On Left F-derivations of d-algebras

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ABSTRACT

Motivated by some results on derivations in rings and derivations of BCI algebras recently we introduce the notion of derivations on d-algebras and f-derivations on d-algebras. In this paper we introduce the notion of left F-derivations of d-algebras and investigate some simple and interesting results.

Keywords: d-algebra, edge d-algebras, derivations, f-derivations, endomorphism, left F-derivations.

Subject Classification: 03G25, 06F35.

1. INTRODUCTION

Y. Imai and K. Iseki introduced two classes of abstract algebras BCK-algebras and BCI-algebras ([1] [2] [3]). It is known that the class of BCK-algebras is a proper sub class of the class of BCI-algebras. In ([4] [5]) Q.P.Hu and X. Li introduced a wide class of abstract algebras. BCH-algebras and have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [6] introduced the notion of d-algebra, which is another generalization of BCK-algebras.

In 2004 Y. B. Jun and X. L. Xin [7] introduced the notion of derivations of BCI-algebras which was motivated from a lot of work done on derivations of rings and near rings. Motivated by the work of Lie and X in recently, we have [8] introduced the notion of derivations and f-derivations [9] on a d-algebras. In this paper we introduce the notion of left F-derivations on d-algebras and study some simple but elegant results.

2. PRELIMINARIES

Definition 2.1 [6] A d-algebra is a non empty set X with a constant 0 and a binary operation * satisfying the following axioms:
1. x * x = 0
2. 0 * x = 0
3. x * y = 0 and y * x = 0 implies x = y.

Definition 2.2 [6] Let S be a non-empty subset of a d-algebra X, then S is called sub algebra of X if x * y ∈ S for all x, y ∈ S.

Definition 2.3 Let X be a d-algebra and I be a subset of X, then I is called d- ideal of X if it satisfies the following conditions:
1. 0 ∈ I
2. x * y ∈ I and y ∈ I implies x ∈ I.
3. x ∈ I and y ∈ X implies x * y ∈ I.

Definition 2.4 [6] Let (X, *, 0) be a d-algebra and x ∈ X.

Define x * X = {x * a | a ∈ X}. X is said to be an edge d-algebra if for any x ∈ X, x * X = {x, 0}.

Properties: In any d-algebra X the following properties hold for all x, y, z ∈ X.

1. (x * y) * z = (x * z) * y.
2. 0 * (x * y) = (0 * x) * (0 * y).
3. (x *(x * y)) * y = 0

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4. \(x \ast (x \ast y) = y.\)
5. \(x \ast (y \ast z) \geq (x \ast y) \ast z.\)
6. \((x \ast y) \ast (x \ast z) \leq (z \ast y).\)
7. \((x \ast z) \ast (y \ast z) \ast (x \ast y) = 0.\)
8. \(x \ast 0 = 0 \Rightarrow x = 0.\)
9. \(x \ast a = x \ast b \Rightarrow a = b.\)
10. \(a \ast x = b \ast x \Rightarrow a = b.\)
11. If \(x \leq y \Rightarrow x \ast z \leq y \ast z\) and \(z \ast y \leq z \ast x.\)

**Definition 2.6** For any \(x_0 \in X,\) the set \(A(x_0) = \{x \in X \mid x_0 \leq x\}\) is known as the branch of \(X\) determined by \(x_0.\) Each branch \(A(x_0) \neq \emptyset\) at \(x_0 \ast x_0 = 0\) \(\Rightarrow x_0 \in A(x_0).\)

Clearly we observe that \(A(x_0)\) contains all those elements of \(X\) that succeed \(x_0.\)

**Properties.** Let \(X\) be a \(d\)-algebra. The following properties hold.

1. If \(x \leq y,\) then \(x\) and \(y\) are contained in the same branch of \(X.\)
2. If \(x \in A(x_0),\) \(y \in A(y_0)\) then \(x \ast y \in A(x_0 \ast y_0).\)
3. Let \(x_0, y_0 \in X\) and \(y \in A(y_0)\) then \(x_0 \ast y = x_0 \ast y_0.\)
4. Let \(x_0, y_0 \in X\) and \(x \in A(x_0)\) then \(x \ast y_0 = x_0 \ast y_0.\)
5. If \(A(x_0) \subseteq X,\) then \(x, y \in A(x_0) \Rightarrow x \ast y, y \ast x \in X.\)

**Definition 2.6** Let \(X\) be a \(d\)-algebra. A map \(\theta : X \rightarrow X\) is a left-right derivations (briefly \((l,r)\)-derivation) of \(X\) if it satisfies the identity \(\theta(x \ast y) = (\theta(x) \ast y \land x \ast \theta(y))\) for all \(x, y \in X.\)

If \(\theta\) satisfies the identity \(\theta(x \ast y) = (x \ast \theta(y) \land \theta(x) \ast y)\) for all \(x, y \in X,\) then \(\theta\) is a right-left derivation (briefly \((r,l)\)-derivation) of \(X.\) Moreover if \(\theta\) is both a \((l,r)\)-derivation and \((r,l)\)-derivation, then \(\theta\) is a derivation of \(X.\)

**Definition 2.7** A mapping \(f\) of a \(d\)-algebra \(X\) into itself is called an endomorphism if \(f(x \ast y) = f(x) \ast f(y).\) Note that \(f(0) = 0.\)

3. LEFT F-DERIVATIONS

In this section we introduce the notion of left \(F\)-derivation of a \(d\)-algebra and give some examples to explain the theory of left \(F\)-derivation in \(d\)-algebras and prove some simple but elegant properties.

**Definition 3.1** Let \(X\) be a \(d\)-algebra. By a left \(F\)-derivation of \(X,\) we mean a self map \(\theta_F\) of \(X\) satisfying the identity \(\theta_F(x \ast y) = (\theta_F(x) \ast F(y) \land x \ast \theta_F(y))\) for all \(x, y \in X.\)

**Definition 3.2** A left \(F\)-derivation \(\theta_F\) of a \(d\)-algebra \(X\) is said to be regular if \(\theta_F(0) = 0.\) Otherwise it is called an irregular left \(F\)-derivation.

**Example 3.3** Let \(X = \{0, 1, 2, 3\}\) be a \(d\)-algebra with the following Cayley table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a self map \(\theta_F : X \rightarrow X\) as follows \(\theta_F(x) = 3\) if \(x = 0, 1, 2, 3.\)

Define an endomorphism \(F : X \rightarrow X\) as follows \(F(x) = 0\) if \(x = 0, 1, 2\) and \(F(x) = 2\) if \(x = 3.\)

Then it is easily checked that \(\theta_F\) is a left \(F\)-derivation of \(X.\)

**Theorem 3.4** Let \(\theta_F\) be a regular left \(F\)-derivation of a \(d\)-algebra \(X.\) If \(F(x)\) and \(\theta_F(x)\) belong to the same branch of \(X,\) then \(\theta_F(x) = F(x).\)
Proof: Since \( F(x) \) and \( \theta_F(x) \) belong to the same branch of \( X \),

\[
F(x) \leq \theta_F(x) \tag{1}
\]

Since \( \theta_F \) is a regular left derivation of \( X \), \( \theta_F(0) = 0 \).

Now \( \theta_F(0) = \theta_F(x^*x) \)

\[
= (\theta_F(x) * F(x)) \wedge (\theta_F(x) * F(x)) \tag{using left F-derivation}
\]

\[
= (\theta_F(x) * F(x)) \wedge ((\theta_F(x) * F(x)) \wedge (\theta_F(x) * F(x))) \tag{using x \wedge y = y \wedge (y \wedge x)}
\]

\[
\theta_F(x) * F(x) \tag{using x \wedge (x \wedge y) = y}
\]

Since \( \theta_F(0) = 0 \), \( \theta_F(x) * F(x) = 0 \).

Hence \( \theta_F(x) \leq F(x) \tag{2} \)

From (1) and (2), it follows \( \theta_F(x) = F(x) \).

Theorem 3.5 Let \( \theta_F \) be a self map and \( A(x_0) \) be any branch of a \( d \)-algebra \( X \). If for any \( x \in A(x_0) \), \( \theta_F(x) = F(x_0) \), then \( \theta_F \) is a left \( F \)-derivation.

Proof: Let \( \theta_F \) be a self map and \( A(x_0) \) be any branch of a \( d \)-algebra \( X \) determined by \( x_0 \).

According to given condition for any \( x \in A(x_0) \), \( \theta_F(x) = F(x_0) \tag{1} \)

Now for \( x, y \in X \) following two cases arise.

Case 1: Both \( x \) and \( y \) belongs to the same branch of \( X \).

Case 2: \( x \) and \( y \) belongs to different branches of \( X \).

Case 1: Let \( x, y \in A(x_0) \), So \( x_0 \leq x \) and \( x_0 \leq y \).

Then \( x \wedge y \in A(x_0 \wedge x_0) = A(0) \). (By result)

So using (1), \( \theta_F(x \wedge y) = F(0) = 0 \tag{2} \)

Also \( x_0 \leq y \Rightarrow x_0 \wedge y = 0 \) and \( x_0 \leq x \Rightarrow x_0 \wedge x = 0 \tag{3} \)

Further, as \( F \) is an endomorphism,

\[
0 = F(0) = F(x_0 \wedge y) = F(x_0) \wedge F(y) \quad \text{and} \quad 0 = F(0) = F(x_0 \wedge x) = F(x_0) \wedge F(x)
\]

Now

\[
(\theta_F(x) * F(y) \wedge (\theta_F(y) * F(x))) = (F(x_0) \wedge F(y)) \wedge (F(x_0) \wedge F(x)) \tag{using (1)}.
\]

\[
= 0 \wedge 0 \wedge (0 \wedge 0) = 0 \wedge 0 = 0.
\]

That is, \( (\theta_F(x) * F(y)) \wedge (\theta_F(y) * F(x)) = 0 \wedge \theta_F(x \wedge y) \) (using (2))

which implies \( \theta_F \) is a left \( F \)-derivation.

Case 2: Let \( x \in A(x_0) \) and \( y \in A(y_0) \).

Then \( x \wedge y \in A(x_0 \wedge y_0) \) (By result)

So using (1) \( \theta_F(x \wedge y) = F(x_0 \wedge y_0) \tag{4} \)

Now

\[
(\theta_F(x) * F(y)) \wedge (\theta_F(y) * F(x)) = (F(x_0) \wedge F(y)) \wedge (F(y_0) \wedge F(x))
\]

\[
= (F(y_0) \wedge F(x)) \wedge ((F(y_0) \wedge F(x)) \wedge (F(x_0) \wedge F(y))
\]

\[
= F(x_0) \wedge F(y)
\]

\[
= F(x_0 \wedge y) \quad \text{(Since \( F \) is an endomorphism)}
\]
\[ F(x_0 * y_0) = (\theta_1(x) * y) \]

which implies \( \theta_1 \) is a left F-derivation.

This completes the proof.

**Lemma 3.6** Let \( \theta_1 \) be a self map of a d-algebra \( X \). If \( \theta_1 \) is a left F-derivation of \( X \). Then the following hold.

1. \( \theta_1(x) * F(x) = \theta_1(y) * F(y) \).
2. \( \theta_1(x) = \theta_1(x) \land \theta_1(0) \).

**Proof:**

1. Let \( x, y \in X \). Then

\[
\theta_1(0) = \theta_1(x^*x) = (\theta_1(x) * F(x)) \land (\theta_1(0) * F(x)) = \theta_1(x) * F(x)
\]

Similarly \( \theta_1(0) = \theta_1(y^*y) = \theta_1(y) * F(y) \)

From (1) and (2), it follows that \( \theta_1(x) * F(x) = \theta_1(y) * F(y) \).

2. Let \( x \in X \). Then

\[
\theta_1(x) = \theta_1(x^*0) = (\theta_1(x) * F(0)) \land (\theta_1(0) * F(x)) = \theta_1(x) \land (\theta_1(0) * F(x)) = \theta_1(x) \land (\theta_1(0) * F(x))
\]

Thus \( \theta_1(x) \leq \theta_1(0) * (\theta_1(0) * \theta_1(x)) \leq \theta_1(0) \).

Therefore \( \theta_1(x) = \theta_1(0) * (\theta_1(0) * \theta_1(x)) \), which implies that \( \theta_1(x) = \theta_1(x) \land \theta_1(0) \).

**Theorem 3.7** A self map \( \theta_1 \) of a d-algebra \( X \), defined as \( \theta_1(x) = F(x) \) for all \( x \in X \) is a left F-derivation of \( X \), where \( F \) is an endomorphism of \( X \).

**Proof:** Let \( \theta_1 \) be a self map of a d-algebra \( X \), where \( F \) is an endomorphism of \( X \) defined as follows

\[
\theta_1(x) = F(x) \quad \text{for all} \quad x \in X
\]

As for \( x, y \in X \), \( x \leq y \).

Therefore \( \theta_1(x^*y) = F(x^*y) = F(x) * F(y) \)

Now \( (\theta_1(x) * F(y)) \land (\theta_1(y) * F(x)) = (F(x) * F(y)) \land (F(y) * F(x)) \) (using (1))

\[
= (F(y) * F(x)) \land (F(y) * F(x)) \land (F(x) * F(y))
\]

Thus \( (\theta_1(x) * F(y)) \land (\theta_1(y) * F(x)) = \theta_1(x * y) \).

which implies that \( \theta_1 \) is a left F-derivation.

**Theorem 3.8.** Let \( \theta_1 \) be a left F-derivation of a d-algebra \( X \) where \( F \) is an endomorphism of \( X \). Then

1. \( x \leq y \) implies \( \theta_1(x) \) and \( \theta_1(y) \) belongs to the same branch of \( X \).
2. \( y \leq x \) implies \( \theta_1(y) \) and \( \theta_1(x) \) belongs to the same branch of \( X \).
Proof:
1. Let $\theta_F$ be a left $F$-derivation of a $d$-algebra $X$, where $F$ is an endomorphism of $X$.

Since $X$ is a $d$-algebra, $x \leq y$ implies $x \ast y = 0$.

when $\theta_F$ is a left $F$-derivation

\[
\theta_F(x) = \theta_F(y \ast (y \ast x)) = (\theta_F(y) \ast F(y) \ast F(x)) \ast (\theta_F(x) \ast F(y)) = \theta_F(y \ast x) \ast F(y) \ast (\theta_F(y) \ast x) = \theta_F(y) \ast F(y \ast x).
\]

which implies $\theta_F(x) = \theta_F(y) \ast (F(y) \ast F(x))$ (Since $F$ is an endomorphism) \hspace{1cm} (1)

Since $x \leq y$ implies $x \ast y = 0$.

Therefore $0 = F(0) = F(x \ast y) = F(x) \ast F(y)$.

That is $F(x) \ast F(y) = 0$ implies $F(x) \leq F(y)$.

As $F(x) \ast F(y) = 0$, So $F(y) \ast F(x) \neq 0$,

Otherwise because of property of $d$-algebra $F(x) \ast F(y) = 0 = F(y) \ast F(x)$.

$\Rightarrow F(x) = F(y)$, a contradiction.

(1) $\Rightarrow \theta_F(x) = \theta_F(y) \ast (F(y) \ast F(x))$.

\[\theta_F(x) \ast \theta_F(y) = \theta_F(y) \ast (F(y) \ast (\theta_F(x) \ast F(y))) = \theta_F(x) \ast \theta_F(y) \ast F(y) = 0 \ast (\theta_F(x) \ast F(y)) = 0\]

Which implies $\theta_F(x) \leq \theta_F(y)$.

By property, it follows that $\theta_F(x)$ and $\theta_F(y)$ belong to the same branch of $X$.

2. Interchanging the role of $x$ and $y$ in (1), we have $y \leq x$ implies $\theta_F(y) \leq \theta_F(x)$.

This implies $\theta_F(y)$ and $\theta_F(x)$ belong to the same branch of $X$.

Definition 3.9 Let $X$ be a $d$-algebra and $\theta_F$, $\theta'_F$ be two self maps of $X$. We define $\theta_F \circ \theta'_F : X \rightarrow X$ as

$\theta_F \circ \theta'_F(x) = \theta_F(\theta'_F(x))$ for all $x \in X$.

Notation: $\text{Der}(X)$ denotes the set of all $F$-derivations (both right $F$-derivation and left $F$-derivation) on $X$.

Definition 3.10. Let $\theta_F$, $\theta'_F \in \text{Der}(X)$. Define the binary operation $\land$ as

$(\theta_F \land \theta'_F)(x) = \theta_F(x) \land \theta'_F(x)$.

Lemma 3.11. Let $X$ be $d$-algebra. $\theta_F$ and $\theta'_F$ are left $F$-derivation of $X$. Then $\theta_F \land \theta'_F$ is also a left $F$-derivation of $X$.

Proof: Let $X$ be a $d$-algebra, $\theta_F$ and $\theta'_F$ are left $F$-derivation of $X$. Then

$(\theta_F \land \theta'_F)(x \ast y) = \theta_F(x \ast y) \land \theta'_F(x \ast y)$

$= [\{ \theta_F(x) \ast F(y) \ast (\theta'_F(x) \ast F(y)) \} \land \{ \theta'_F(x) \ast F(y) \ast (\theta_F(x) \ast F(y)) \}]

= [\theta_F(x) \ast F(y) \land (\theta'_F(x) \ast F(y)) \land (\theta_F(x) \ast F(y))]

= \theta_F(x) \ast F(y)

= (\theta'_F(x) \ast ((\theta_F(x) \ast \theta'_F(x)) \ast F(y))

= (\theta_F(x) \land \theta'_F(x)) \ast F(y)

= (\theta_F(x) \land \theta'_F(x)) \ast F(y)

= (\theta_F(x) \land \theta'_F(x)) \ast (\theta_F(x) \ast F(y))

= ((\theta_F \land \theta'_F)(x) \ast F(y)) \ast ((\theta_F \land \theta'_F)(x) \ast F(y))

= ((\theta_F \land \theta'_F)(x) \ast F(y)) \land ((\theta_F \land \theta'_F)(x) \ast F(y))$
This shows that \((\theta_F \wedge \theta'_F)\) is a left F-derivation of X.

This completes the proof.

**Theorem 3.12** Let X be a d-algebra and \(\theta_F, \theta'_F, \theta''_F\) are left F-derivations of X. Then

\[
(\theta_F \wedge \theta'_F) \wedge \theta''_F = \theta_F \wedge (\theta'_F \wedge \theta''_F).
\]

**Proof:** Let \(\theta_F, \theta'_F\) and \(\theta''_F\) \(\in\) \(\text{Der}(X)\).

Then by definition

\[
((\theta_F \wedge \theta'_F) \wedge \theta''_F)(x \ast y) = (\theta_F \wedge \theta'_F)(x \ast y) \wedge \theta''_F(x \ast y)
\]

Also consider the following

\[
(\theta_F \wedge (\theta'_F \wedge \theta''_F))(x \ast y) = \theta_F(x \ast y) \wedge (\theta'_F \wedge \theta''_F)(x \ast y)
\]

From (1) and (2) it follows that

\[
((\theta_F \wedge \theta'_F) \wedge \theta''_F)(x \ast y) = (\theta_F \wedge (\theta'_F \wedge \theta''_F))(x \ast y)
\]

Put \(y = 0\), we have \(((\theta_F \wedge \theta'_F) \wedge \theta''_F)(x) = (\theta_F \wedge (\theta'_F \wedge \theta''_F))(x)\).

which implies that \((\theta_F \wedge \theta'_F) \wedge \theta''_F = \theta_F \wedge (\theta'_F \wedge \theta''_F)\).

Thus \(\text{Der}(X)\) form a semi group.

**4. REFERENCES**


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