COMMUTANT OF COMPOSITE INTEGRAL OPERATORS

Anupama Gupta*

Govt. College for Women, Parade. Jammu, J&K, India

(Received on: 29-09-12; Revised & Accepted on: 10-11-12)

ABSTRACT

In this paper we study the composite integral operators on $L^p$-spaces. The conditions for composite integral operators to be bounded are investigated. The commutants of composite integral operators and Volterra composition operators are computed.

Keywords: Randon-Nikodym derivative, conditional expectation operator, Commutant, Contraction, Fixed point

AMS Subject Classification (2005): Primary 47B38, Secondary 47B399.

1. INTRODUCTION AND PRELIMINARIES

Let $(X, S, \mu)$ be a $\sigma$-finite measure space and let $\phi: X\rightarrow X$ be a non-singular measurable transformation ($\mu(E) = 0 \Rightarrow \mu\phi^{-1}(E) = 0$). Then a composition transformation $C_\phi: L^p(\mu) \rightarrow L^p(\mu)$ is defined by the equation

$$C_\phi f = f\circ \phi$$

for every $f \in L^p(\mu)$. In case $C_\phi$ is continuous, we call it a composition operator induced by $\phi$.

For each $f \in L^p(\mu)$, $1 \leq p < \infty$, there exists a unique $\phi^{-1}(S)$ measurable function $E(f)$ such that

$$\int gfd\mu = \int gE(f) d\mu$$

for every $\phi^{-1}(S)$ measurable function $g$ for which the left integral exits. The function $E(f)$ is called conditional expectation of $f$ with respect to the sub $\sigma$-algebra $\phi^{-1}(S)$. $E$ has the property that for $f \in L^p(\mu)$, $E(f) = g\phi$ for exactly one $S$-measurable function $g$. We shall write $g = E(f)\phi^{-1}$, which is well-defined measurable function. For more details about expectation operator, we refer to Parathasarthy [8].

Let $K: X \times X \rightarrow \mathbb{C}$ be a measurable function. Then a linear transformation $I: L^p(\mu) \rightarrow L^p(\mu)$ defined by

$$(I f)(x) = \int K(x, y) f(y) d\mu(y)$$

for all $f \in L^p(\mu)$ is known as integral operator. The composite integral operator $I_\phi$ is a bounded linear operator $I_\phi: L^p(\mu) \rightarrow L^p(\mu)$ defined by

$$(I_\phi f)(x) = \int K(x, y) f(\phi(y)) d\mu(y)$$

(1)

The equation (1) can also be written as

$$(I_\phi f)(x) = \int E(K_x)\phi^{-1}(y) f_\circ(y) d\mu(y),$$

where $K_x(y) = K(x, y)$ and $f_\circ = \frac{d\mu\phi^{-1}}{d\mu}$, the Randon-Nikodym derivative of the measure $\mu\phi^{-1}$ with respect to the measure $\mu$.

Corresponding author: Anupama Gupta*, Govt. College for Women, Parade. Jammu, J&K, India
The Volterra composition operator is a composition of Volterra integral operator \( V \) and a composition operator \( C_\phi \) defined as
\[
(V_\phi f)(x) = (V f) \circ \phi(x) = \int_0^x f(\phi(t))dt \quad \text{for every } f \in L^p[0,1],
\]
where \( \phi : [0,1] \to [0,1] \) is a measurable function.

An intensive study of composition operators is made over the past several decades. To worth mention, few of them are Singh ([10], [11]), Singh and Kumar [12], Singh and Komal [13], Singh and Manhas [14], Ridge [9] and Campbell [2].

The integral operators and composite integral operators in particular Volterra integral operator on \( L^p(\mu) \) have received considerable attention in recent years. The theory of integral operators is the source of all modern functional analysis and operator theory. Mathematicians like Halmos and Sunder [6], Setpanov ([15], [16]), Bloom and Kermen [1] have done great deal of work on integral operators. Gupta and Komal ([3], [4], [5]) also studied composite integral operators. Whitley [17] established the Lyubic’s conjecture [7] and generalized it to Volterra composition operators on \( L^p[0,1] \).

In this paper we have make an effort to explore a commutant of composite integral operators.

2. BOUNDED COMPOSITE INTEGRAL OPERATORS

In this section we study bounded composite integral operators.

**Theorem 2.1:** Suppose \( 1 \leq p, q < \infty \). Suppose \( I_\phi : L^p(\mu) \to L^q(\mu) \) is a linear transformation. Then \( I_\phi \) is continuous.

**Proof:** Let \( f_n \to f \) in \( L^p(\mu) \) and \( I_\phi f_n \to g \) in \( L^q(\mu) \). Then there exists a dominated subsequence \( \{ f'_n \} \) of \( \{ f_n \} \) such that
\[
f'_n(x) \to f(x) \quad \text{a.e.} \quad (i)
\]
Again since \( I_\phi f_n \to g \) in \( L^q(\mu) \), we can select a dominated subsequence \( \{ f''_n \} \) of \( \{ f'_n \} \) such that
\[
(I_\phi f''_n)(x) \to g(x) \quad \text{a.e.}
\]
or that
\[
\int K_\phi(x, y) f''_n(y) \, d\mu(y) \to g(x) \quad \text{a.e.}
\]
Also \( |f''_n| \leq h \) for some \( h \in L^p(\mu) \).

It follows from (i) that
\[
K_\phi(x, y) f''_n(y) \to K_\phi(x, y) f(y) \quad \text{a.e.} \quad (ii)
\]
and
\[
|K_\phi(x, y) f''_n(y)| \leq |K_\phi(x, y) h(y)| \quad \text{for almost every } y.
\]

But the dominated subsequence \( \{ K_\phi f''_n \} \) converges to \( \{ K_\phi f \} \) almost everywhere. By the Lebesgue dominated convergence theorem,
\[
\int K_\phi(x, y) f''_n(y) \, d\mu(y) \to \int K_\phi(x, y) f(y) \, d\mu(y) \quad (iii)
\]
From (ii) and (iii), we conclude that
\[
(I_\phi f)(x) = \int K_\phi(x, y) f(y) \, d\mu(y) = g(x)
\]
which proves that the graph of \( I_\phi \) is closed. Hence, by the closed graph theorem, \( I_\phi \) is continuous.

In the following theorem, we take \( r \) such that and \( \frac{1}{p} + \frac{1}{r} = \frac{1}{q} \) and
\[
S(x) = \|K_\phi(x, \cdot)\|_{L^q}
\]
Theorem 2.2: For $1 \leq p, q < \infty$, let $S \in L^{r/q}(\mu)$. Then $I_{\phi}: L^p(\mu) \to L^q(\mu)$ is a bounded composite integral operator.

Proof: For $f \in L^p(\mu)$, consider

$$|| I_{\phi}f ||^q = \int_X | I_{\phi}f |^q dx$$

$$\leq \int_X \{ (\int_X | K_\phi(x, y)|^{r/q} dy)^{r/q} (\int |f(y)|^{p/q} dy)^{p/q} \} dx$$

$$= || S(x) ||_q^{q/r} \cdot || f ||_p^p$$

This proves that $I_{\phi}$ is bounded composite integral operator.

In the next result we make an attempt to use composite integral operators to solve the integral equations.

Theorem 2.3: If $K_\phi \in L^2(\mu \times \mu)$ and $g \in L^2[0, 1]$, then the integral equation

$$f(x) = g(x) + \lambda \int_0^1 K_\phi(x, y) f(y) d\mu(y) \quad (1)$$

has unique solution for sufficiently small values of scalar $\lambda$.

Proof: Define $I_{\phi}: L^2[0, 1] \to L^2[0, 1]$ as $I_{\phi}f = h$

where $h(x) = g(x) + \lambda \int_0^1 K_\phi(x, y) f(y) d\mu(y)$.

We first show that

$$\psi(x) = \int K_\phi(x, y) f(y) d\mu(y) \quad \text{for every } f \in L^2[0, 1].$$

Consider

$$\int_0^1 K_\phi(x, y) f(y) d\mu(y) \leq \left( \int_0^1 | K_\phi(x, y) |^2 d\mu(y) \right)^{1/2} \left( \int_0^1 |f(y)|^2 d\mu(y) \right)^{1/2} \quad \text{(by using Holder’s inequality)}$$

Therefore,

$$\int_0^1 |\psi(x)|^2 dx \leq \int_0^1 \left( | K_\phi(x, y) |^2 d\mu(y) \right) dx \cdot \int_0^1 \left( |f(y)|^2 d\mu(y) \right) dy$$

$$< \infty.$$ 

Now

$$|| I_{\phi}f - I_{\phi}f_1 || = || \lambda \{ \int_0^1 K_\phi(x, y) [f(y) - f_1(y)] d\mu(y) \} ||$$

$$\leq |\lambda| \left( \int_0^1 \int_0^1 | K_\phi(x, y) |^2 d\mu(x) d\mu(y) \right)^{1/2} \left( \int_0^1 |f(y) - f_1(y)|^2 d\mu(y) \right)^{1/2}$$

$$\leq M || f - f_1 ||,$$

where $M = |\lambda| \left( \int_0^1 \int_0^1 | K_\phi(x, y) |^2 d\mu(x) d\mu(y) \right)^{1/2}$.

This proves that $I_{\phi}$ is a contraction and hence it has a unique fixed point, say $f^*$. Thus $f^*$ is a unique solution of eq. (1).

3. COMMUTANT OF COMPOSITE INTEGRAL OPERATOR

In this section we have made an attempt to compute the commutant of composite integral operator.

Theorem 3.1: Let $I_{\phi} \in B(L^p(\mu))$. Then $M_0$ commutes with $I_{\phi}$ if and only if $\theta = 0 \circ \phi \quad \text{a.e.}$
Proof: For \( f \in L^p (\mu) \),

\[
(I_\varphi M_\theta f)(x) = \int K(x, y) (M_\theta f)(y) \, d\mu(y)
\]

and

\[
(M_\theta I_\varphi f)(x) = \theta(x) (I_\varphi f)(x)
\]

In view of (i) and (ii)

\[
(M_\theta I_\varphi f)(x) - (I_\varphi M_\theta f)(x) = \int f(y) E(K_x \circ \varphi^{-1})(y) \left[ \theta(y) - \theta(x) \right] \, d\mu(y).
\]

Hence, the result.

In the next theorem we characterize multiplication operators which commute with Volterra composite operators.

**Theorem 3.2:** Let \( M_\theta \in B(L^2(\mu)) \). Suppose \( \varphi \) is an injective map. Then \( M_\theta \) commutes with \( V_{\varphi} \) if and only if \( \theta = \theta \circ \varphi \) a.e.

**Proof:** For \( f \in L^2(\mu) \), we have

\[
(M_\theta V_{\varphi} f)(x) = (\theta \circ V_{\varphi} f)(x)
\]

Also we have

\[
(V_{\varphi} M_\theta f)(x) = \int \chi_{[0,x]}(t) \, d\mu(t)
\]

as \( \varphi \) is injective, \( C_{\varphi} \) has dense range

\[
\chi_{[0,\varphi]}(t) \left[ \theta(x) - \theta \circ \varphi(t) \right] = 0.
\]

Hence the result follows using the given condition.

**Corollary:** There is a composition operator \( C_{\varphi} \in L^2(\mu) \) such that \( V C_{\varphi} = C_{\varphi} V \)

**Proof:** For \( f \in L^2(\mu) \), we have

\[
(V C_{\varphi} f)(x) = \int C_{\varphi} f(t) \, dt = \int f(t) \, dt
\]

Hence, the result follows.
REFERENCES


Source of support: Nil, Conflict of interest: None Declared