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GENERALIZED ROUGH PROBABILITY IN TOPOLOGICAL SPACES

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ABSTRACT

Rough set theory has been introduced by Pawlak [4]. It is considered as a base for all researches in the rough set area introduced after this date. Most of these researches concentrated on developing results and techniques based on Pawlak's results [4]. In this paper we shall introduce generalized rough probability from topological view. The basic concepts of generalized open, generalized rough, and generalized exact sets are introduced and sufficiently illustrated. Moreover, proved results, examples and counter examples are provided. The topological structure which suggested in this paper opens up the way for applying rich amount of topological facts and methods in the process of granular computing.

Keywords: *Topological space; Generalized rough set, Generalized rough probability.*

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1. INTRODUCTION:

One of the most powerful notions in system analysis is the concept of topological structures [2] and their generalizations. Rough set theory, introduced by Pawlak in 1982 [4], is a mathematical tool that supports also the uncertainty reasoning but qualitatively. In this paper, we shall integrate some ideas in terms of concepts in topology. Topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics, but also in many real life applications. We believe that topological structure will be an important base for modification of knowledge extraction and processing.

2. PRELIMINARIES:

A topological space [2] is a pair (X,τ) consisting of a set X and family τ of subsets of X satisfying the following conditions:

(T1) $\phi \in \tau$ and $X \in \tau$.
(T2) τ is closed under arbitrary union.
(T3) τ is closed under finite intersection.

Throughout this paper (X,τ) denotes a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, the complement of the subsets of X belonging to τ are called closed sets in the space, and the family of all open sets of (X,τ) is denoted by τ and the family of all closed sets of (X,τ) is denoted by $C(X)$.

For a subset A of a space (X,τ), $Cl(A)$ denote the closure of A and is given by $Cl(A) = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in C(X)\}$. Evidently, $Cl(A)$ is the smallest closed subset of X which contains A. Note that A is closed iff $A = Cl(A)$. $Int(A)$ denote the interior of A and is given by $Int(A) = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}$ Evidently, $Int(A)$ is the largest open subset of X which contained in A. Note that A is open iff $A = Int(A)$.

We shall recall some concepts about some near open sets which are essential for our present study.

Definition: 2.1.[3] A subset A of a space (X,τ) is said to be generalized closed (briefly, g-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).

The complement of a g-closed set is called g-open The family of all g-open sets of (X,τ) is denoted by $gO(X)$. The family of all g-closed sets of (X,τ) is denoted by $gC(X)$. The generalized interior (briefly g-interior) of A is denoted by ${}_g Int(A)$ and is defined by ${}_g Int(A) = \cup\{G \subseteq X : G \subseteq A, G \text{ is a } g\text{-open}\}$.

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The generalized closure (briefly g -closure) of A is denoted by $_{g}Cl(A)$ and is defined by $_{g}Cl(A)=\cap\{F\subseteq X\ :A\subseteq F,F$ is a g -closed set$\}$.

3. PAWLAK'S APPROACH:
Consider the approximation space $K=(U,R)$, where U is a set called the universe and R is an equivalence relation. The order triple $S=(U,R,p)$ is called the stochastic approximation space [5], where p is a probability measure; any subset of U will be called an event. The probability measure p has the following properties:

$p(\phi)=0,\ \ p(U)=1$ and if $A=\bigcup_{i=1}^{n}X_i$ is an observable set in K, then $p(A)=\sum_{i=1}^{n}p(X_i)$.

It is clear that A is a union of disjoint sets, since R is an equivalence relation. Pawlak introduced the definitions of the lower and upper probabilities of an event A in the stochastic approximation space $S=(U,R,p)$. These definitions are:

- The lower probability of A is denoted by $\underline{p}(A)$ and is given by $\underline{p}(A)=p(\underline{R}A)$.
- The upper probability of A is denoted by $\overline{p}(A)$ and is given by $\overline{p}(A)=p(\overline{R}A)$.

Clearly, $0\leq\underline{p}(A),\ \ \overline{p}(A)\leq1$.

4. GENERALIZED ROUGH PROBABILITY IN TOPOLOGICAL SPACES:
Definition: 4.1. [1]. Let $K=(X,R)$ be an approximation space with general relation R and τ_k is the topology associated to K. Then the triple (X,R,τ_k) is called a topologized approximation space.

Definition: 4.2 [1]. Let $K=(U,R)$ be an approximation space with general relation R and τ_K is the topology associated to K. Then the order 4-tuples $S=(U,R,p,\tau_K)$ is called the topologized stochastic approximation space.

4.1. GENERALIZED ROUGH PROBABILITY:
Definition: 4.1.1. Let A be an event in the topologized stochastic approximation space $S=(U,R,p,\tau_K)$. Then the generalized lower (resp. generalized upper) probability of A is given by $_{g}\underline{p}(A)=p(_{g}Int(A))$ (resp. $_{g}\overline{p}(A)=p(_{g}Cl(A))$).

Definition: 4.1.2. Let A be an event in the topologized stochastic approximation space $S=(U,R,p,\tau_K)$. Then the generalized rough probability of A is denoted by $_{g}p^{*}(A)$ and is given by

$$_{g}p^{*}(A)=\left\langle {}_{g}\underline{p}(A),\ {}_{g}\overline{p}(A)\right\rangle.$$

4.2. GENERALIZED ROUGH DISTRIBUTION FUNCTION:
The distribution function of a random variable X gives the probability that X does not exceed x. We shall define the generalized lower and the generalized upper distribution functions of a random variable X.

Definition: 4.2.1. Let X be a random variable in the topologized stochastic approximation space $S=(U,R,p,\tau_K)$. Then the generalized lower distribution (resp. generalized upper distribution) function of X is given by

$$_{g}\underline{F}(x)={}_{g}\underline{p}(X\leq x)\ \text{(resp. } {}_{g}\overline{F}(x)={}_{g}\overline{p}(X\leq x)\text{)}.$$

Definition: 4.2.2. Let X be a random variable in the topologized stochastic approximation space $S=(U,R,p,\tau_K)$. Then the generalized rough distribution function of X is denoted by $_{g}F^{*}(x)$ and is given by $_{g}F^{*}(x)=\left\langle {}_{g}\underline{F}(x),\ {}_{g}\overline{F}(x)\right\rangle$.

Example: 4.2.1. Consider the experiment of choosing one from four cards have numbered from one to four. The collection of the five elements forms the outcome space. Hence,

$$U=\{1,2,3,4\},$$

Let R be a binary relation defined on U such that

$$R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(3,4),(4,2)\}.$$

Thus $U/R = \{\{1,2,3\},\{3\},\{3,4\},\{2\}\}$. Let $K = (U, R)$ be an approximation space and τ_K is the topology associated to K. Thus,

$$\tau_K = \{U, \phi, \{2\},\{3\},\{2,3\},\{3,4\},\{1,2,3\},\{2,3,4\}\}$$

Define the random variable X to be the number on the chosen card. We can construct Table 4.2.1. which contains the generalized lower and the generalized upper probabilities of a random variable $X = x$.

Table: 4.2.1.

X	1	2	3	4
${}_g\underline{p}(X = x)$	0	$\frac{1}{4}$	$\frac{1}{4}$	0
${}_g\overline{p}(X = x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{1}{4}$

Then the generalized lower distribution function of X is

$${}_g\underline{F}(x) = \begin{cases} 0 & -\infty < x < 2, \\ \frac{1}{4} & 2 \le x < 3, \\ \frac{2}{4} & 3 \le x < \infty. \end{cases}$$

And the generalized upper distribution function of X is

$${}_g\overline{F}(x) = \begin{cases} 0 & -\infty < x < 1, \\ \frac{1}{4} & 1 \le x < 2, \\ \frac{3}{4} & 2 \le x < 3, \\ \frac{6}{4} & 3 \le x < 4, \\ \frac{7}{4} & 4 \le x < \infty. \end{cases}$$

Therefore ${}_gF^*(4) = \left\langle \frac{2}{4}, \frac{7}{4} \right\rangle$

4.3. GENERALIZED ROUGH EXPECTATION:

The expectation of a random variable X is the average of all possible values of X weighted by their probabilities. We shall define the generalized lower and the generalized upper expectations of a random variable X.

Definition: 4.3.1. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, p, \tau_K)$. Then the generalized lower (resp. generalized upper) expectation of X is given by ${}_g\underline{\mu} = {}_g\underline{E}(X) = \sum_{k=1}^{n} x_k \; {}_g\underline{p}(X = x_k)$

$$(\text{resp. } {}_g\overline{\mu} = {}_g\overline{E}(X) = \sum_{k=1}^{n} x_k \; {}_g\overline{p}(X = x_k)).$$

Definition: 4.3.2. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, p, \tau_K)$. Then the generalized rough expectation of X is denoted by ${}_gE^*(X)$ and is given by ${}_gE^*(X) = \left\langle {}_g\underline{E}(X), {}_g\overline{E}(X) \right\rangle$.

The generalized rough expectation of X also denoted by ${}_g\mu^* = \left\langle {}_g\underline{\mu}, {}_g\overline{\mu} \right\rangle$.

4.4. GENERALIZED ROUGH VARIANCE AND GENERALIZED ROUGH STANDARD DEVIATION:

We shall define the generalized lower and the generalized upper variances of a random variable X.

Definition: 4.4.1. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, p, \tau_K)$. Then the generalized lower (resp. generalized upper) variance of X is given by ${}_g\underline{V}(X) = {}_g\underline{E}(X - {}_g\underline{\mu})^2$ (resp. ${}_g\overline{V}(X) = {}_g\overline{E}(X - {}_g\overline{\mu})^2$).

Definition: 4.4.2. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, p, \tau_K)$. Then the generalized rough variance of X is denoted by ${}_gV^*(X)$ and is given by ${}_gV^*(X) = \left\langle {}_g\underline{V}(X), {}_g\overline{V}(X) \right\rangle$.

Definition: 4.4.3. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, p, \tau_K)$. Then the generalized lower (resp. generalized upper) standard deviation of X is given by

$${}_g\underline{\sigma}(X) = \sqrt{{}_g\underline{V}(X)} \quad (\text{resp. } {}_g\overline{\sigma}(X) = \sqrt{{}_g\overline{V}(X)}).$$

Definition: 4.4.4. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, p, \tau_K)$. Then the generalized rough standard deviation of X is denoted by ${}_g\sigma^*(X)$ and is given by ${}_g\sigma^*(X) = \left\langle {}_g\underline{\sigma}(X), {}_g\overline{\sigma}(X) \right\rangle$.

Example:4.4.1. Consider the same experiment as in Example 4.2.1. From Table 4.2.1 it is easy to see the following:

- Neither of the lower and upper probabilities summed to one.
- The generalized lower and generalized upper expectations of X are:

$${}_g\underline{\mu} = {}_g\underline{E}(X) = \frac{5}{4}, \quad {}_g\overline{\mu} = {}_g\overline{E}(X) = \frac{18}{4}.$$

- The generalized rough mean (or generalized rough expectation) of X equals:

$${}_g\mu^* = \left\langle \frac{5}{4}, \frac{18}{4} \right\rangle$$

- The generalized lower and generalized upper variances of X are:

$${}_g\underline{V}(X) = \frac{29}{32}, \quad {}_g\overline{V}(X) = \frac{127}{16}.$$

- The generalized rough variance of X equals:

$$ {}_gV^*(X) = \left\langle \frac{29}{32}, \frac{127}{16} \right\rangle . $$

- Finally, the rough standard deviation of X is given by

$$ {}_g\sigma^*(X) = \langle 0.952,\ 2.817 \rangle . $$

4.5 GENERALIZED ROUGH MOMENTS:

We shall define the generalized lower and the generalized upper moments of a random variable X.

Definition: 4.5.1. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, p, \tau_K)$. Then the generalized lower (resp. generalized upper) r^{th} moment of X about the generalized lower mean ${}_g\underline{\mu}$ (resp. the generalized upper mean ${}_g\overline{\mu}$), also called the generalized lower (resp. generalized upper) r^{th} central moment, is defined as

$$ {}_g\underline{\mu_r} = {}_g\underline{E}(X - {}_g\underline{\mu})^r \quad (\text{resp. } {}_g\overline{\mu_r} = {}_g\overline{E}(X - {}_g\overline{\mu})^r), \text{ where } r = 0,1,2,\ldots $$

We have

$$ {}_g\underline{\mu_r} = \sum_{k=1}^{n} \left(x_k - {}_g\underline{\mu}\right)^r {}_g\underline{p}(X = x_k) \text{ and } {}_g\overline{\mu_r} = \sum_{k=1}^{n} \left(x_k - {}_g\overline{\mu}\right)^r {}_g\overline{p}(X = x_k) . $$

The r^{th} generalized lower (resp. generalized upper) moment of X about origin is defined as

$$ {}_g\underline{\mu_r'} = {}_g\underline{E}(X^r) \quad (\text{resp. } {}_g\overline{\mu_r'} = {}_g\overline{E}(X^r)), \text{ where } r = 0,1,2,\ldots $$

Definition: 4.5.2. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, p, \tau_K)$. Then the generalized rough r^{th} moment of X is denoted by ${}_g\mu_r^*$ and is defined by ${}_g\mu_r^*(X) = \left\langle {}_g\underline{\mu_r},\ {}_g\overline{\mu_r} \right\rangle$.

We shall introduce the definition of the generalized moment generating function of a random variable X.

Definition: 4.5.3. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, p, \tau_K)$. Then the generalized lower (resp. generalized upper) moment generating function of X is defined by:

$$ {}_g\underline{M_X}(t) = {}_g\underline{E}\left(e^{tX}\right) = \sum_{k=1}^{n} e^{tx_k}\ {}_g\underline{p}(X = x_k) $$

$$ (\text{resp. } {}_g\overline{M_X}(t) = {}_g\overline{E}\left(e^{tX}\right) = \sum_{k=1}^{n} e^{tx_k}\ {}_g\overline{p}(X = x_k)). $$

Definition: 4.5.5. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, p, \tau_K)$. Then the generalized rough moment generating function of X is defined by

$$ {}_gM_X^*(t) = \left\langle {}_g\underline{M_X}(t),\ {}_g\overline{M_X}(t) \right\rangle . $$

Example: 4.5.1. Consider the same experiment as in Example 4.2.1. From Table 4.3.1 it is easy to see the following:

- The $g-$lower r^{th} moment of X about the $gs-$lower mean ${}_g\underline{\mu}$ is

$$ {}_g\underline{\mu_r} = {}_g\underline{E}(X - {}_g\underline{\mu})^r = \sum_{k=1}^{4} \left(x_k - {}_g\underline{\mu}\right)^r {}_g\underline{p}(X = x_k) $$

$$= \frac{1}{4}\left[(\frac{3}{4})^r + (\frac{7}{4})^r \right] = \left[\frac{3^r + 7^r}{4^{r+1}} \right], \text{ where } r = 0,1,2,...$$

- The $g-$upper r^{th} moment of X about the $gs-$upper mean ${}_g\overline{\mu}$ is

$${}_g\overline{\mu_r} = {}_g\overline{E}(X - {}_g\overline{\mu})^r = \sum_{k=1}^{n} \left(x_k - {}_g\overline{\mu} \right)^r {}_g\overline{p}(X = x_k)$$

$$= \frac{1}{4}\left[\frac{(-7)^r + 2(-5)^r + 3(-3)^r + (-1)^r}{(2)^r} \right], \text{ where } r = 0,1,2,...$$

- The r^{th} $g-$lower moment of X about origin is

$${}_g\underline{\mu_r'} = {}_g\underline{E}(X^r) = \frac{1}{4}(2^r + 3^r), \text{ where } r = 0,1,2,...$$

- The r^{th} $g-$upper moment of X about origin is

$${}_g\overline{\mu_r'} = {}_g\overline{E}(X^r) = \frac{1}{4}\left[1 + (2)^{r+1} + (3)^{r+1} + (4)^r \right], \text{ where } r = 0,1,2,...$$

- The $g-$lower moment generating function of X is

- $${}_g\underline{M_X}(t) = {}_g\underline{E}\left(e^{tX} \right) = \sum_{k=1}^{n} e^{t x_k} \, {}_g\underline{p}(X = x_k) = \frac{1}{4}(e^{2t} + e^{3t})$$

- The $g-$upper moment generating function of X is

$${}_g\overline{M_X}(t) = {}_g\overline{E}\left(e^{tX} \right) = \sum_{k=1}^{n} e^{t x_k} \, {}_g\overline{p}(X = x_k)$$

$$= \frac{1}{4}(e^t + 2e^{2t} + 3e^{3t} + e^{4t})$$

5. CONCLUSIONS:

In this paper, we used topological concepts to introduce definitions to generalized rough probability, generalized rough distribution function, generalized rough expectation, …etc. The topological applications which introduced help for measuring generalized rough probability, generalized rough expectation, …etc.

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