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SHARPER BOUNDS FOR ZEROS OF COMPLEX POLYNOMIALS

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ABSTRACT

We prove some extensions of the classical results concerning Enestrom-Kakeya theorem and related analytic functions. Besides several consequences, our results considerably improve the bounds by relaxing and weakening the hypothesis in some cases.

Mathematics Subjects Classification: *26C10, 30C10, 30C15.*

Keywords: *Polynomials, Zeros, Enestrom - Kakeya theorem & the sharper bounds.*

1. INTRODUCTION

The following result due to Enestrom & Kakeya [8], page 136 is well known in the theory of distribution of zeros of polynomials.

Theorem A (a): If P (z) $=\sum_0^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots\ldots \geq a_1 \geq a_0 > 0, \quad a_j \in R \qquad 1(a)$$

Then P (z) has all its zeros in $|z| \leq 1$

A. Joyal et al [7] extended theorem to the polynomials whose coefficient are montonic but not necessarily non negative and proved the following:

Theorem A (b): If P (z) $=\sum_0^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots\ldots \geq a_1 \geq a_0 \quad , \quad a_j \in R$$

Then all the zeros of P(z) lie in

$$|z| \leq (a_n - a_0 + |a_0|) \div |a_n|. \qquad 1(b)$$

Theorem A(c): If P(z) $=\sum_0^n a_j z^j$ be a polynomial of degree n such that for some $\lambda \geq 1$,

$$\lambda a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots\ldots \geq a_1 \geq a_0 \quad , \quad \lambda, a_j \in R,$$

then all the zeros of P(z) lie in

$$|z + \lambda - 1| \leq (\lambda a_n - a_0 + |a_0|) \div |a_n|. \qquad 1(c)$$

Among other authors besides Joyal et al [7], Dewan & Govil[3] and Aziz & Zarger[1] also extended Theorem A(1) to the polynomials whose coefficients are monotonic but not necessarily non negative.

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2. THE POLYNOMIALS WITH COMPLEX COEFFICIENTS

Govil and Mc Tume [6] extended the results of Aziz and Zarger[1] to the polynomials with complex coefficients given by:

Theorem B: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, For $j = 0,1,2\ldots..n$. If for some $\lambda \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \ldots\ldots \geq \alpha_1 \geq \alpha_0, \quad \lambda, a_j \in R,$$

then all the zeros of P(z) lie in

$$|z + \lambda - 1| \leq (\lambda\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_0^n |\beta_j|) \div |a_n| \quad 2(a)$$

Recently Rather and Shakeel [10] on the lines of Govil & Mc Tume[6] obtained the following result:

Theorem C: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, For $j = 0, 1, 2$-----n. If for some $\lambda \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \ldots\ldots \geq \alpha_1 \geq \alpha_0, \qquad \lambda, a_j \in R,$$

then all the zeros of P(z) lie in

$$|z + (\lambda - 1)\frac{\alpha_n}{|a_n|}| \leq (\lambda\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_0^n |\beta_j|) \div |a_n| \quad 2(b)$$

Generalizing the above result, Rather & Shakeel also proved the following result:

Theorem D: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. If for some $\lambda \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \ldots\ldots \geq \alpha_1 \geq \alpha_0,$$

$$\lambda\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \ldots\ldots \geq \beta_1 \geq \beta_0$$

Then all the zeroes of P(z) lie in

$$|z + \lambda - 1| \leq [\lambda(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + |a_0|] \div |a_n| \quad (3)$$

Recently, B. L. Raina et al [9] have generalized the above result and proved the following:

Theorem E: If $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n and if m^{th} mean is associated to some λ & $\mu \geq 1$, such that

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \ldots\ldots \geq \alpha_1 \geq \alpha_0,$$

$$\mu\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \ldots\ldots \geq \beta_1 \geq \beta_0$$

and if $k = \frac{\lambda + \mu}{m}$ for $m \in R^+$, (the set of all positive real numbers),

Then all the zeroes of P(z) lie in

$$|z + k - 1| \leq [k(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + |a_0|] \div |a_n| \quad (4)$$

Theorem F: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2\ldots\ldots n$. If for some $\lambda \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \ldots\ldots \geq \alpha_1 \geq \alpha_0,$$

then all the zeros of P(z) lie in the disc:

$$|z + (\lambda - 1)\frac{\alpha_n}{|a_n|}| \leq [\, b + \sqrt{2}\sqrt{a^2 + b^2}\,] \div |a_n|, \quad (5)$$

where $a = \lambda|\alpha_n| + |\beta_n|$ and $b = |\alpha_{n-1}| + |\beta_{n-1}|$ (6)

Theorem G: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2\ldots\ldots n$. If for some $\lambda \geq 1$ and $t>0$,

$$\lambda t^n \alpha_n \geq t^{n-1}\alpha_{n-1} \geq \alpha_{n-2} \geq \ldots\ldots \geq t\alpha_1 \geq \alpha_0,$$

then all the zeros of P(z) lie in the disc.

$$|z + \frac{(\lambda-1)t\alpha_n}{|a_n|}| \leq [(t^{n-1}\alpha_{n-1} + \beta_{n-1}) + \{2(\lambda t^n \alpha_n + \beta_n)^2 + (t^{n-1}\alpha_{n-1} + \beta_{n-1})^2\}^{1/2]} \div |a_n| t^{n-1} \quad (7)$$

Theorem H: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. If for some $\mu \geq 1$ and $t>0$,

$$\mu t^n \beta_n \geq t^{n-1}\beta_{n-1} \geq \beta_{n-2} \geq \ldots\ldots \geq t\beta_1 \geq \beta_0,$$

then all the zeros of P(z) lie in the disc.

$$|z + \frac{(\mu-1)t\beta_n}{|a_n|}| \leq [(\alpha_{n-1} + t^{n-1}\beta_{n-1}) + \{2(\alpha_n + \mu t^n \beta_n)^2 + (\alpha_{n-1} + t^{n-1}\beta_{n-1})^2\}^{1/2}] \div |a_n| t^{n-1} \quad (8)$$

In this paper we consider the generalization of the above theorem and discuss certain properties given by the following:

Theorem 1.1: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients such that $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. and If for some λ and $\mu \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \ldots\ldots\ldots \geq \alpha_1 \geq \alpha_0$$

$$\mu\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \ldots\ldots \geq \beta_1 \geq \beta_0 \quad \text{(i)}$$

then all the zeros of P(z) lie in the disc:

$$|z + \frac{(\lambda-1)\alpha_n + i(\mu-1)\beta_n}{a_n}| \leq \frac{1}{|a_n|}(B + (A-B)\cos\alpha + (A+B)\sin\alpha) \quad \text{(ii)}$$

Where $A = \lambda|\alpha_n| + \mu|\beta_n|$ and $B = |\alpha_{n-1}| + |\beta_{n-1}|$ (iii)

Proof: Consider the polynomial

$$F(z) = (1-z)P(z) = (1-z)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots + a_{n-1}z^{n-1} + a_n z^n)$$

$$= (a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \text{----} + a_{n-1}z^{n-1} + a_n z^n - a_0 z - a_1 z^2 - a_2 z^3 - \ldots\ldots - a_{n-1}z^n - a_n z^{n+1})$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \sum_{j=0}^{j=n-1}(a_j - a_{j-1})z^j \quad (\text{let } a_{-1} = 0)$$

$$= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + i(\beta_n - \beta_{n-1})z^n + \sum_{j=0}^{j=n-1}(\alpha_j - \alpha_{j-1})z^j + i\sum_{j=0}^{j=n-1}(\beta_j - \beta_{j-1})z^j$$

$$= -a_n z^{n+1} - (\lambda\alpha_n - \alpha_n)z^n + (\lambda\alpha_n - \alpha_{n-1})z^n - i(\mu\beta_n - \beta_n)z^n + i(\mu\beta_n - \beta_{n-1})z^n + \sum_{j=0}^{j=n-1}(\alpha_j - \alpha_{j-1})z^j + i\sum_{j=0}^{j=n-1}(\beta_j - \beta_{j-1})z^j$$

Let $|z|>1$. Then

$$|F(z)| \geq |-z^n\{a_n z + (\lambda-1)\alpha_n + i(\mu-1)\beta_n\} - \{(\lambda\alpha_n - \alpha_{n-1}) + i(\mu\beta_n - \beta_{n-1})\} - \sum_{j=0}^{j=n-1}(\alpha_j - \alpha_{j-1})z^{j-n} - i\sum_{j=0}^{j=n-1}(\beta_j - \beta_{j-1})z^{j-n}\}|$$

$$= |z|^n |[F_1(\lambda, \mu, \alpha, \beta, z) - \{F_2(\lambda, \mu, \alpha, \beta) + F_3(\alpha, z) + F_4(\beta, z)\}]|, \quad \text{(iv)}$$

where,
$F_1(\lambda, \mu, \alpha, \beta, z) = [a_n z + (\lambda-1)\alpha_n + i(\mu-1)\beta_n]$

$F_2(\lambda, \mu, \alpha, \beta) = (\lambda\alpha_n - \alpha_{n-1}) + i(\mu\beta_n - \beta_{n-1})$

$F_3(\alpha, z) = \sum_{j=0}^{j=n-1}(\alpha_j - \alpha_{j-1})z^{j-n}$

$F_4(\beta, z) = i\sum_{j=0}^{j=n-1}(\beta_j - \beta_{j-1})z^{j-n}$ (v)

By using the lemma due to Govil & Rehman[5] given as:

Lemma: If $|\arg a_j-\beta|\leq \alpha \leq \pi/2$ for some $t>0$, $|ta_j|\geq|a_{j-1}|$, then

$$|ta_j - a_{j-1}|\leq \{(|ta_j|-|a_{j-1}|)\cos\alpha + (|ta_j|+|a_{j-1}|)\sin\alpha\} \quad \text{(vi)}$$

From eq(iv), $|F(z)|\geq |z|^n[|F_1(\lambda, \mu, \alpha, \beta, z)|-|F_5(\lambda, \mu, \alpha, \beta, z)|$, (by T. inequality) (vii)

where
$F_5 (\lambda, \mu, \alpha, \beta, z) = F_2 (\lambda, \mu, \alpha, \beta) + F_3 (\alpha, z) + F_4 (\beta, z)$

By triangular inequality, we've

$$|F_5 (\lambda, \mu, \alpha, \beta, z)| = |F_2 (\lambda, \mu, \alpha, \beta)| + |F_3 (\alpha, z)| +|F_4 (\beta, z)| \quad \text{(viii)}$$

Using (v), we have

$|F_2(\lambda,\mu,\alpha,\beta)| \leq|(\lambda\alpha_n-\alpha_{n-1})| +|\mu\beta_n-\beta_{n-1}|$

$\leq \{(|\lambda\alpha_n|-|\alpha_{n-1}|) \cos\alpha + (|\lambda\alpha_n|+|\alpha_{n-1}|)\sin\alpha\}+\{(|\mu\beta_n|-|\beta_{n-1}|)\cos\alpha + (|\mu\beta_n|+|\beta_{n-1}|)\sin\alpha\}$ (using lemma)

$\leq \{(|\lambda\alpha_n|-|\alpha_{n-1}|+|\mu\beta_n|-|\beta_{n-1}|) \cos\alpha + (|\lambda\alpha_n|+|\alpha_{n-1}|+|\mu\beta_n|+|\alpha_{n-1}|)\sin\alpha\}$ (ix)

Also $| F_3(\alpha,z)| \leq \sum_{j=0}^{j=n-1} |(\alpha_j - \alpha_{j-1})||z|^{j-n}$

$\leq|\alpha_{n-1}|$. (by Triangular inequality & eq(i)& $|z|^{j-n} < 1, |\alpha_{-1}| = 0$) (x)

Similarly $| F_4(\alpha, z)| \leq |\beta_{n-1}|$. (let $|\beta_{-1}|=0$) (xi)

Therefore, from eq(viii), taking in to the account of the result of the equations (ix),(x),(xi),

We write eq(vii) as

$$|F(z)| \geq |z|^n[|a_nz+(\lambda-1)\alpha_n +i(\mu-1)\beta_n |]-\{(A-B)\cos\alpha+(A+B)\sin\alpha+B\} , \quad \text{(xii)}$$

where $A = \lambda|\alpha_n|+ \mu|\beta_n|$ and $B=|\alpha_{n-1}|+|\beta_{n-1}|$

Thus for $|z|>1$, $|F(z)|>0$ only if

$$|a_nz+ (\lambda-1)\alpha_n +i(\mu-1)\beta_n |> (B+(A-B)\cos\alpha+(A+B)\sin\alpha)$$

Which gives

$$|z+\frac{(\lambda-1)\alpha_n+i(\mu-1)\beta_n}{a_n}| > (B+ (A-B)\cos\alpha+(A+B)\sin\alpha)\div|a_n| \quad \text{(xiii)}$$

Above equation shows that the zeros of F(z) having modulii greater than 1 lie in the circle

$$|z+\frac{(\lambda-1)\alpha_n+i(\mu-1)\beta_n}{a_n}| \leq \frac{1}{|a_n|}(B + (A-B)\cos\alpha + (A+B)\sin\alpha) \quad \text{(xiv)}$$

It can also be verified that the zeros of F(z) whose modulus is less than or equal to one also lie in the circle defined by equation(ii) of Theorem 1.1 and therefore all the zeros of P(z) lying in the disc given by equation(ii)

Hence above theorem is proved.

Corollary: We note here that since max (acosα +b sinα) $=\sqrt{a^2+b^2}$, therefore the above abound can alternatively expressed by:

$$|z+\frac{(\lambda-1)\alpha_n+i(\mu-1)\beta_n}{a_n}| \leq \frac{1}{|a_n|} [B + \sqrt{2}\sqrt{A^2+B^2}] \quad \text{(xv)}$$

where $A = \lambda|\alpha_n|+\mu|\beta_n|$ and $B = |\alpha_{n-1}|+|\beta_{n-1}|$

Which is independent of α and is therefore not as sharper bound as given by above equation (xiii)

Remark: If we take μ=1 in Theorem 1.1, then the above theorem coincides with Theorem F which gives the sharper bounds than otherwise given by Govil & Mctume[6], Dewan & Govil [3] and Rather & Shakeel[10] as discussed by B.L. Raina et al[9]

Corollary 1: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with complex co-efficents with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. and If for some λ and $\mu \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \ldots. \geq \alpha_1 \geq \alpha_0 > 0$$

$$\mu\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \ldots. \geq \beta_1 \geq \beta_0 > 0$$

then all the zeros of P(z), (independent of α)lie in

$$\left|z+\frac{(\lambda-1)\alpha_n+i(\mu-1)\beta_n}{a_n}\right| \leq \frac{1}{|a_n|}\left[\alpha_{n-1}+\beta_{n-1}+\sqrt{2}\left[\{(\lambda\alpha_n+\mu\beta_n)^2+(\alpha_{n-1}+\beta_{n-1})^2\}^{1/2}\right]\right] \quad \text{(xvi)}$$

Corollary 2: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with complex co-efficients with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. and If for some λ and $\mu \geq 1$, and t>0 such that

$$\lambda t^n\alpha_n \geq t^{n-1}\alpha_{n-1} \geq t^{n-2}\alpha_{n-2} \geq \ldots\ldots \geq t\alpha_1 \geq \alpha_0$$

$$\mu t^n\beta_n \geq t^{n-1}\beta_{n-1} \geq t^{n-2}\beta_{n-2} \geq \ldots\ldots. \geq t\beta_1 \geq \beta_0$$

then all the zeros of P(z) lie in

$$\left|z+\frac{(\lambda-1)t\alpha_n+i(\mu-1)t\beta_n}{a_n}\right| \leq \frac{1}{|a_n|t^{n-1}}\left[t^{n-1}(\alpha_{n-1}+\beta_{n-1})+\sqrt{2}\ \{(\lambda t^n\alpha_n+\mu t^n\beta_n)^2+(t^{n-1}\alpha_{n-1}+t^{n-1}\beta_{n-1})^2\}^{1/2}\right]$$

Illustration: Now we give some examples to show that the present estimate given by our main Theorem 1.1 are sharper as compared to the other authors. We therefore construct a polynomial $P(z) = \sum_0^n a_j z^j$ correspoding to n=2, 3 & 4 and compare the bounds obtained by other authors with our present bounds and thereby give the location of zeros of the polynomials corresponding to these values of n.

n	$a_j = \alpha_j + i\beta_j$	Approximate zeros of polynomials $P_n(z)$	Different values of λ and μ	Bounds for the zeros of the polynomials by the present estimate	Comparison of present estimate with Raina et al [9] where $k = \frac{\lambda+\mu}{m}$ (by Th-E)	Comparison of present estimate with other authors
2	$a_2=(2,3)$, $a_1= (-2,-2)$, $a_0 =(-5,-5)$ with constraint $\lambda\alpha_2 \geq \alpha_1 \geq \alpha_0$ and $\mu\beta_2 \geq \beta_1 \geq \beta_0$	$z_1= 3.17-0.905i$ $z_2= 2.5 + 0.75i$	Case-(i) λ=3, μ=3	$\|z\| \leq 9.972$ from Th-1.1	$\|z\| \leq 18.057$ for m=1. $\|z\| \leq 10.896$ for m=2.	$\|z\| \leq 10.986$ (even without any constraint on β_i's) from Th-B $\|z\| \leq 11.096$ (even without any constraint on β_i's) from Th-C. $\|z\| \leq 10.896$ from Th-D
			Case-(ii) λ=3, μ=2	$\|z\| \leq 8.013$ from Th-1.1	$\|z\| \leq 15.67$ for m=1 $\|z\| \leq 9.702$ for m=2	$\|z\| \leq 10.986$ (even without any constraint on β_i's) from Th-B $\|z\| \leq 11.096$ (even without any constraint on β_i's) from Th-C.
			Case-(ii) λ=2, μ=3	$\|z\| \leq 8.659$ from Th-1.1	$\|z\| \leq 15.67$ for m=1 $\|z\| \leq 9.702$ for m=2	$\|z\| \leq 10.43$ from Th-B $\|z\| \leq 9.98$ from Th-C

n	$a_j = \alpha_j + i\beta_j$	Approximate zeros of polynomials $P_n(z)$	Different values of λ and μ	Bounds for the zeros of the polynomials by the present estimate	Comparison of present estimate with Raina et al[9] $k = \frac{\lambda+\mu}{m}$ (From Theorem E)	Comparison of present estimate with other authors
3	a_3=2+3i , a_2=2+4i, a_1= -9-9i a_0 =-10-10i with constraint $\lambda\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \alpha_0$, $\mu\beta_3 \geq \beta_2 \geq \beta_1 \geq \beta_0$	z_1= -2+0i z_2 = 3.17-0.905i z_3 = 2.5+0.75i	Case-(i) When λ=3, μ=3	$\|z\| \leq 10.77$ from Th-1.1	$\|z\| \leq 22.79$ for m=1 $\|z\| \leq 15.63$ for m=2 $\|z\|>10.77$ for m≥3	$\|z\| \leq 23.64$ from Th-B $\|z\| \leq 22.74$ from Th-C. $\|z\| \leq 15.63$ from Th-D
			Case-(ii) When λ=3, μ=2	$\|z\| \leq 8.87$ from Th-1.1	$\|z\| \leq 20.40$ for m=1 $\|z\| \leq 14.43$ for m=2 $\|z\|>8.87$ for m≥3	$\|z\| \leq 23.64$ from Th-B $\|z\| \leq 22.74$ from Th-C
			Case-(ii) When λ=2, μ=3	$\|z\| \leq 9.48$ from Th-1.1	$\|z\| \leq 20.40$ for m=1 $\|z\| \leq 14.43$ for m=2 $\|z\|>9.48$ for m≥3	$\|z\| \leq 22.08$ from Th-B $\|z\| \leq 21.63$ from Th-C

n	$a_j = \alpha_j + i\beta_j$	Approximate zeros of polynomials $P_n(z)$	Different values of λ and μ	Bounds for the zeros of the polynomials by the present estimate	Comparison of present estimate with Raina et al[9] $k = \frac{\lambda+\mu}{m}$ (From Theorem E)	Comparison of present estimate with other authors
4	a_3=1+0i , a_2=0+0i, a_1= 0+0i a_0 =-i with constraint $\lambda\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \alpha_0$ $\mu\beta_3 \geq \beta_2 \geq \beta_1 \geq \beta_0$	z_1= 0.9+0.4i z_2 = 0.38+0.92i z_3 =-0.38+0.92i z_4= -0.92+0.38i	Case-(i) When λ=3, μ=3	$\|z\| \leq 6.24$ from Th-1.1	$\|z\| \leq 8$ for m=1 $\|z\| \leq 7$ for m=2	$\|z\| \leq 7$ from Th-B $\|z\| \leq 7$ from Th-C. $\|z\| \leq 7$ from Th-D
			Case-(ii) When λ=3, μ=2	$\|z\| \leq 6.24$ from Th-1.1	$\|z\| \leq 11$ for m=1 $\|z\| \leq 6$ for m=2	$\|z\| \leq 7$ from Th-B $\|z\| \leq 7$ from Th-C
			Case-(iii) When λ=2, μ=3	$\|z\| \leq 3.8$ from Th-1.1	$\|z\| \leq 11$ for m=1 $\|z\| \leq 6$ for m=2	$\|z\| \leq 5$ from Th-B $\|z\| \leq 5$ from Th-C

Remark: From the above table one can easily find that the present estimates are sharper for different values of λ and μ in all the cases.

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