infty,\ as\ well\ as\ k \to \infty \quad for\ some\ \rho > 0\}.$$

$$\ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) = \{\bar{x} = (x_k)_{-\infty}^{\infty}: x_k \in X,\ k \in \mathbb{Z},\ and\ \sum_{-\infty}^{\infty}\left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k} < \infty,\ \ for\ some\ \rho > 0\}.$$

Throughout the paper we denote $t_k = \left|\frac{\lambda_k}{\mu_k}\right|$.

**Definition 1.3:** Let $X$be a linear space. A mapping$g: X \to \mathbb{R}$ is called a paranorm if it satisfies
**(i)** $g(\theta) = 0$;
**(ii)** $g(x) = g(-x)$;
**(iii)** $g(x+y) \leq g(x) + g(y)$;
**(iv)** if$(\alpha_n)$ is a sequenceof scalars with $\alpha_n \to \alpha$and $(x_n)$is a sequence in $X$ with $g(x_n - x) \to 0$then $g(\alpha_n x_n - \alpha x) \to 0$(continuity of scalar multiplication). The paranorm is called total if
**(v)**$g(x) = 0$ implies $x = 0$ , see [9].

In this paper our aim is to investigate results concerning the above defined classes with the help of Orlicz function $M$ .

## 2. CONTAINMENT

**Lemma 2.1:** $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) \subset c_\circ(\mathbb{Z},X,M,\bar{\mu},\bar{p})$ if and only if

$$\lim_{k\to-\infty} \inf_k t_k > 0\ and\ \lim_{k\to\infty} \inf_k t_k > 0 \text{ with } l = \inf_k p_k \leq p_k.$$

**Proof:** For the sufficiency of the condition suppose that $\lim_{k\to-\infty} \inf_k t_k > 0$ and$\lim_{k\to\infty} \inf_k t_k > 0$ and$\bar{x} = (x_k)_{-\infty}^{\infty} \in c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ Then there exists a real number $m > 0$ such that $m < |\frac{\lambda_k}{\mu_k}$for all sufficiently large values of $|k|$. Thus $m||\mu_k x_k|| < ||\lambda_k x_k||$, for all sufficiently large values of $|k|$. Also $\bar{x} \in c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ so we can find some $\rho_1 > 0$ such that $\left(M(\frac{||\lambda_k x_k||}{\rho_1})\right)^{p_k} \to 0$. Let us choose $\rho$ such that $\rho_1 < m\rho$. Since $M$ is non-decreasing, we have

$$\left(M(\frac{||\mu_k x_k||}{\rho})\right)^{p_k} \leq \left(M(\frac{||\lambda_k x_k||}{m\rho})\right)^{p_k} < \left(M(\frac{||\lambda_k x_k||}{\rho_1})\right)^{p_k} \to 0$$

and hence $\bar{x} \in c_\circ(\mathbb{Z},X,M,\bar{\mu},\bar{p})$ and hence $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) \subset c_\circ(\mathbb{Z},X,M,\bar{\mu},\bar{p})$.

For the necessity, let $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) \subset c_\circ(\mathbb{Z},X,M,\bar{\mu},\bar{p})$ but either $\lim_{k\to-\infty} \inf t_k = 0$or $\lim_{k\to\infty} \inf t_k = 0$. Let us take $\lim_{k\to\infty} \inf t_k = 0$. Then there exists a sequence $(k(n))$such that $k(n+1) > k(n) \geq 1, n \geq 1$, for which $n^2|\lambda_{k(n)}| < |\mu_{k(n)}|$. Now the bilateral sequence $\bar{x} = (x_k)_{-\infty}^{\infty™}$ defined by

$$x_k = \begin{cases} \lambda_{k(n)}^{-1} n^{-1} z & \text{if } k = k(n), n \geq 1, \text{ and} \\ \theta, & \text{otherwise} \end{cases}$$

where$z \in X$ and $||z|| = 1$. Then $||\lambda_{k(n)} x_{k(n)}|| = \frac{1}{n}$. Which implies that$||\lambda_{k(n)} x_{k(n)}|| \to 0$as $n \to \infty$.

Therefore$\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{l} \to 0$ as $n \to \infty$ for any fixed $l$.

Hence

$$\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{p_{k(n)}} \to 0 \text{as} n \to \infty \text{ since } l \leq p_k$$

i.e., $\bar{x} \in c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$. But $||\mu_{k(n)} x_{k(n)}|| > n$, implies that

$$\left(M(\frac{||\mu_{k(n)} x_{k(n)}||}{\rho})\right)^{p_{k(n)}/l} > 1 \quad \text{for all } n \geq 1 \text{ and for some fixed } l \leq p_k,$$

or $\left(M(\frac{||\mu_{k(n)} x_{k(n)}||}{\rho})\right)^{p_{k(n)}} > 1$ for all $n \geq 1$ and for some $\rho$.

Which implies $\bar{x} \notin c_\circ(\mathbb{Z},X,M,\bar{\mu},\bar{p})$, a contradiction.

Similar proof can be given in the case when $\lim_{k\to-\infty}\inf_k t_k > 0$. This completes the proof.

**Lemma 2.2:**$c_\circ(\mathbb{Z},X,M,\bar{\mu},\bar{p}) \subset c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ if and only if $\lim_{k\to-\infty}\sup_k t_k < \infty$ *and* $\lim_{k\to\infty}\sup_k t_k < \infty$ with $l = \inf_k p_k \le p_k$.

**Proof:** Sufficiency is straightforward. On the other hand for the necessity suppose that $(\mathbb{Z},X,M,\bar{\mu},\bar{p}) \subset c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ but either $\lim_{k\to-\infty}\sup_k t_k = \infty$or $\lim_{k\to\infty}\sup_k t_k = \infty$. Let $\lim_{k\to\infty}\sup_k t_k = \infty$. Then there exists a sequence$(k(n))$, $k(n) \ge 1$ such that for each $n \ge 1$, $|\lambda_{k(n)}| > n^2|\mu_{k(n)}|$ Now define the bilateral sequence $\bar{x} = (x_k)_{-\infty}^{\infty}$ by

$$x_k = \begin{cases} \mu_{k(n)}^{-1} n^{-1} z & \text{if } k = k(n), n \ge 1 \text{ and,} \\ \theta, & \text{otherwise} \end{cases}$$

where $z \in X$ and $||z|| = 1$ . Then $||\mu_{k(n)}x_{k(n)}|| = \frac{1}{n}$ $n \ge 1$ and $||\mu_k x_k|| = 0$ , otherwise. This implies that $\frac{||\mu_{k(n)}x_{k(n)}||}{\rho} \to 0$ as $n \to \infty$ for some $\rho$. Thus by the property of Orlicz function, we have$\left(M(\frac{||\mu_{k(n)}x_{k(n)}||}{\rho})\right)^{l} \to 0$ as $n \to \infty$for some $\rho$ and for any fixed $l$.

Therefore $\left(M(\frac{||\mu_{k(n)}x_{k(n)}||}{\rho})\right)^{p_k} \to 0$ as $n \to \infty$since $l = \inf p_k < p_k$. This shows that $\bar{x} \in c_\circ(\mathbb{Z},X,M,\bar{\mu},\bar{p})$. But $||\lambda_{k(n)}x_{k(n)}|| > n$ implies that $||\lambda_{k(n)}x_{k(n)}|| \to \infty$ as $n \to \infty \left(M(\frac{||\lambda_{k(n)}x_{k(n)}||}{\rho})\right)^{p_k/l} \to \infty$ for arbirary large $n$ and $l \le p_k$. This shows that$\left(M(\frac{||\lambda_{k(n)}x_{k(n)}||}{\rho})\right)^{p_k} \to \infty$as $n \to \infty$;i.e.,$\bar{x} \notin c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$, which is a contradiction.

Similar proof can be given in the case when $\lim_{k\to-\infty}\sup_k t_k = \infty$. This completes the proof.

On combining Lemma 2.1 and Lemma 2.2, we get the following theorem:

**Theorem 2.3:**For $l = \inf p_k \le p_k$, $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) = c_\circ(\mathbb{Z},X,M,\bar{\mu},\bar{p})$ if and only if

$$0 < \lim_{k\to-\infty}\inf t_k \le \lim_{k\to-\infty}\sup t_k < \infty \; and,$$
$$0 < \lim_{k\to\infty}\inf t_k \le \lim_{k\to\infty}\sup t_k < \infty.$$

**Corollary 2.4:**For $l = \inf p_k \le p_k$,

**(i)**$c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) \subset c_\circ(\mathbb{Z},X,M,\bar{p})$if and only if
$$\lim_{k\to-\infty}\inf|\lambda_k|^{p_k} > 0 \; and \; \lim_{k\to\infty}\inf|\lambda_k|^{p_k} > 0;$$

**(ii)**$c_\circ(\mathbb{Z},X,M,\bar{p}) \subset c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$if and only if
$$\lim_{k\to-\infty}\sup|\lambda_k|^{p_k} < \infty \; and \; \lim_{k\to\infty}\sup|\lambda_k|^{p_k} < \infty \;;$$

**(iii)**$c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) = c_\circ(\mathbb{Z},X,M,\bar{p})$if and only if
$$0 < \lim_{k\to-\infty}\inf|\lambda_k|^{p_k} \le \lim_{k\to-\infty}\sup|\lambda_k|^{p_k} < \infty \; and$$
$$0 < \lim_{k\to\infty}\inf|\lambda_k|^{p_k} \le \lim_{k\to\infty}\sup|\lambda_k|^{p_k} < \infty.$$

**Proof:**
**(i)** Take $\mu_k = 1$, for all $k$ in Lemma 2.1,
**(ii)**Take $\mu_k = 1$, for all $k$ in Lemma 2.2,
**(iii)**Take $\mu_k = 1$, for all $k$ in Theorem 2.3.

**Lemma 2.5:**$c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) \subset c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{q})$ if and only if

$$\lim_{k\to-\infty}\inf\frac{q_k}{p_k} > 0 \; and \; \lim_{k\to\infty}\inf\frac{q_k}{p_k} > 0 \text{with} l = \inf_k p_k \le p_k.$$

**Proof:** For the sufficiency condition, suppose $\lim_{k\to-\infty}\inf\frac{q_k}{p_k} > 0$ and $\lim_{k\to\infty}\inf\frac{q_k}{p_k} > 0$and $\bar{x} = (x_k)_{-\infty}^{\infty} \in c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$. Then there exists a real number $m > 0$ such that $q_k > mp_k$ for all sufficiently large values of $|k|$. Further since $\bar{x} \in c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ we have $\left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k} < 1$, for all sufficiently large values of $|k|$ and hence $(M(\frac{||\lambda_k x_k||}{\rho}))^{q_k} < [(M(\frac{||\lambda_k x_k||}{\rho}))^{p_k}]^m < 1$, for all sufficiently large values of $|k|$. This implies that $\bar{x} \in c_{\circ}(\mathbb{Z}, X, M, \bar{\lambda}, \bar{q})$ and hence $c_{\circ}(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p}) \subset c_{\circ}(\mathbb{Z}, X, M, \bar{\lambda}, \bar{q})$.

For the necessity of the condition, suppose that inclusion holds but either $\lim_{k\to-\infty} \inf\frac{q_k}{p_k} = 0$or $\lim_{k\to\infty} \inf\frac{q_k}{p_k} = 0$ . Here we prove the result for the case when $\lim_{k\to\infty} \inf\frac{q_k}{p_k} = 0$, then there exists a sequence $(k(n)), k(n) \geq 1$ such that foreach $n \geq 1 n q_{k(n)} < p_{k(n)}$. Now taking $z \in X$ and $||z|| = 1$ for the bilateral sequence $\bar{x} = (x_k)_{-\infty}^{\infty}$ defined by

$$x_k = \begin{cases} \lambda_{k(n)}^{-1} n^{-1} z & \text{if } k = k(n),\ n \geq 1 \text{ and,} \\ \theta, & \text{otherwise} \end{cases}$$

Then $\bar{x} = c_{\circ}(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$as $||\lambda_{k(n)} x_{k(n)}|| = \frac{1}{n}$. This implies that $||\lambda_{k(n)} x_{k(n)}|| \to 0$as $n \to \infty$. Thus by the definition of Orlicz function, we have

$$M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho}) \to 0 \text{as} n \to \infty \text{ for some } \rho > 0$$

or $\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^l \to 0$ as $n \to \infty$ for some $\rho$ and some fixed $l$.

or $\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{p_{k(n)}} \leq \left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^l$ since $l \leq p_k$ .

This implies that $\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{p_{k(n)}} \to 0$ as $n \to \infty$ for some $\rho$. Therefore $\bar{x} = (x_k)_{-\infty}^{\infty} \in c_{\circ}(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$. But

$$\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{p_{k(n)}} < \left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{n q_{k(n)}}$$

Now we can choose some $\rho_1 > \rho$ such that $\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho_1})\right)^{p_{k(n)}}$ does not converge to zero.

Therefore $\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{n q_{k(n)}}$ does not converge to zero for some $\rho_1 > \rho$ which shows that $\bar{x} \notin c_{\circ}(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$,a contrdiction.

Similar proof can be given for the case when $\lim_{k\to-\infty} \inf\frac{q_k}{p_k} = 0$. This completes the proof.

**Lemma 2.6:**$c_{\circ}(\mathbb{Z}, X, M, \bar{\lambda}, \bar{q}) \subset c_{\circ}(\mathbb{Z}, X, M, \bar{\lambda}, \bar{p})$ if and only if

$$\lim_{k\to-\infty} \sup \frac{q_k}{p_k} < \infty \text{ and } \lim_{k\to\infty} \sup \frac{q_k}{p_k} < \infty$$

with l = $\inf_k q_k \leq q_k$.

**Proof:** Sufficiency is straightforward. On the other hand for the necessity, let the inclusion holds but $\lim_{k\to-\infty} \sup\frac{q_k}{p_k} = \infty$or $\lim_{k\to\infty} \sup\frac{q_k}{p_k} = \infty$. Here we prove the result for the case when $\lim_{k\to\infty} \sup\frac{q_k}{p_k} = \infty$ then there exists a sequence $(k(n)), k(n) \geq 1$ such that $q_{k(n)} > n p_{k(n)}$ for all $n \geq 1$. Thus, the bilateral sequence $\bar{x} = (x_k)_{-\infty}^{\infty}$ defined by

$$x_k = \begin{cases} \lambda_{k(n)}^{-1} n^{-1} z & \text{if } k = k(n), n \geq 1 \text{ and,} \\ \theta, & \text{otherwise} \end{cases}$$

where $z \in X$, $||z|| = 1$. We see that, $||\lambda_{k(n)} x_{k(n)}|| \to 0$ as $n \to \infty$ .Thus by the definition of Orlicz function we have $\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^l \to 0$as$n \to \infty$ for some $\rho$ and some fixed $l$ and

$\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{q_{k(n)}} \leq (M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho}))^l$, since $l \leq q_k$.

Therefore $\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{q_{k(n)}} \to 0$ for some $\rho$ implies that $\bar{x} \in c_{\circ}(M, X, \lambda, q)$. But $\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{n p_{k(n)}}$ will not surely converge to zero for each $n \geq 1$ as

$$\left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{q_{k(n)}} < \left(M(\frac{||\lambda_{k(n)} x_{k(n)}||}{\rho})\right)^{n p_{k(n)}}$$

and we can choose some $\rho_1 < \rho$ such that $\left(M(\frac{||\lambda_{k(n)}x_{k(n)}||}{\rho_1})\right)^{q_{k(n)}} \to \infty$. Therefore

$$\left(M(\frac{||\lambda_{k(n)}x_{k(n)}||}{\rho})\right)^{q_{k(n)}} < \left(M(\frac{||\lambda_{k(n)}x_{k(n)}||}{\rho_1})\right)^{q_{k(n)}} < \left(M(\frac{||\lambda_{k(n)}x_{k(n)}||}{\rho_1})\right)^{np_{k(n)}}$$

Implies $\left(M(\frac{||\lambda_{k(n)}x_{k(n)}||}{\rho})\right)^{p_{k(n)}}$ will not converge to zero for some $\rho > \rho_1$. Therefore $\bar{x} \notin c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$, which is a contrdiction.

Similar proof can be given for the case when $\lim_{k\to-\infty}\sup\frac{q_k}{p_k} = \infty$. This completes the proof.

On combining Lemma 2.5 and Lemma 2.6 we get the following theorem:

**Theorem 2.7:** $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) = c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{q})$

$0 < \lim_{k\to-\infty}\inf\frac{q_k}{p_k} < \lim_{k\to-\infty}\sup\frac{q_k}{p_k} < \infty,\ \ and \ \ 0 < \lim_{k\to\infty}\inf\frac{q_k}{p_k} < \lim_{k\to\infty}\sup\frac{q_k}{p_k} < \infty.$

**Lemma 2.8:** $\ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) \subset \ell(\mathbb{Z},X,M,\bar{\mu},\bar{p})$ if and only if

$$\lim_{k\to-\infty}\inf t_k > 0 \ and\ \lim_{k\to\infty}\inf t_k > 0.$$

**Proof:** Suppose $\lim_{k\to-\infty}\inf t_k > 0\ and\ \lim_{k\to\infty}\inf t_k > 0$and $\bar{x} = (x_k)_{-\infty}^{\infty} \in \ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p}$. Then there exists a real number $m > 0$ such that $m|\mu_k| < |\lambda_k|$ for all sufficiently large values of $|k|$. Thus $m||\mu_k x_k|| < ||\lambda_k x_k||$ for all sufficiently large values of $|k|$. Since $\bar{x} \in \ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ so there exists $\rho_1 > 0$ such that $\sum_{-\infty}^{\infty}\left(M(\frac{||\lambda_k x_k||}{\rho_1})\right)^{p_k} < \infty$. Let us choose$\rho > 0$ such that $\rho_1 < m\rho$. Since $M$ is non-decreasing therefore

$$\sum_{-\infty}^{\infty}\left(M(\frac{||\mu_k x_k||}{\rho})\right)^{p_k} < \sum_{-\infty}^{\infty}\left(M(\frac{||\lambda_k x_k||}{m\rho})\right)^{p_k} < \sum_{-\infty}^{\infty}\left(M(\frac{||\lambda_k x_k||}{\rho_1})\right)^{p_k} < \infty$$

for some $\rho > 0$. Hence $\bar{x} \in \ell(\mathbb{Z},X,M,\bar{\mu},\bar{p})$ and this implies that

$$\ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) \subset \ell(\mathbb{Z},X,M,\bar{\mu},\bar{p}).$$

Conversely, let the inclusion holds but either $\lim_{k\to-\infty}\inf t_k = 0$ or $\lim_{k\to\infty}\inf t_k = 0$. Here we take $\lim_{k\to\infty}\inf t_k = 0$, then there exists a sequence $(k(n))$, $k(n) \geq 1$ such that $n^2|\lambda_{k(n)}| < |\mu_{k(n)}|\ forall n \geq 1$. Now we see that $\bar{x} = (x_k)_{-\infty}^{\infty}$ defined by

$$x_k = \begin{cases}\lambda_{k(n)}^{-1}n^{-1}z & \text{if } k = k(n), n \geq 1 \text{ and,}\\ \theta, & \text{otherwise}\end{cases}$$

where $z \in X$ , $||z|| = 1$ is in $\ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ but not in $\ell(\mathbb{Z},X,M,\bar{\mu},\bar{p})$ as $||\lambda_{k(n)}x_{k(n)}|| = \frac{1}{n}$ . Therefore$\left(M(\frac{||\lambda_{k(n)}x_{k(n)}||}{\rho})\right)^{l} \to 0$ as $n \to \infty$ for some $\rho$ and for any fixed $l$. Hence$\sum_{-\infty}^{\infty}\left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k} < \infty$for $\ell = \inf_k p_k \leq p_k$ .But $||\mu_{k(n)}x_{k(n)}|| = |\frac{\mu_{k(n)}}{\lambda_{k(n)}}|\frac{1}{n} > n$,implies that$\left(M(\frac{||\mu_{k(n)}x_{k(n)}||}{\rho})\right)^{p_{k(n)}/l} > 1$ for $\rho > 0$ and for some fixed $l \leq p_k$,or$\sum_{-\infty}^{\infty}\left(M(\frac{||\mu_{k(n)}x_{k(n)}||}{\rho})\right)^{p_{k(n)}} > 1$

Hence $\bar{x} \notin \ell(\mathbb{Z},X,M,\bar{\mu},\bar{p})$, which is a contrdiction.

Similar proof can be given for the case when we take $\lim_{k\to-\infty}\inf t_k = 0$. This completes the proof.

**Lemma 2.9:**$\ell(\mathbb{Z},X,M,\bar{\mu},\bar{p}) \subset \ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ if and only if

$$\lim_{k\to-\infty}\sup_k t_k < \infty\ and\ \lim_{k\to\infty}\sup_k t_k < \infty \quad \text{with } l = \inf_k p_k \leq p_k.$$

**Proof:** Sufficiency is straightforward. On the other hand suppose that

$\ell(\mathbb{Z},X,M,\bar{\mu},\bar{p}) \subset \ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$but either$\lim_{k\to-\infty}\sup_k t_k = \infty$ or $\lim_{k\to\infty}\sup_k t_k = \infty$. Let $\lim_{k\to\infty}\sup_k t_k = \infty$. Then there exists a sequence$(k(n))$, $k(n) \geq 1$ such that for each $n \geq 1$, $|\lambda_{k(n)}| > n|\mu_{k(n)}|$. Now define the bilateral sequence $\bar{x} = (x_k)_{-\infty}^{\infty}$ by

$$x_k = \begin{cases} \mu_{k(n)}^{-1} n^{-2} z & \text{if } k = k(n), \quad n \geq 1 \text{ and,} \\ \theta, & \text{otherwise} \end{cases}$$

where$z \in X$ and $||z|| = 1$. Then $\bar{x} \in \ell(\mathbb{Z},X,M,\bar{\mu},\bar{p})$ since $||\mu_{k(n)}x_{k(n)}|| = \frac{1}{n^2}$ i.e., $||\mu_{k(n)}x_{k(n)}|| \to 0$as $n \to \infty$. Therefore $\left(M(\frac{||\mu_{k(n)}x_{k(n)}||}{\rho})\right)^{l} \to 0$ as $n \to \infty$ for any fixed $l = \inf_k p_k$. Hence $\sum_{-\infty}^{\infty}\left(M(\frac{||\mu_k x_k||}{\rho})\right)^{p_k} < \infty$for $l = \inf_k p_k \leq p_k$.

But $||\lambda_{k(n)}x_{k(n)}|| = |\frac{\lambda_{k(n)}}{\mu_{k(n)}}|.n^2 > n$, implies that $M(\frac{||\lambda_{k(n)}x_{k(n)}||}{\rho}) > 1$. Therefore $\sum_{-\infty}^{\infty}\left(M(\frac{||\lambda_{k(n)}x_{k(n)}||}{\rho})\right)^{p_{k(n)}} > 1$for arbitrary large $n$. Hence $\bar{x} \notin \ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$, which is a contrdiction.

Similar proof can be given for the case when we take $\lim_{k\to-\infty}\sup t_k = \infty$. This completes the proof.

On combining above two Lemmas 2.8 and 2.9 we easily get:

**Theorem 2.10:**$\ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p}) = \ell(\mathbb{Z},X,M,\bar{\mu},\bar{p})$ if and only if

$$0 < \lim_{k\to-\infty}\inf t_k < \lim_{k\to-\infty}\sup t_k < \infty$$
$$\text{and } 0 < \lim_{k\to\infty}\inf t_k < \lim_{k\to-\infty}\sup t_k < \infty.$$

## 3. LINEARITY

As far as linear space structures of $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ and $\ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ are concerned, here also we take co-ordinate-wise addition and scalar multiplication in what follows for $\bar{p} = (p_k)_{-\infty}^{\infty} \in \ell_\infty(\mathbb{Z})$ we shall use the notation$H = max(1, \sup_k p_k)$.

**Theorem 3.1:**$c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ forms a linear space over the field $\mathbb{C}$.

**Proof:** Let $\bar{x},\bar{y} \in c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ and $\alpha,\beta \in \mathbb{C}$ therefore there exist some positive $\rho_1$ and $\rho_2$ such that $\left(M(\frac{||\lambda_k x_k||}{\rho_1})\right)^{p_k} \to 0$ as $k \to -\infty$ as well as $k \to \infty$ and $\left(M(\frac{||\lambda_k y_k||}{\rho_2})\right) \to 0$ as $k \to -\infty$ as well as $k \to \infty$. In order to prove the result, we need to find some $\rho_3 > 0$ such that,

$\left(M(\frac{||\alpha\lambda_k x_k + \beta\lambda_k y_k||}{\rho_3})\right)^{p_k} \to 0$ as $k \to -\infty$ as well as $k \to \infty$.

Consider $\rho_3 = max(2|\alpha|\rho_1, 2|\beta|\rho_2)$ i.e., $\frac{|\alpha|}{\rho_3} \leq \frac{1}{2\rho_1}$and $\frac{|\beta|}{\rho_3} \leq \frac{1}{2\rho_2}$. Then we have

$$\begin{aligned}
\left(M(\frac{||\alpha\lambda_k x_k + \beta\lambda_k y_k||}{\rho_3}\right)^{p_k} &\leq \left(M(\frac{||\alpha\lambda_k x_k||}{\rho_3} + \frac{||\beta\lambda_k y_k||}{\rho_3})\right)^{p_k} \\
&\leq \left(M(\frac{||\lambda_k x_k||}{2\rho_1} + \frac{||\lambda_k y_k||}{2\rho_2})\right)^{p_k} \\
&\leq \frac{1}{2^{p_k}}\left(M(\frac{||\lambda_k x_k||}{\rho_1}) + M(\frac{||\lambda_k y_k||}{\rho_2})\right)^{p_k} \\
&< \left(M(\frac{||\lambda_k x_k||}{\rho_1}) + M(\frac{||\lambda_k y_k||}{\rho_2})\right)^{p_k} \\
&\leq c\left(M(\frac{||\lambda_k x_k||}{\rho_1})\right)^{p_k} + c\left(M(\frac{||\lambda_k y_k||}{\rho_2})\right)^{p_k} \to 0
\end{aligned}$$

as$k \to -\infty$ as well as $k \to \infty$,

where $c = max(1, 2^{H-1})$. This proves that $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ forms a linear space over $\mathbb{C}$.

**Theorem 3.2:**$\ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ forms a linear space over the field $\mathbb{C}$.

**Proof:** We can prove the theorem on the lines of Theorem 3.1.

 

## 4. PARANORMED SPACE STRUCTURE

We define

**(4.1)** $P(\bar{x}) = \inf\left\{\rho^{p_n/H} : \sup_k \left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k/H} \leq 1,\ n \in \mathbb{Z}^+\right\}$ and

**(4.2)** $Q(\bar{x}) = \inf\left\{\rho^{p_n/H} : \left(\sum_{-\infty}^{\infty} \left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k}\right)^{\frac{1}{H}} \leq 1, \quad n \in \mathbb{Z}^+\right\}$

where $H = (1, \sup_k p_k)$

**Theorem 4.1:** $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ is a total paranormed space with paranorm defined by (4.1).

**Proof :(i)** Clearly $\mathrm{P}(\bar{x}) = P(-\bar{x})$.

**(ii)** $P(\bar{x}+\bar{y}) \leq P(\bar{x}) + P(\bar{y})$follows by putting $\alpha = \beta = 1$ in Theorem 3.1 .

**(iii)** If $\bar{x} = \theta$ then $P(\theta) = 0$ follows easily since

$\sup_k \left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k/H} = 0$for all $k$

Conversely suppose $P(\bar{x}) = 0$ then

$$\inf\left\{\rho^{p_n/H} : \sup_k \left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k/H} \leq 1,\ n \in \mathbb{Z}^+\right\} = 0$$

In such a case, for given $\epsilon > 0$, there exists some $\rho_\epsilon$, $0 < \rho_\epsilon < \epsilon$ such that $\sup_k \left(M(\frac{||\lambda_k x_k||}{\rho_\epsilon})\right)^{p_k/H} \leq 1$. Thus, $\sup_k \left(M(\frac{||\lambda_k x_k||}{\epsilon})\right)^{p_k/H} \leq \sup_k \left(M(\frac{||\lambda_k x_k||}{\rho_\epsilon})\right)^{p_k/H} \leq 1$.

Suppose, $x_{n_m} \neq 0$, for some m. Let $\epsilon \to 0$, then $(\frac{||x_{n_m}||}{\epsilon}) \to \infty$. It follows that

$$\sup_m \left(M(\frac{||\lambda_m x_{n_m}||}{\epsilon})\right)^{p_m/H} \to \infty$$

which is a contradiction. Therefore $x_{n_m} = 0$ for each $m$.

**(iv)** Finally, we prove that scalar multiplication is continuous. Let $\mu$ be any number. By definition,

$P(\mu\bar{x}) = \inf\left\{\rho^{p_n/H} : \sup_k \left(M(\frac{||\mu\lambda_k x_k||}{\rho})\right)^{p_k/H} \leq 1, n \in \mathbb{Z}^+\right\}$.

Then $P(\mu\bar{x}) = \inf\left\{(\mu r)^{p_n/H} : \sup_k \left(M(\frac{||\lambda_k x_k||}{r})\right)^{p_k/H} \leq 1, n \in \mathbb{Z}^+\right\}$,

where $r = \frac{\rho}{\mu}$. Since $|\mu|^{p_k} \leq max(1, |\mu|^H)$. Therefore $|\mu|^{p_k/H} \leq (max(1, |\mu|^H))^{1/H}$.

Hence, $P(\mu\bar{x}) \leq (max(1, |\mu|^H))^{1/H} \inf\left\{(r)^{p_n/H} : \sup_k \left(M(\frac{||\lambda_k x_k||}{r})\right)^{p_k/H} \leq 1, n \in \mathbb{Z}^+\right\}$
$= (max(1, |\mu|^H))^{1/H} P(\bar{x})$,

which converges to zero as $P(\bar{x})$ converges to zero in $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$. Now suppose $\mu_n \to 0$and $\bar{x} \in c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$. For arbitrary $\epsilon > 0$, let $N$ be a positive integer such that $\left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k} < \frac{\epsilon}{2}$, $k \in \mathbb{Z}\backslash\mathbb{Z}(-N,N)$ for some $\rho > 0$. This implies thatsup$_k \left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k/H} \leq \frac{\epsilon}{2}$, $k \in \mathbb{Z}\backslash\mathbb{Z}(-N,N)$.

Let $0 < |\mu| < 1$, then by convexity of M, we get

$$\left(M(\frac{||\mu\ \lambda_k x_k||}{\rho})\right)^{p_k} < \left(|\mu| M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k} < (\frac{\epsilon}{2})^H, \quad k \in \mathbb{Z}\backslash\mathbb{Z}(-N,N).$$

 

Since $M$ is continuous everywhere in $[0,\infty)$, then $f(t)=\left(M(\frac{||t\lambda_k x_k||}{\rho})\right)$, $k\in\mathbb{Z}\backslash\mathbb{Z}(-N,N)$ is continuous at 0. So there is $1>\delta>0$ such that $|f(t)|<\frac{\epsilon}{2}$, $0<t<\delta$. Let $K$ be such that $|\mu_n|<\delta$ for all $n>K$, then for $n>K$

$$\left(M(\frac{||\mu_n\lambda_k x_k||}{\rho})\right)^{p_k/H}<\frac{\epsilon}{2},\quad k\in\mathbb{Z}(-N,N),$$

Hence $\sup_k\left(M(\frac{||\mu_n\lambda_k x_k||}{\rho})\right)^{p_k/H}<\frac{\epsilon}{2}$, $k\in\mathbb{Z}(-N,N)$. Thus

$$\sup_k\left(M(\frac{||\mu_n\lambda_k x_k||}{\rho})\right)^{p_k/H}<\epsilon,\ \text{for}\ n>K,\ k\in Z(-N,N).$$

This completes the proof.

**Theorem 4.2:**Let $1\le p_k<\infty$. Then $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ is a complete paranormed space with paranorm

$$P(\bar{x})=\inf\left\{\rho^{p_n/H}:\sup_k\left(M(\frac{||\lambda_k x_k||}{\rho})\right)^{p_k/H}\le 1,\ for\ some\ \rho\ and\ n\in\mathbb{Z}\right\}$$

**Proof:** Let $(\bar{x}^{(i)})$ be a Cauchy sequence in $c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$. Let $r$ and $x_\circ$ be fixed positive real numbers with $M(\frac{rx_0}{2})>1$. Then for each $\frac{\epsilon}{rx_0}>0$ there exists a positive integer $N$ such that

**(4.3)** $\quad P(\bar{x}^{(i)}-\bar{x}^{(j)})<\frac{\epsilon}{rx_0}$ for all$i,j\ge N$.

Using definition of paranorm, we get

**(4.4)** $\quad \sup_k\left(M(\frac{||\lambda_k x_k^{(i)}-\lambda_k x_k^{(j)}||}{P(\bar{x}^{(i)}-\bar{x}^{(j)})})]^{p_k/H}\right)\le 1$ for all $i,j\ge N$,and $k\in\mathbb{Z}$.

Thus

$$\left(M(\frac{||\lambda_k x_k^{i}-\lambda_k x_k^{j}||}{P(\bar{x}^{(i)}-\bar{x}^{(j)})})\right)^{p_k}\le 1\quad \text{for all}\ i,j\ge N\ \text{and}\ k\in\mathbb{Z}.$$

Since $1\le p_k<\infty$, it implies that

$$\left(M(\frac{||\lambda_k x_k^{(i)}-\lambda_k x_k^{(j)}||}{P(\bar{x}^{(i)}-\bar{x}^{(j)})})\right)\le 1\quad \text{for all}\ k\ge 1\ \text{and for all}\ i,j\ge N.$$

But $M(\frac{rx_0}{2})>1$. Therefore

$$M(\frac{||\lambda_k x_k^{(i)}-\lambda_k x_k^{(j)}||}{P(\bar{x}^{(i)}-\bar{x}^{(j)})})<M(\frac{rx_0}{2}).$$

But $M$ is non-decreasing therefore

$$\frac{||\lambda_k x_k^{(i)}-\lambda_k x_k^{(j)}||}{P(\bar{x}^{(i)}-\bar{x}^{(j)})}<\frac{rx_0}{2}$$

or, $\quad ||\lambda_k x_k^{(i)}-\lambda_k x_k^{(j)}||<\frac{rx_0}{2}.[P(\bar{x}^{(i)}-\bar{x}^{(j)})]$

or, $\quad ||\lambda_k x_k^{(i)}-\lambda_k x_k^{(j)}||<\frac{rx_0}{2}.\frac{\epsilon}{rx_0}=\frac{\epsilon}{2}.$

Hence $(x_k^{(i)})$ is a Cauchy sequence in $X$ for all $k\in\mathbb{Z}$ and therefore convergent. But $X$ is complete, therefore $x_k^{(i)}\to x_k$ (say) as $i\to\infty$. Let us choose $\rho>0$ such that $P((\bar{x}^{(i)}-\bar{x}^{(j)}))<\rho<\epsilon$ for all $i,j\ge N$.Since $M$ is non decreasing we have by (4.4)

$$\sup_k\left(M(\frac{||\lambda_k x_k^{(i)}-\lambda_k\lim_{j\to\infty}x_k^{(j)}||}{\rho})\right)^{p_k/H}\le\sup_k\left(M(\frac{||\lambda_k x_k^{(i)}-\lambda_k x_k^{(j)}||}{P(\bar{x}^{(i)}-\bar{x}^{(j)})})]^{p_k/H}ght)\le 1,$$

for all $i,j\ge N$

Letting $j \to \infty$ and using continuity of $M$, we get

$$\sup_k \left( M\left(\frac{||\lambda_k x_k^{(i)} - \lambda_k \lim_{j\to\infty} x_k^{(j)}||}{\rho}\right)\right)^{p_k/H} \leq 1 \quad \text{for all } k \in \mathbb{Z}(-N,N).$$

Thus $$\sup_k \left( M\left(\frac{||\lambda_k x_k^{(i)} - \lambda_k x_k||}{\rho}\right)\right)^{p_k/H} \leq 1 \quad \text{for all } k \in \mathbb{Z}(-N,N).$$

Taking infimum of such $\rho$'s, we get

$$P(\bar{x}^{(i)} - \bar{x}) = \inf\left\{\rho^{p_n/H} : \sup_k \left( M\left(\frac{||\lambda_k x_k^{(i)} - \lambda_k x_k||}{\rho}\right)\right)^{p_k/H} \leq 1 \quad for\ all\ i \geq N\right\}$$

$$\leq \rho < \epsilon.$$

Hence $P(\bar{x}^{(i)} - \bar{x}) < \epsilon$ for all $i \geq N$.

Since $(\bar{x}^{(i)}) \in c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ and $M$ is continuous, it follows that $\bar{x} \in c_\circ(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$. This completes the proof.

**Theorem 4.3:** $\ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ is a total paranormed space with

$$Q(\bar{x}) = \inf\left\{\rho^{p_n/H} : \left(\sum_{-\infty}^{\infty} \left( M\left(\frac{||\lambda_k x_k||}{\rho}\right)\right)^{p_k}\right)^{\frac{1}{H}} \leq 1, \quad n \in \mathbb{Z}^+\right\}$$

where $H = (1, \sup_k p_k)$.

**Proof:** The theorem can be proved on the lines of Theorem 4.1

**Theorem 4.4:** Let $1 \leq p_k < \infty$. Then $\ell(\mathbb{Z},X,M,\bar{\lambda},\bar{p})$ is a complete paranormed space with respect to paranorm

$$Q(\bar{x}) = \inf\left\{\rho^{p_n/H} : \left(\sum_{-\infty}^{\infty} \left( M\left(\frac{||\lambda_k x_k||}{\rho}\right)\right)^{p_k}\right)^{\frac{1}{H}} \leq 1, \quad n \in \mathbb{Z}^+\right\}$$

**Proof:** We can prove this theorem on the lines of Theorem 4.2.

**REFERENCES**

[1] Krasnoselskii , M. A and Rutitsky, Y. B. (1961). Convex Functions and Orlicz Spaces. Groningen, Netherlands.

[2] Lindberg, K. (1973). On Subspaces of Orlicz Sequence Spaces.Studia Math., 45:119{146.

[3] Lindenstrauss , J and Tzafriri , L. (1971). On Orlicz Sequence Spaces. IsraelJour. Math., 10(3): 379 -390.

[4] Lindenstrauss , J and Tzafriri , L. (1972). On Orlicz Sequence Spaces ll. Israel Jour. Math., 11(4) :355 - 379.

[5] Lindenstrauss , J and Tzafriri , L. (1973). On Orlicz Sequence Spaces lll.Israel Jour. Math., 14(4): 368 -389.

[6] Lindenstrauss , J and Tzafriri , L. (1977). Classical Banach Spaces. Springer- Verlag, New York/ Berlin.

[7] Simon, N. C and Marko, L. W. (2011). Limit Operators, Collective Compactnessand the Spectral Theory of Infinite Matrices. Memoirs, Amer. Math. Soc., 210.

[8] Srivastava, J. K and Agrawal, R. (2012). Banach Space X-Valued Bilateral Sequence Spaces $c_\circ(\mathbb{Z},X,\bar{\lambda},\bar{p})$ and $C(\mathbb{Z},X,\bar{\lambda},\bar{p})$. Submitted for Publication.

[9] Wilansky, A. (1978). Modern Methods in Topological Vector Spaces. McGraw-HillBook Co. Inc. New York.

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