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ON MAXIMAL INDEPENDENT SETS IN A GRAPH

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ABSTRACT

In this paper, the concepts of maximal independent set, maximal edge independent set of a graph are discussed. Further the famous graphs P_n, C_n and K_n are studied with respect to these concepts.

Key words: *Non – empty, simple, finite graph; maximal independent set; maximal edge independent set.*

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1. INTRODUCTION AND PRELIMINARIES

In literature, the concepts of independent set, (order) maximum independent set of a graph have significant use. Now, we introduce the concepts of maximal independent set, edge independent set and observe that these concepts are more general than the already known one. Some results pertaining to these concepts are presented.

The significant graphs P_n, C_n and K_n are considered with respect to these concepts. For terminology and notation, we refer Bondy and Murthy [1].

Definition 1.1[1]. A subset S of the vertex set V of a graph G is said to be an independent set of G if and only if (iff) no two elements of S are adjacent in G.

Remark 1.2:
(i) Since any single vertex set is itself an independent set of G, interest lies only in finding out a maximum independent set of G.
(ii) There is no significance when G is empty (no edges) or has loops or has multiple edges.

So, unless or otherwise stated, by a graph G we mean a simple, non – empty, finite graph.

Definition 1.3[1]. An independent set S of a graph G, with vertex set V is said to be a maximum independent set of G iff there is no independent set S^1 of G with $|S^1| > |S|$ (| | denotes the number of elements).

Now, we introduce the analogous concept of a maximal independent set of a graph G with respect to the set – inclusion relation as follows:

Definition 1.4 [3]. An independent set S of a graph G is said to be a maximal independent set of G iff there is no independent set S^1 of G with $S \subset S^1$ ($\subset$ denotes properly contained).

Observation 1.5. A maximum independent set is clearly a maximal independent set; but the converse is false.

Counter example 1.6. Consider the graph given under:

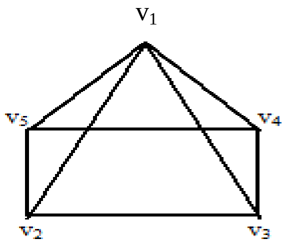

Figure: 1

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$\{v_1\}$ is a maximal independent set of G, since any other vertex of G is adjacent with v_1. But it is not an order maximum independent set, since $\{v_2,v_4\}$, $\{v_2,v_5\}$ are independent sets with two elements.

Thus the maximal independent set is more general concept than the usual maximum independent set in a graph. Hence the study of this concept is worth considerable.

Definition 1.7 [1]. The number of elements in an order maximum independent set of a (finite) graph G is called the independence number of G and is denoted by $\alpha(G)$.

Definition 1.8[1]: A covering of a graph G is a subset K of the vertex set V of G such that every edge of G has atleast one end in K.

A Covering K of a graph G is a minimum covering of G iff G has no covering K^1 with $|K^1| < |K|$.

A Covering K of a graph G is a minimal covering of G iff G has no covering K^1 with $K^1 \subset K$ (see [2]).

The number of elements in a minimum covering of G is called the covering number of G and is denoted by $\beta(G)$.

Result 1.9 [1]. In any graph G with n vertices, $\alpha(G) + \beta(G) = n$.

Theorem 1.10. If G is a graph with n vertices then there is a maximal independent set S_0 and a minimal covering K_0 of G wit$| S_0 | + | K_0 | = n$.

Proof. By definition, every order maximum independent set S of G has $\alpha(G)$ elements and every order minimum covering K of G has $\beta(G)$ elements follows that $| S | + | K | = n$.

Since order concepts $\Rightarrow$ set – inclusion concepts, it follows that such S_0 and K_0 exist with $| S_0 | + | K_0 | = n$.

Observation 1.11. For the path P_n $(n \geq 2)$

$$\alpha(P_n) = \begin{cases} n/2 & \text{if n is even,} \\ (n+1)/2 & \text{if n is odd.} \end{cases}$$

Proof. Let the vertex set of P_n $(n \geq 2)$ be $V = \{v_1, v_2\ldots v_n\}$. Clearly $\alpha(P_2) = 1 = 2/2$ and $\alpha(P_3) = 2 = (3+1)/2$.

Case (i). Let n be even; say $n = 2m$ $(m \geq 2)$.

Now $\{v_1,v_3, \ldots, v_{2m-1}\}$ or $\{v_2,v_4, \ldots, v_{2m}\}$ is an independent set of of P_{2m} with m elements. We observe that no subset of V with > m elements is an independent set of $P_{2m,}$ since there will be atleast two vertices in that set which are adjacent (in P_{2m}). So $\alpha(P_{2m}) = m = n/2$.

Case (ii). Let n be odd; say $n = 2m+1$ $(m \geq 2)$

Now $\{v_1, v_3, \ldots, v_{2m+1}\}$ is a vertex covering of P_{2m+1} with m+1 elements. Clearly a subset of V with > m+1 elements contains atleast two vertices that are adjacent (in P_{2m+1}) So $\alpha(P_{2m+1}) = \{(2m+1)+1\}/2 = (n+1)/2$.

Observation 1.12. For the cycle C_n $(n \geq 3)$,

$$\alpha(C_n) = \begin{cases} n/2 \text{ if n is even,} \\ (n-1)/2 \text{ if n is odd.} \end{cases}$$

Proof. Let the vertex set of C_n be $V = \{v_1, v_2,\ldots v_n\}$.

Case (i). Let n be even; say $n = 2m$ $(m \geq 2)$.

$\{v_1,v_3,\ldots v_{2m-1}\}$ or $\{v_2,v_4,\ldots v_{2m}\}$ is an independent set of C_{2m} with m elements. Clearly any subset of V with > m elements is not an independent set of C_{2m}. Hence $\alpha(C_{2m}) = m = n/2$.

Case (ii). Let n be odd. Clearly $\alpha(C_3) = 1 = (3 – 1)/2$.

Let $n \geq 5$. So $n = 2m+1$ $(m \geq 2)$.

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Now $\{v_2, v_4, \ldots v_{2m}\}$ is an independent set of C_{2m+1} with m elements. Since v_{2m+1} is adjacent with v_{2m} and v_1, it follows that any subset of V with > m elements is not an independent set of C_{2m+1}.

Hence $\alpha(C_{2m+1}) = m = \{(2m+1)-1\}/2 = (n-1)/2$.

Observation 1.13. For the complete graph K_n $(n \geq 2)$,

$$\alpha(K_n) = 1.$$

Proof. Since every vertex in K_n is adjacent with all the other vertices, it follows that any single vertex set is an independent set of K_n.

Hence $\alpha(K_n) = 1$.

2. MAIN RESULTS.

Theorem 2.1. In the path P_n $(n \geq 2)$ any maximal independent set is an order maximum independent set and vice versa iff n is even.

Proof. Let the vertex set of P_n be $V = \{v_1, v_2. . . v_n\}$. In P_2, clearly $\{v_1\}$ and $\{v_2\}$ are the only maximal as well as order maximum independent sets.

Case (i). Let n be even and >2; say n = 2m $(m \geq 2)$.

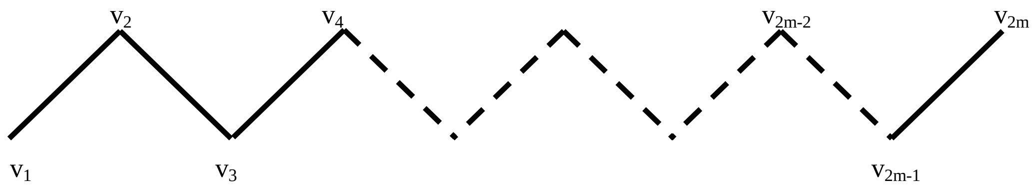

Figure: 2 (Graph P_{2m})

Now $\{v_1, v_3\ldots v_{2m-1}\}$, $\{v_2, v_4, \ldots, v_{2m}\}$ are two independent sets, each containing m elements. Considering the set with the first r elements from the first set and with the last (m – r) elements from the second set (r = 1… (m – 1)) We get an independent set.

Further the inclusion of any other vertex in any of these 2 + (m – 1) = m + 1 sets results in having atleast two adjacent vertices in that set. Thus these are the only maximal independent sets. Since each set has m elements, by observation (1.11), it follows that each of them is an order maximum independent set. Thus every maximal independent set is an order maximum independent set and vice versa when n is even.

Case (ii). Let n be odd; say n = 2m+1 $(m \geq 1)$

In P_3, clearly $\{v_1,v_3\}$ is an order maximum independent set ($\Rightarrow$ maximal independent set), where as $\{v_2\}$ is a maximal independent set but not an order maximum independent set. (Since $\alpha(P_3) = 2$).

Now assume that $n \geq 5$ (i.e. $m \geq 2$).

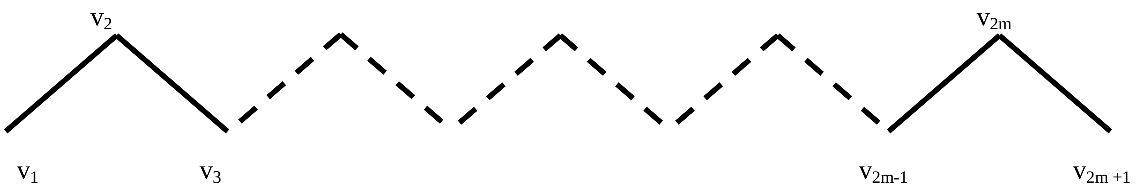

Figure: 3 (Graph P_{2m+1})

Since $\alpha(P_{2m+1}) = m + 1$, it follows that $\{v_1, v_3, \ldots, v_{2m+1}\}$ is the only order maximum independent set.

Clearly $\{v_2, v_4\ldots v_{2m}\}$ is an independent set of P_{2m+1}. Since the inclusion of any other vertex, say v_{2i+1} (i = 0, 1… (m-1)) results in having two adjacent vertices follows that the above one is a maximal independent set of P_{2m+1}. As it has m (<m+1) elements, by observation (1.11) follows that this is not an order maximum independent set of P_{2m+1}.

This completes the proof of the Theorem.

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Theorem 2.2. In the cycle $C_n(n \geq 3)$ any maximal independent set is an order maximum independent set and vice versa.

Proof. Let the vertex set of C_n be $V = \{v_1, v_2, v_3,\ldots,v_n\}$.

Case (i). Let n be even; say $n = 2m$ $(m \geq 2)$.

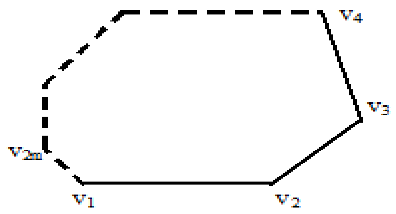

Figure: 4

C_{2m} has 2m vertices and these vertices can be partitioned into two pairwise disjoint subsets, each having m elements (vertices); namely $\{v_1, v_3,\ldots v_{2m-1}\}$and $\{v_2, v_4,\ldots v_{2m}\}$. Both saturate all the edges of C_{2m}. So, both of them are maximal independent sets. Further there is no maximal independent set that does not saturate any of the edges of C_{2m}, since there are even number of edges. Thus there are only two maximal independent sets which are both order maximum independent sets, since each has m elements and $\alpha\,(C_{2m}) = m$.

Case (ii). Let n be odd; say $n = 2m+1$ $(m \geq 1)$.

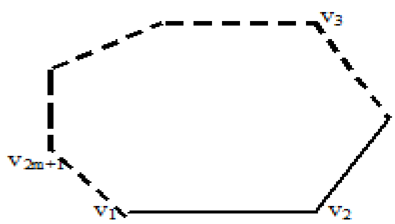

Figure: 5

In C_3, clearly $\{v_1\}$, $\{v_2\}$, $\{v_3\}$ are the only maximal as well as maximum independent sets.

So, we take $m \geq 2$. Now consider the pairwise disjoint subsets $\{v_1, v_3\ldots v_{2m-1}\}$ and $\{v_2, v_4\ldots v_{2m}\}$.

The first one saturates all the edges except $v_{2m}v_{2m+1}$. The vertex v_{2m} is adjacent with v_{2m-1} and v_{2m+1} is adjacent with v_1. The second set does not saturate the edge $v_{2m+1}v_1$ and v_{2m+1} is adjacent with v_{2m} and v_1 is adjacent with v_2. Hence follows that both of them are maximal independent sets and also maximum independent sets since $\alpha\,(C_{2m+1}) = m$.

Taking the first r elements from the first set and last m – r elements from the second set (r = 1, 2… (m – 1)) We get a maximal independent set that does not saturate a single edge whose ends are adjacent with two vertices of the set.

Further $\{v_3, v_5,\ldots v_{2m+1}\}$ is an independent set that does not saturate the edge v_1v_2. So it is a maximal independent set as well as an order maximum independent set.

Removing the first r elements from the first set and including the first r elements from the second set ($r \in (1, 2, \ldots, (m-1)$) we get a maximal independent set that does not saturate exactly one edge. So these are also maximal independent sets which are also order maximum independent sets. Further, we will not get any order maximum independent sets except these.

This completes the proof of the theorem.

Theorem 2.3. In the complete graph K_n $(n \geq 2)$ any maximal independent set is an order maximum independent set and vice versa.

Proof. Since any two vertices in K_n are adjacent, it follows that any single vertex set is a maximal independent set as well as order maximum independent set.

This completes the proof of the theorem.

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Open problem 2.4. Are there graphs other than $P_{2m}(m \geq 1)$, $C_n(n \geq 3)$ and $K_n(n \geq 2)$ for which any maximal independent set is same as order maximum independent set ?

3. EDGE ANALOGUE OF INDEPENDENT SETS.

Definition 3.1 [1]. A subset M of the edge set E of a graph G is said to be an edge covering of G iff no two elements of M are adjacent in G ($\Leftrightarrow$ M is a matching in G).

The results in matching are discussed in [3].

Definition 3.2[1]. The number of edges in a (an order) maximum matching of a graph G is denoted by α^1 (G) and is called the edge independence number of G.

Definition 3.3[1]. An edge covering L is a subset L of the edge set E such that each vertex of G is an end of some edge in L.

An edge covering L of a graph G is an order minimum edge covering iff there is no edge covering L^1 of G with $|L^1| < |L|$.

An edge covering L of a graph G is a (set-inclusion minimum) minimal edge covering of G iff there is no edge covering L^1 of G with $L^1 \subset L$. (see [2])

The number of edges in a (an order) minimum edge covering of a graph G is denoted by β^1 (G) and is called the edge covering number of G.

Result 3.4[1]. If G is a non – empty graph with n vertices then $\alpha^1(G) + \beta^1(G) = n$.

Theorem 3.5. If G is a non-empty graph with n vertices then there is a maximal matching M_0 and a minimal edge covering N_0 of G with $|M_0| + |N_0| = n$.

Proof. Since order concepts $\Rightarrow$ set – inclusion concepts, by Result (3.4), this theorem follows.

Result 3.6 [1]. In a non – empty, bipartite graph G the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering of G.

Theorem 3.7. Let G be a non – empty bipartite graph then there is a maximal independent set S_0 of G and a minimal edge covering L_0 of G such that $|S_0| = |L_0|$.

Proof. Obvious; follows from the above result.

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