International Journal of Mathematical Archive-2(3), Mar. - 2011, Page: 393-398
IJMA Available online through www.ijma.info ISSN 2229 – 5046

A GENERAL THEOREM ON LAPLACE TRANSFORM AND ITS APPLICATIONS

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(Received on: 26-02-11; Accepted on: 05-03-11)

ABSTRACT

In this paper we obtain a general theorem on Laplace transform in terms of Fox-Wright hypergeometric function. Laplace transforms of some composite functions are also obtained as special cases.

2010 Mathematics Subject Classification: *Primary 44A10; Secondary 44A20.*

Keywords and Phrases: *Generalized hypergeometric functions; Fox-Wright hypergeometric functions; Fox's H-function; Pochhammer's symbol; Gauss-Legendre multiplication theorem, Laplace transform.*

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES:

In the usual notation, the Pochhammer's symbol or shifted factorial or generalized factorial function $(b)_k$ is defined by

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k \in \{1,2,3,\ldots\} \\ 1 & ; \text{ if } k=0 \\ k! & ; \text{ if } b=1, k \in \{1,2,3,\ldots\} \end{cases}$$

where b is neither zero nor a negative integer and the notation Γ stands for Gamma function.

$$(\lambda)_{mn} = m^{mn}\left(\frac{\lambda}{m}\right)_n \left(\frac{\lambda+1}{m}\right)_n \left(\frac{\lambda+2}{m}\right)_n \cdots \left(\frac{\lambda+m-1}{m}\right)_n$$

where m is a positive integer, n is a non negative integer and λ may be real or complex number.

The convenient notation $\Delta(N;b)$ is used to denote the array of N number of parameters given by $\frac{b}{N}, \frac{b+1}{N}, \frac{b+2}{N}, \cdots, \frac{b+N-1}{N}$, where b is a real or complex number and N is a positive integer.
The generalized Gaussian hypergeometric function ${}_AF_B$ of one variable is defined by

$${}_AF_B\left[\begin{matrix} a_1, a_2, \ldots, a_A ; \\ b_1, b_2, \ldots, b_B ; \end{matrix} \; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k \, z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k \, k!} \tag{1.1}$$

where denominator parameters $b_1, b_2, \ldots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

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If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z (real or complex).
If $A = B+1$, then series ${}_AF_B$ is convergent for $|z|<1$.

In 1933, Wright (or Fox-Wright) defined a more interesting generalized hypergeometric function of one variable[12,pp.50-51(1.5.21), p.179(34iii),p.395(23);4,p.11(1.7.8)] in the following forms:

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),(\alpha_2,A_2),\ldots,(\alpha_p,A_p); \\ (\beta_1,B_1),(\beta_2,B_2),\ldots,(\beta_q,B_q)\ ;\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+nA_1)\Gamma(\alpha_2+nA_2)\cdots\Gamma(\alpha_p+nA_p)\,z^n}{\Gamma(\beta_1+nB_1)\Gamma(\beta_2+nB_2)\cdots\Gamma(\beta_q+nB_q)\,n!} \tag{1.2}$$

$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_p)}{\Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_q)}\ {}_p\Psi_q^*\left[\begin{matrix}(\alpha_1,A_1),(\alpha_2,A_2),\ldots,(\alpha_p,A_p); \\ (\beta_1,B_1),(\beta_2,B_2),\ldots,(\beta_q,B_q)\ ;\end{matrix}\ z\right]$$

$$= H^{1,\,p}_{p,q+1}\left[-z\left|\begin{matrix}(1-\alpha_1,A_1),(1-\alpha_2,A_2),\ldots,(1-\alpha_p,A_p) \\ (0,1),(1-\beta_1,B_1),(1-\beta_2,B_2),\ldots,(1-\beta_q,B_q)\end{matrix}\right.\right] \tag{1.3}$$

$$ {}_p\Psi_q^*\left[\begin{matrix}(\alpha_1,A_1),(\alpha_2,A_2),\ldots,(\alpha_p,A_p); \\ (\beta_1,B_1),(\beta_2,B_2),\ldots,(\beta_q,B_q)\ ;\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_{nA_1}(\alpha_2)_{nA_2}\cdots(\alpha_p)_{nA_p}\,z^n}{(\beta_1)_{nB_1}(\beta_2)_{nB_2}\cdots(\beta_q)_{nB_q}\,n!} \tag{1.4}$$

where the coefficients $A_1,A_2,\ldots,A_p,B_1,B_2,\ldots,B_q$ are positive real numbers and $\alpha_1,\alpha_2,\ldots,\alpha_p,\beta_1,\beta_2,\ldots,\beta_q$ are complex parameters.

Convergence Conditions:

(i) The Fox H -function makes sense when
$\delta \equiv (1+B_1+B_2+\cdots+B_q)-(A_1+A_2+\cdots+A_p)>0$ and $0<|z|<\infty;\ z\neq 0$

(ii) The equality holds true only for suitably constrained values of $|z|$ or appropriately bounded values of $|z|$ i.e. $\delta = 0$ and $0<|z|<R\equiv A_1^{-A_1}\,A_2^{-A_2}\cdots A_p^{-A_p}B_1^{B_1}\,B_2^{B_2}\cdots B_q^{B_q}$.

Suppose given function $F(t)$ is well defined for all values of $t>0$, then Laplace transform of $F(t)$ is given by

$$\mathrm{L}\{F(t)\} = \int_0^{\infty} e^{-st}\,F(t)\,\mathrm{d}t = f(s) \tag{1.5}$$

$$\mathrm{L}^{-1}\{f(s)\} = \frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} e^{st}\,f(s)\,\mathrm{d}s = F(t),\ \ h>0 \tag{1.6}$$

provided that above integrals exist and s is a real or complex number.

$$\int_0^{\infty} e^{-sx}\,x^{\alpha-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)}{s^{\alpha}} \tag{1.7}$$

$$\big((\mathrm{Re}(s)>0,\mathrm{Re}(\alpha)>0)\ \text{or}\ (\mathrm{Re}(s)=0,\ 0<\mathrm{Re}(\alpha)<1)\big)$$

$$\int_0^{\infty} e^{-\frac{\alpha}{x}}\,x^{s-1}\,\mathrm{d}x = \alpha^{s}\,\Gamma(-s) \tag{1.8}$$

$$\big(\mathrm{Re}(s)<0,\mathrm{Re}(\alpha)>0\big)$$

$$\int_0^{\infty} (e^{-x}-1)\,x^{s-1}\,\mathrm{d}x = \Gamma(s) \tag{1.9}$$

$$\big(-1<\mathrm{Re}(s)<0\big)$$

Hankel's Contour Integral:

$$\int_{h-i\infty}^{h+i\infty} e^{t}\, t^{-A}\, \mathrm{d}t = \frac{2\pi i}{\Gamma(A)} \qquad (1.10)$$

$$\left(\mathrm{Re}(A) > 0,\ h > 0\right)$$

Fractional Derivative:

The fractional derivative of t^{P-1} with respect to t, λ times is given by the following formula

$$\frac{\mathrm{d}^{\lambda}}{\mathrm{d}t^{\lambda}}(t^{P-1}) = \frac{\Gamma(P)}{\Gamma(P-\lambda)} t^{P-\lambda-1} \qquad (1.11)$$

where λ is arbitrary complex number. Any values of λ and P leading to the result which do not make sense, are tacitly excluded.

2. GENERAL THEOREM:

Since Pochhammer's symbol is associated with Gamma function and Gamma function is undefined for zero and negative integers, therefore arguments, numerator and denominator parameters are adjusted in such a way that each term of following results is completely well defined and meaningful then without any loss of convergence, we have

$$\mathrm{L}\left\{ t^{\nu} \frac{\mathrm{d}^{\lambda}}{\mathrm{d}t^{\lambda}} \left(t^{\mu}\ {}_{A}F_{B}\left[\begin{matrix} (a_A)\ ; \\ (b_B)\ ; \end{matrix}\ xt^{k} \right] \right) \right\} = \frac{\Gamma(\mu+1)\Gamma(\nu+\mu-\lambda+1)}{s^{\nu+\mu-\lambda+1}\,\Gamma(\mu-\lambda+1)} \times$$

$$\times\ {}_{2+A}\Psi^{*}_{1+B}\left[\begin{matrix} (a_1,1),(a_2,1),\ldots,(a_A,1),(\mu+1,k),(\nu+\mu-\lambda+1,k)\ ; \\ (b_1,1),(b_2,1),\ldots,(b_B,1),(\mu-\lambda+1,k)\ ; \end{matrix}\ \frac{x}{s^{k}} \right] \qquad (2.1)$$

$$\left(\mathrm{Re}(\nu+\mu-\lambda) > -1;\ k \text{ is a positive real number}\right)$$

Proof:

$$\mathrm{L}\left\{ t^{\nu} \frac{\mathrm{d}^{\lambda}}{\mathrm{d}t^{\lambda}} \left(t^{\mu}\ {}_{A}F_{B}\left[\begin{matrix} (a_A)\ ; \\ (b_B)\ ; \end{matrix}\ xt^{k} \right] \right) \right\} = \mathrm{L}\left\{ t^{\nu} \sum_{r=0}^{\infty} \frac{[(a_A)]_r\, x^{r}}{[(b_B)]_r\, r!} \frac{\mathrm{d}^{\lambda}}{\mathrm{d}t^{\lambda}} t^{\mu+kr} \right\}$$

$$= \mathrm{L}\left\{ t^{\nu} \sum_{r=0}^{\infty} \frac{[(a_A)]_r\, x^{r}\, \Gamma(\mu+kr+1)\, t^{\mu+kr-\lambda}}{[(b_B)]_r\, r!\,\Gamma(\mu+kr-\lambda+1)} \right\}$$

$$= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)} \sum_{r=0}^{\infty} \frac{[(a_A)]_r\, x^{r}\, (\mu+1)_{kr}}{[(b_B)]_r\, r!\, (\mu-\lambda+1)_{kr}} \mathrm{L}\left\{ t^{\nu+\mu+kr-\lambda} \right\}$$

$$= \frac{\Gamma(\mu+1)\Gamma(\nu+\mu-\lambda+1)}{\Gamma(\mu-\lambda+1)\, s^{\nu+\mu-\lambda+1}} \sum_{r=0}^{\infty} \frac{[(a_A)]_r\, (\mu+1)_{kr}\, (\nu+\mu-\lambda+1)_{kr}\, x^{r}}{[(b_B)]_r\, (\mu-\lambda+1)_{kr}\, s^{kr}\, r!}$$

Now writing above series in hypergeometric notation (1.4) of Fox-Wright, we get the right hand side of (2.1).

3. SOME DEDUCTIONS OF (2.1):

If k is a positive integer, then (2.1) reduces to

$$\mathrm{L}\left\{ t^{\nu} \frac{\mathrm{d}^{\lambda}}{\mathrm{d}t^{\lambda}} \left(t^{\mu}\ {}_{A}F_{B}\left[\begin{matrix} (a_A)\ ; \\ (b_B)\ ; \end{matrix}\ xt^{k} \right] \right) \right\}$$

$$= \frac{\Gamma(\mu+1)\Gamma(\nu+\mu-\lambda+1)}{s^{\nu+\mu-\lambda+1}\,\Gamma(\mu-\lambda+1)}\ {}_{A+2k}F_{B+k}\left[\begin{matrix} (a_A),\Delta(k;\mu+1),\Delta(k;\nu+\mu-\lambda+1)\ ; \\ (b_B),\Delta(k;\mu-\lambda+1)\ ; \end{matrix}\ \frac{xk^{k}}{s^{k}} \right] \qquad (3.1)$$

$$\left(\mathrm{Re}(\nu+\mu-\lambda)>-1;\ A+k\le B+1\right)$$

In (3.1), setting $\lambda=\mu=0$ and then adjusting parameters and variables suitably, we get on simplification

$$\mathrm{L}\left\{t^{\nu-1}\ {}_AF_B\left[\begin{matrix}(a_A)\ ; \\ (b_B)\ ;\end{matrix}\ (xt)^k\right]\right\}=\frac{\Gamma(\nu)}{s^{\nu}}\ {}_{A+k}F_B\left[\begin{matrix}(a_A),\Delta(k;\nu)\ ; \\ (b_B)\qquad\qquad ;\end{matrix}\ \frac{(xk)^k}{s^k}\right] \tag{3.2}$$

$$\left(\mathrm{Re}(\nu)>0;\mathrm{Re}(s)>0;\ A+k\le B+1\right)$$

$$\mathrm{L}\left\{t^{k\sigma-1}\ {}_AF_{B+k}\left[\begin{matrix}(a_A)\qquad\qquad ; \\ (b_B),\Delta(k;k\sigma)\ ;\end{matrix}\ \frac{\lambda^k t^k}{k^k}\right]\right\}=\frac{\Gamma(k\sigma)}{s^{k\sigma}}\ {}_{A+1}F_B\left[\begin{matrix}1,\ (a_A)\ ; \\ (b_B)\quad ;\end{matrix}\left(\frac{\lambda}{s}\right)^k\right] \tag{3.3}$$

$$\left(\mathrm{Re}(\sigma)>0;\mathrm{Re}(s)>0;\ A\le B\right)$$

4. APPLICATIONS:

Making suitable adjustments of parameters and variables in (3.2) and (3.3), we obtain following results

$$\mathrm{L}\{t^n\sin(bt)\}=\frac{b\,\Gamma(n+2)}{s^{n+2}}\ {}_2F_1\left[\begin{matrix}\frac{n+2}{2},\frac{n+3}{2}\ ; \\ \frac{3}{2}\qquad\qquad ;\end{matrix}\ \frac{-b^2}{s^2}\right];\ n>-2,\ \mathrm{Re}(s)>|\mathrm{Im}(b)| \tag{4.1}$$

$$\mathrm{L}\{t^n\cos(bt)\}=\frac{\Gamma(n+1)}{s^{n+1}}\ {}_2F_1\left[\begin{matrix}\frac{n+1}{2},\frac{n+2}{2}\ ; \\ \frac{1}{2}\qquad\qquad ;\end{matrix}\ \frac{-b^2}{s^2}\right];\ n>-1,\ \mathrm{Re}(s)>|\mathrm{Im}(b)| \tag{4.2}$$

$$\mathrm{L}\left\{\sin(\sqrt{t})\right\}=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\exp\left(-\frac{1}{4s}\right);\ \mathrm{Re}(s)>0 \tag{4.3}$$

$$\mathrm{L}\left\{\frac{\sin(\sqrt{t})}{\sqrt{t}}\right\}=\frac{1}{s}\ {}_1F_1\left[\begin{matrix}1\ ; \\ \frac{3}{2};\end{matrix}\ -\frac{1}{4s}\right] \tag{4.4}$$

$$\mathrm{L}\left\{\frac{\cos(\sqrt{t})}{\sqrt{t}}\right\}=\sqrt{\frac{\pi}{s}}\exp\left(-\frac{1}{4s}\right) \tag{4.5}$$

$$\mathrm{L}\left\{\sqrt{t}\cos(\sqrt{t})\right\}=\frac{1}{2s\sqrt{\pi s}}\ {}_1F_1\left[\begin{matrix}\frac{3}{2}\ ; \\ \frac{1}{2};\end{matrix}\ -\frac{1}{4s}\right] \tag{4.6}$$

$$\mathrm{L}\left\{J_1(t)\right\}=1-\frac{s}{\sqrt{s^2+1}} \tag{4.7}$$

$$\mathrm{L}\left\{\sqrt{t}\,J_1(\sqrt{t})\right\}=\frac{1}{2s^2}\exp\left(-\frac{1}{4s}\right) \tag{4.8}$$

$$\mathrm{L}\left\{\sqrt{t}\,J_1(t)\right\}=\frac{3}{8s^2}\sqrt{\frac{\pi}{s}}\ {}_2F_1\left[\begin{matrix}\frac{5}{4},\frac{7}{4}\ ; \\ 1\qquad ;\end{matrix}\ -\frac{25}{16s^2}\right] \tag{4.9}$$

$$\mathrm{L}\left\{J_0(\sqrt{t})\right\}=\frac{1}{s}\exp\left(-\frac{1}{4s}\right) \tag{4.10}$$

$$\mathrm{L}\left\{\frac{1}{t^{\frac{3}{2}}}\exp\left(-\frac{1}{4t}\right)\right\} = 2\sqrt{\pi}\exp(-\sqrt{s}) \tag{4.11}$$

$$\mathrm{L}\left\{\frac{1}{\sqrt{t}}\exp\left(-\frac{1}{4t}\right)\right\} = \sqrt{\frac{\pi}{s}}\exp(-\sqrt{s}) \tag{4.12}$$

$$\mathrm{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}} \tag{4.13}$$

$$\mathrm{L}\{I_0(t)\} = \frac{1}{\sqrt{s^2-1}} \tag{4.14}$$

$$\mathrm{L}\left\{\frac{\exp(-2t)}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s+2}} \tag{4.15}$$

$$\mathrm{L}\{\mathrm{erf}(\sqrt{t})\} = \frac{1}{s\sqrt{s+1}} \tag{4.16}$$

$$\mathrm{L}\left\{\mathrm{erfc}\left(\frac{1}{\sqrt{t}}\right)\right\} = \frac{\exp(-2\sqrt{s})}{s} \tag{4.17}$$

$$\mathrm{L}\left\{t^2\ {}_0F_2\left[\begin{matrix} ---\ ; \\ \frac{4}{3},\frac{5}{3}\ ; \end{matrix}\ -\frac{t^3}{27}\right]\right\} = \frac{2}{1+s^3} \tag{4.18}$$

$$\mathrm{L}\left\{{}_0F_2\left[\begin{matrix} --\ ; \\ 1,1\ ; \end{matrix}\ -t\right]\right\} = \frac{1}{s}\ {}_0F_1\left[\begin{matrix} --\ ; \\ 1\ ; \end{matrix}\ -\frac{1}{s}\right] = \frac{1}{s}J_0\left(\frac{2}{\sqrt{s}}\right) \tag{4.19}$$

$$\mathrm{L}\left\{t^{m+n-1}\ {}_1\Psi_2\left[\begin{matrix} (1,1)\ ; \\ (2,2),(m+n,2m)\ ; \end{matrix}\ -b^2 t^{2m}\right]\right\} = \frac{1}{b\,s^n}\sin\left(\frac{b}{s^m}\right) \tag{4.20}$$

$$\mathrm{L}\left\{t^{n-1}\ {}_1\Psi_2\left[\begin{matrix} (1,1)\ ; \\ (1,2),(n,2m)\ ; \end{matrix}\ -b^2 t^{2m}\right]\right\} = \frac{1}{s^n}\cos\left(\frac{b}{s^m}\right) \tag{4.21}$$

$$\mathrm{L}\left\{t^{m+n-1}\ {}_2\Psi_2\left[\begin{matrix} (1,1),(1,2)\ ; \\ (2,2),(m+n,2m)\ ; \end{matrix}\ -b^2 t^{2m}\right]\right\} = \frac{1}{b\,s^n}\tan^{-1}\left(\frac{b}{s^m}\right) \tag{4.22}$$

$$\mathrm{L}\left\{t^{m+n-1}\ {}_2\Psi_2\left[\begin{matrix} (1,1),(1,1)\ ; \\ (2,1),(m+n,m)\ ; \end{matrix}\ -b\,t^{m}\right]\right\} = \frac{1}{b\,s^n}\ln\left(1+\frac{b}{s^m}\right) \tag{4.23}$$

$$\mathrm{L}\left\{\frac{\sin(bt)}{t}\right\} = \tan^{-1}\left(\frac{b}{s}\right),\ \mathrm{Re}(s) > |\mathrm{Im}(b)| \tag{4.24}$$

$$\mathrm{L}\left\{\frac{\sin(\sqrt{t})}{t}\right\} = \pi\ \mathrm{erf}\left(\frac{1}{2\sqrt{s}}\right) \tag{4.25}$$

$$\mathrm{L}\{\sinh(\sqrt{t})\} = \frac{\sqrt{\pi}}{2\,s^{\frac{3}{2}}}\exp\left(\frac{1}{4s}\right);\ \mathrm{Re}(s) > 0 \tag{4.26}$$

$$\mathrm{L}\{I_0(\sqrt{t})\} = \frac{1}{s}\exp\left(\frac{1}{4s}\right) \tag{4.27}$$

where m is a positive real number and standard notations have their usual meanings.

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