g**-Quasi – FH-closed spaces and g**-Quasi – CH-closed spaces

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(Received on: 12-11-12; Revised & Accepted on: 21-12-12)

ABSTRACT

In this paper we have introduced τ - g** boundary, g**-Quasi-FH-closed, g**-FH-closed, g**-Quasi-CH-closed, g**-CH-closed, g**-meager and few results on these new definitions have been established.

Key Words: τ - g** boundary, g**-Quasi-FH-closed, g**-FH-closed, g**-Quasi-CH-closed, g**-CH-closed, g**-meager.

1. INTRODUCTION

Levine [2] and M.K.R.S. Veerakumar [7] introduced the class of g-closed sets and g* -closed sets in the year 1970 and 1991 respectively. T.R. Hamlett and D. Jankovic [1] have introduced the class of Quasi-H-closed and H-closed sets in the year 1990. R.L. Newcomb [5] have introduced Quasi-H-closed modulo I in the year 1967. We have introduced g**-closed sets [4], separation axioms via g**-closed sets [5] and strongly g**-regular and strongly g**-normal spaces [6] and investigated many of their properties and in this paper we have introduced g**-Quasi-FH-closed and g**-Quasi-CH-closed and discussed their characteristics.

2. PRELIMINARIES

Definition 2.1: [9] A subset A of a topological space (X, τ) is called
1) generalized closed (briefly g-closed) [2] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).
2) generalized star closed (briefly g*-closed) [12] if cl(A) ⊆ U whenever A ⊆ U and U is g-open in (X, τ).
3) generalized star star closed (briefly g**-closed) [6] if cl(A) ⊆ U whenever A ⊆ U and U is g* -open in (X, τ).

Definition 2.2: [9] A topological space (X, τ) is said to be g**-Lindelof if every g**-open cover has a countable sub cover.

Definition 2.3: [7] The topological space (X, τ) is said to be g**-additive if arbitrary union of g**-closed sets is g**-closed. Equivalently arbitrary intersection of g**-open sets is g**-open.

Definition 2.4: [7] A topological space (X, τ) is said to be g** multiplicative if arbitrary intersection of g**-closed sets is g**-closed . Equivalently arbitrary union of g**-open sets is g**-open.

Definition 2.5: [9] The topological space (X, τ) is said to be g**- finitely additive if finite union of g**-closed sets is g**-closed.

Definition 2.6: [3] An ideal I on a non empty set X is a collection of subsets of X which satisfies the following properties:(i) A ∈ I, B ∈ I ⇒ A ∪ B ∈ I (ii) A ∈ I, B ⊆ A ⇒ B ∈ I. A topological space (X, τ) with an ideal I on X is called an ideal topological space and is denoted by (X, τ, I).

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International Journal of Mathematical Archive- 3 (12), Dec. – 2012

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Definition 2.7: [8] An ideal topological space \((X, \tau, I)\) is said to be \(g^{**}\)-compact modulo \(I\) if for every \(g^{**}\)-open covering \(\{U_a\}_{a \in \Delta}\) of \(X\), there exists a finite subset \(\Delta_0\) of \(\Delta\) such that \(X - \bigcup_{a \in \Delta_0} U_a \in I\).

Definition 2.8: [10] An ideal topological space \((X, \tau)\) is said to be \(g^{**}\)-Lindelöf modulo \(I\) if for every \(g^{**}\)-open cover \(\{U_a\}_{a \in \Omega}\) of \(X\), there exists a countable subset \(\Omega_0\) such that \(X - \bigcup_{a \in \Omega_0} U_a \in I\).

Definition 2.9: [7] A topological space \((X, \tau)\) is said to be a \(g^{**}\)-T2 space if for every pair of distinct points \(x, y\) in \(X\) there exists disjoint \(g^{**}\)-open sets \(U\) and \(V\) in \(X\) such that \(x \in U\) and \(y \in V\).

Definition 2.10: [8] A topological space \((X, \tau)\) is said to be \(g^{**}\)-compact if every \(g^{**}\)-open covering of \(X\) contains a finite sub collection that also covers \(X\). A subset \(A\) of \(X\) is said to be \(g^{**}\)-compact if every \(g^{**}\)-open covering of \(A\) contains a finite sub collection that also covers \(A\).

Definition 2.11: [7] Let \(A\) be a subset of \(X\). A point \(x \in X\) is said to be a \(g^{**}\)-limit point of \(A\) if every \(g^{**}\)-neighbourhood of \(x\) contains a point of \(A\) other than \(x\).

Definition 2.12: [7] Let \(A\) be a subset of a topological space \((X, \tau)\). \(g^{**}cl(A)\) is defined to be the intersection of all \(g^{**}\)-closed sets containing \(A\).

If \((X, \tau)\) is \(g^{**}\)-multiplicative then \(g^{**}cl(A)\) is \(g^{**}\)-closed.

Definition 2.13: [9] A subset \(A\) of a space \((X, \tau)\) is said to be \(g^{**}\)-dense in \(X\), if \(g^{**}cl(A) = X\).

Definition 2.14: [7] Let \((X, \tau)\) be a topological space and \(A\) be a subset of \(X\). A point \(x \in A\) is said to be \(g^{**}\)-interior point of \(A\) if there exists a \(g^{**}\)-open set \(U\) such that \(x \in U \subseteq A\).

Definition 2.15: [7] Let \(A\) be a subset of a topological space \((X, \tau)\). \(g^{**}int(A)\) is defined to be the union of all \(g^{**}\)-open sets contained in \(A\).

If \((X, \tau)\) is \(g^{**}\)-multiplicative then \(g^{**}int(A)\) is \(g^{**}\)-open.

Definition 2.16: [11] A topological space \((X, \tau)\) is said to be \(g^{**}\)-space if it is \(g^{**}\)-finitely additive and \(g^{**}\)-multiplicative.

Definition 2.17: [1] A topological space \((X, \tau)\) is said to be Quasi-H-closed or QHC if every open cover of a space contains a finite sub collection whose closures cover the space \(X\).

Definition 2.18: [5] Let \((X, \tau)\) be a topological space and \(I\) be an ideal on \(X\). Then \(X\) is said to be Quasi-H-closed modulo \(I\) if for every open cover \(U = \{U_\alpha / \alpha \in \Omega\}\) of \(X\) there exists a finite sub family \(\{U_\alpha / i = 1, 2, \ldots, n\}\) of \(U\) such that \(X - \bigcup_{i=1}^{n} cl(U_\alpha) \in I\).

Definition 2.19: [1] A T2 space which is QHC is said to be H-closed.

Definition 2.20: [5] A space \((X, \tau, I)\) is called \(\tau\)-boundary if \(\tau \cap I = \{\emptyset\}\).

Example 2.21: [5] Let \(X\) be an infinite set, \(\tau = \{A \subseteq X / A^c \text{ is finite}\}\). \(I_F = \{\text{ finite subsets of } X\}\). Then \(\tau \cap I_F = \emptyset\).

3. \(g^{**}\)-Quasi – FH-closed spaces and \(g^{**}\)-Quasi – CH-closed spaces

Definition 3.1 Given an ideal topological space \((X, \tau, I)\), \(I\) is called \(g^{**}\)-codense in \(X\) if \(G^{**}O(X) \cap I = \{\emptyset\}\)
Example 3.2: Let $X$ be an infinite set with cofinite topology and $I_F = \{\text{finite subsets of } X\}$. Then $g^{**}O(X) \cap I_F = \{\emptyset\}$

Definition 3.3: A space $(X, \tau)$ is said to be $g^{**}$-Quasi-FH-closed (briefly $g^{**}$-QFHC) if every $g^{**}$-open cover of $X$ contains a finite sub collection whose $g^{**}$-closures cover $X$.

Definition 3.4: A $g^{**}$-T$_2$ space which is $g^{**}$-QFHC is said to be $g^{**}$-FH-closed.

Definition 3.5: A space $(X, \tau)$ is said to be $g^{**}$-Quasi-CH-closed (briefly $g^{**}$-QCHC) if every $g^{**}$-open cover of $X$ contains a countable sub collection whose $g^{**}$-closures cover $X$.

Definition 3.6: A $g^{**}$-T$_2$ space which is $g^{**}$-QCHC is said to be $g^{**}$-CH-closed.

Theorem 3.7: Every $g^{**}$-compact space is $g^{**}$-QFHC.

Proof: Let $\{U_\alpha / \alpha \in \Omega\}$ be a $g^{**}$-open cover for $X$. Then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X = \bigcup_{i=1}^{n} U_{\alpha_i}$ and this implies $X = \bigcup_{i=1}^{n} U_{\alpha_i}$. Therefore $X$ is $g^{**}$-QFHC.

Note: Any finite space is $g^{**}$-compact and hence $g^{**}$-QFHC.

Theorem 3.8: Every $g^{**}$-Lindelof space is $g^{**}$-QCHC.

Proof: similar to the above.

Example 3.9: An infinite cofinite topological space $(X, \tau)$ is $g^{**}$-compact and hence $g^{**}$-QFHC.

Example 3.10: In infinite indiscrete topological space $(X, \tau)$, all subsets are $g^{**}$-open and $g^{**}$-closed. $\{\{x\}/x \in X\}$ is a $g^{**}$-open cover which has no finite sub cover such that $g^{**}$-closures can cover $X$ and hence $(X, \tau)$ is $g^{**}$-T$_2$ but not $g^{**}$-QFHC and hence not $g^{**}$-FH.

Example 3.11: A finite indiscrete topological space $(X, \tau)$ is $g^{**}$-T$_2$ and $g^{**}$-QFHC and hence it is $g^{**}$-FH-closed.

Example 3.12: An infinite, cofinite topological space is $g^{**}$-QFHC but not $g^{**}$-T$_2$ and hence it is not $g^{**}$-FH-closed.

Remark 3.13: Any $g^{**}$-QFHC space is $g^{**}$-QCHC space and $g^{**}$-FH-closed space is $g^{**}$-CH-closed. Converse need not be true as seen in the following example.

Example 3.14: A countably infinite indiscrete topological space is $g^{**}$-QCHC and $g^{**}$-CH-closed but not $g^{**}$-QFHC and $g^{**}$-FH-closed.

Definition 3.15: A subset $A$ of $(X, \tau)$ is said to be nowhere $g^{**}$-dense if $g^{**}\text{int}[g^{**}\text{cl}(A)] = \emptyset$.

Theorem 3.16: If $A$ is $g^{**}$-closed then $A$ is nowhere $g^{**}$-dense if and only if $A^c$ is $g^{**}$-dense.

Proof: Let $x \in X$ and $g^{**}\text{cl}(A) = A$. $g^{**}\text{int}[g^{**}\text{cl}(A)] = g^{**}\text{int}[A] = \emptyset$. Therefore every $g^{**}$-open set containing $x$ should intersect $A^c$. $A^c$ is $g^{**}$-dense in $X$. Let $A^c$ be $g^{**}$-dense in $X$. $g^{**}\text{cl}(A^c) = X$. For $x \in X$, every $g^{**}$-open set containing $x$ should intersect $A^c$. $g^{**}\text{int}(A) = \emptyset$. Therefore $A$ is nowhere $g^{**}$-dense.

Theorem 3.17: Let $(X, \tau)$ be a $g^{**}$-space. Then $g^{**}(I_n) = \{\text{nowhere } g^{**}\text{-dense subsets of } X\}$ is an ideal in $X$.

Proof: Let $A \in g^{**}(I_n)$ and $B \subseteq A$ then $g^{**}\text{int}[g^{**}\text{cl}(B)] \subseteq g^{**}\text{int}[g^{**}\text{cl}(A)] = \emptyset$. $\therefore B \in g^{**}(I_n)$. Let $A, B \in g^{**}(I_n)$. 

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Under the given hypothesis, $g^{**}cl(A \cup B) = g^{**}cl(A) \cup g^{**}cl(B)$. \\
$\therefore g^{**}\text{int}[g^{**}cl(A \cup B) = g^{**}\text{int}[g^{**}cl(A) \cup g^{**}cl(B)] = g^{**}\text{int}[g^{**}cl(A)] \cup [g^{**}\text{int}[g^{**}cl(B)] = \varnothing.$ \\
$\therefore A \cup B \in g^{**}(I_n).$ Therefore $g^{**}(I_n)$ is an ideal.

**Theorem 3.18:** If an ideal topological space $(X, \tau, I)$ is $g^{**}$-multiplicative, $g^{**}$-compact modulo $I$ and $G^{**}O(X) \cap I = \{\varnothing\}$ then $(X, \tau)$ is $g^{**}$-QFHC.

**Proof:** Let $\{U_\alpha / \alpha \in \Omega\}$ be a $g^{**}$-open cover for $X$. Therefore there exists $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ such that $X - \bigcap_{i=1}^n U_{\alpha_i} \in I$.

Case (1): If $X - \bigcap_{i=1}^n U_{\alpha_i} = \varnothing$, then $X = \bigcup_{i=1}^n U_{\alpha_i}$. Therefore $(X, \tau)$ is $g^{**}$-QFHC.

Case (2): If $X - \bigcap_{i=1}^n U_{\alpha_i} \neq \varnothing$, then $X - \bigcup_{\alpha \in I} U_{\alpha_i} \in I$. But $\bigcup_{\alpha \in I} U_{\alpha_i} \in G^{**}O(X)$ (since $X$ is $g^{**}$-multiplicative). $\therefore \bigcup_{\alpha \in I} U_{\alpha_i} \in G^{**}O(X) \cap I = \{\varnothing\}$.

**Theorem 3.19:** Let $(X, \tau)$ be a $g^{**}$-multiplicative space. If $(X, \tau, I)$ is $g^{**}$-Lindeloff modulo $I$ and $G^{**}O(X) \cap I = \{\varnothing\}$ then $(X, \tau)$ is $g^{**}$-QCHC.

**Proof:** Similar to the above proof.

**Theorem 3.20:** Let $(X, \tau)$ be a $g^{**}$-space.

1. Then $(X, \tau)$ is $g^{**}$-compact modulo $g^{**}(I_n)$ if and only if $(X, \tau)$ is $g^{**}$-QFHC.
2. If $(X, \tau)$ is $g^{**}$-T$_2$ then $(X, \tau)$ is $g^{**}$-compact modulo $g^{**}(I_n)$ if and only if $(X, \tau)$ is $g^{**}$-FH-closed.

**Proof:**

1. **Necessity:** (1) Let $(X, \tau)$ be $g^{**}$-compact modulo $g^{**}(I_n)$ and $\{U_\alpha / \alpha \in \Omega\}$ be a $g^{**}$-open cover for $X$. Then there exists $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ such that $X - \bigcap_{i=1}^n U_{\alpha_i} \in g^{**}(I_n)$. Therefore $X - \bigcap_{i=1}^n U_{\alpha_i}$ is nowhere $g^{**}$-dense in $X$ and it is $g^{**}$-closed. Therefore $X - \bigcap_{i=1}^n U_{\alpha_i}$ is nowhere $g^{**}$-dense in $X$. Hence $g^{**}cl(\bigcup_{i=1}^n U_{\alpha_i}) = X$. In a $g^{**}$-space, $g^{**}cl(\bigcup_{i=1}^n U_{\alpha_i}) = \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i})$. Therefore $(X, \tau)$ is $g^{**}$-QFHC.

2. **Sufficiency:** Let $(X, \tau)$ be $g^{**}$-QFHC. Since $(X, \tau)$ is $g^{**}$-space, $g^{**}(I_n)$ is an ideal in $X$. Let $\{U_\alpha / \alpha \in \Omega\}$ be a $g^{**}$-open cover. Then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X = \bigcup_{i=1}^n g^{**}cl(U_{\alpha_i}) = g^{**}cl(\bigcup_{i=1}^n U_{\alpha_i})$. Therefore $X - \bigcap_{i=1}^n U_{\alpha_i}$ is nowhere $g^{**}$-dense in $X$ and it is nowhere $g^{**}$-dense in $X$ by theorem 3.16 and so $X - \bigcap_{i=1}^n U_{\alpha_i} \in g^{**}(I_n)$.

(2) By (1), $(X, \tau)$ is $g^{**}$-compact modulo $g^{**}(I_n)$ if and only if $(X, \tau)$ is $g^{**}$-QFHC. By definition (3.4), $(X, \tau)$ is $g^{**}$-compact modulo $g^{**}(I_n)$ if and only if it is $g^{**}$-FH-closed.
Theorem 3.21: Let \((X, \tau)\) be \(g^{**}\)-multiplicative, \(g^{**}\)-countably additive space. Then
1) \((X, \tau)\) is \(g^{**}\)-QCHC space if and only if \((X, \tau)\) is \(g^{**}\)-Lindeloff modulo \(g^{**}(I_n)\).
2) If \((X, \tau)\) is \(g^{**}\)-T_2 then \((X, \tau)\) is \(g^{**}\)-Lindeloff modulo \(g^{**}(I_n)\) if and only if \((X, \tau)\) is \(g^{**}\)-CH-closed.

**Proof:** Since \((X, \tau)\) is \(g^{**}\)-multiplicative and \(g^{**}\)-countably additive,

\[
\bigcup_{i=1}^{\infty} g^{**}\text{cl}(U_{a_i}) = g^{**}\text{cl}\left(\bigcup_{i=1}^{\infty} U_{a_i}\right)
\]

The rest of the proof is similar to the proof of theorem (3.20).

**Definition 3.24:** A space \((X, \tau)\) is said to be \(g^{**}\)-meager or \(g^{**}\)-first category if it is a countable union of nowhere \(g^{**}\)-dense sets.

**Theorem 3.25:** Let \((X, \tau)\) be a \(g^{**}\)-space and let \(g^{**}(I_m) = \{g^{**}\text{-meager subsets of } X\}\). Then \(g^{**}(I_m)\) is an ideal in \(X\).

**Proof:** Let \(A \in g^{**}(I_m)\) and \(B \subseteq A\). Then \(A = \bigcup G_i\) where each \(G_i\) is nowhere \(g^{**}\)-dense subsets. Now

\[
B = \bigcup_{i=1}^{\infty} (B \cap G_i).
\]

Therefore

\[
g^{**}\text{int}[g^{**}\text{cl}(B \cap G_i)] \subseteq g^{**}\text{int}[g^{**}\text{cl}(B) \cap g^{**}\text{cl}(G_i)] = g^{**}\text{int}[g^{**}\text{cl}(B)] \cap g^{**}\text{int}[g^{**}\text{cl}(G_i)],
\]

Since \(X\) is a \(g^{**}\)-space. Therefore \(B \cap G_i\) is nowhere \(g^{**}\)-dense for all \(i\) and so \(B \in g^{**}(I_m)\). Obviously \(A, B \in g^{**}(I_m) \Rightarrow A \cup B \in g^{**}(I_m)\). Therefore \(g^{**}(I_m)\) is an ideal in \(X\).

**Definition 3.26:** A topological space \((X, \tau)\) is said to be of \(g^{**}\)-second category if it is not of \(g^{**}\)-first category.

**Definition 3.27:** A \(g^{**}\)-space \((X, \tau)\) is said to be a \(g^{**}\)-Baire space if \(\bigcap_{i=1}^{\infty} \bigcup g^{**}O(X) = \{\emptyset\}\).

**Theorem 3.28:** Let \((X, \tau)\) be a \(g^{**}\)-Baire space. Then

1) \((X, \tau)\) is \(g^{**}\)-compact modulo \(g^{**}(I_m)\) if and only if \((X, \tau)\) is \(g^{**}\)-QFHC.
2) In addition if \((X, \tau)\) is \(g^{**}\)-T_2 then \((X, \tau)\) is \(g^{**}\)-compact modulo \(g^{**}(I_m)\) if and only if \((X, \tau)\) is \(g^{**}\)-FH-closed.

**Proof:** (1) Let \((X, \tau)\) be \(g^{**}\)-Baire space and \(g^{**}(I_m)\). Let \(\{U_{a_i}\}\) be a \(g^{**}\)-open cover for \(X\). Then there exists \(U_{a_1}, U_{a_2}, \ldots, U_{a_n}\) such that \(X - \bigcup_{i=1}^{n} U_{a_i} \in g^{**}(I_m)\).

Case (i): \(X - \bigcup_{i=1}^{n} U_{a_i} = \emptyset\) then \(X = \bigcup_{i=1}^{n} g^{**}\text{cl}(U_{a_i})\) and so it is \(g^{**}\)-QFHC.

Case (ii): \(X - \bigcup_{i=1}^{n} U_{a_i} \neq \emptyset\) then \(X - \bigcup_{i=1}^{n} U_{a_i} = \bigcup_{i=1}^{n} U_{a_i} \in G^{**}O(X)\) (since \((X, \tau)\) is a \(g^{**}\)-space).

\[
\therefore X - \bigcup_{i=1}^{n} U_{a_i} \in G^{**}O(X) \cap g^{**}(I_m) = \emptyset. \therefore X = \bigcup_{i=1}^{n} g^{**}\text{cl}(U_{a_i}).
\]

(by case(ii)). Therefore \(X\) is \(g^{**}\)-QFHC.

Conversely, let \((X, \tau)\) be \(g^{**}\)-QFHC. By theorem (3.20), \((X, \tau)\) is \(g^{**}\)-compact modulo \(g^{**}(I_n)\).

This implies that \((X, \tau)\) is \(g^{**}\)-compact modulo \(g^{**}(I_m)\), since \(g^{**}(I_n) \subseteq g^{**}(I_m)\). (2) Follows from (1) and definition of \(g^{**}\)- FH-closed space.

**Theorem 3.29:** Let \((X, \tau)\) be a \(g^{**}\)-Baire space which is \(g^{**}\)-countably additive. Then

1) \((X, \tau)\) is \(g^{**}\)-Lindeloff modulo \(g^{**}(I_m)\) if and only if \((X, \tau)\) is \(g^{**}\)-QCHC.
In addition if \((X, \tau)\) is g**-T2 then \((X, \tau)\) is g**-Lindeloff modulo \(g**(I_m)\) if and only if \((X, \tau)\) is g**-CH-closed.

**Proof:** Similar to the above proof.

### 4. g**-Quasi-H-closed modulo an ideal

**Definition 4.1:** An ideal topological space \((X, \tau, I)\) is said to be g**-QFHC modulo an ideal if for every g**-open cover \(\{U_a / a \in \Omega\}\) of \(X\) there exists a finite sub family \(\{U_a / i = 1,2,\ldots,n\}\) such that \(X - \bigcup_{i=1}^{n} g**\text{cl}(U_a) \in I\).

Such a sub family is said to be proximate g**-sub cover modulo \(I\).

**Definition 4.2:** A subset \(A\) of \((X, \tau)\) is said to be g**-pre-open if \(\text{int}(g**A) \subseteq A\). The collection of all g**-preopen sets is denoted by \(G**\text{PO}(X)\).

**Definition 4.3:** An ideal \(I\) in \((X, \tau)\) is said to be completely g**-codense if \(\cap G**\text{PO}(X) = \varnothing\).

Note: (1) \(G**\text{PO}(X) \subseteq G**\text{PO}(X)\).

(2) \(I \cap G**\text{PO}(X) = \varnothing \Rightarrow I \cap G**\text{PO}(X) = \varnothing\).

Therefore every completely g**-codense ideal is g**-codense. But the converse is not true as seen in the following example.

**Example 4.4:** Consider \(R\) with cofinite topology. A subset is g**-closed if and only if it is finite. Let \(I_c\) be the ideal of all countable subsets. Then \(G**\text{PO}(R) \cap I_c = \varnothing\) and so \(I_c\) is g**-codense.

\(g**\text{cl}(Q) = R, g**\text{int}(g**\text{cl}(Q)) = g**\text{int}(R) = R\). \(\therefore Q \in G**\text{PO}(R)\). \(\therefore Q \in I_c \cap G**\text{PO}(R)\).

Therefore \(I_c\) is not completely g**-codense in this space.

**Theorem 4.5:** For a space \((X, \tau)\), the following statements are equivalent.

1. \((X, \tau)\) is g**-QFHC.
2. \((X, \tau)\) is g**-QFHC modulo \(\varnothing\).
3. \((X, \tau)\) is g**-QFHC modulo \(I_F\).

If \((X, \tau)\) is a g**-space then these are equivalent to

4. \((X, \tau)\) is g**-QFHC modulo \(I\) for every g**-codense ideal \(I\).

**Proof:** (1) \(\iff\) (2) is obvious. (2) \(\Rightarrow\) (3) is obvious.

(3) \(\Rightarrow\) (1): Let \(\{U_a\}\) be a g**-open cover for \(X\). Then there exists \(U_{a_1}, U_{a_2}, \ldots, U_{a_n}\) such that \(X - \bigcup_{i=1}^{n} g**\text{cl}(U_a) \in I_F\). Let \(X - \bigcup_{i=1}^{n} g**\text{cl}(U_a) = \{x_1, x_2, \ldots, x_k\}\). Choose \(U_{\beta_i}\) such that \(x_i \in U_{\beta_i}\). Let \(\Delta_0 = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_k\}\). Then \(\Delta_0\) is finite and \(X = \bigcup_{\alpha \in \Delta_0} g**\text{cl}(U_a)\).

Therefore \(X\) is g**-QFHC.

(1) \(\Rightarrow\) (4) is obvious.

(4) \(\Rightarrow\) (1): Let \(I\) be g**-codense ideal. Let \(\{U_a\}_{a \in \Omega}\) be a g**-open cover in \(X\). Then there exists \(\{U_a / i = 1,2,\ldots,n\}\) such that \(X - \bigcup_{i=1}^{n} g**\text{cl}(U_a) \in I\). But \(X - \bigcup_{i=1}^{n} g**\text{cl}(U_a) \in G**\text{PO}(X)\), since \(X\) is a g**-space. But \(G**\text{PO}(X) \cap I = \varnothing\). \(\therefore X = \bigcup_{i=1}^{n} g**\text{cl}(U_a)\). \(\therefore (X, \tau)\) is g**-QFHC.

**Theorem 4.6:** For a topological space \((X, \tau)\) the following statements are equivalent.

1. \((X, \tau)\) is g**-QCHC.
2. \((X, \tau)\) is g**-QCHC modulo \(\varnothing\).
3. \((X, \tau)\) is g**-QCHC modulo \(I_c\).
If $(X, \tau)$ is a g**-multiplicative and g**-countably additive then these are equivalent to

(4) $(X, \tau)$ is g**-QCHC modulo $I$ for every g**-codense ideal $I$.

**Proof:** Similar to the above proof, since $(X, \tau)$ is g**-countably additive implies $\bigcup_{i=1}^{\infty} g**cl(U_{a_i})$ is g**-closed.

**Remark 4.7:** In theorem (4.6) the condition $I$ is g**-codense is necessary as seen in the following example.

**Example 4.8:** Consider $R$ with indiscrete topology $\tau$. Let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a finite subset of $R$. Then $I = \varphi(R - A)$ is an ideal in $R$. Since all subsets are g**-open, $g**O(X) \cap I \neq \varphi$.

Therefore $I$ is not g**-codense. Let $\{U_{a_1} / \alpha \in \Omega\}$ be a g**-open cover for $X$. Choose $U_{a_1}$ such that $a_i \in U_{a_i}$. Then $R - \bigcup_{i=1}^{n} U_{a_i} \in I$. But $R - \bigcup_{i=1}^{n} g**cl(U_{a_i}) \subseteq R - \bigcup_{i=1}^{n} U_{a_i}$.

Therefore $R$ is g**-QFHC modulo $I$. $\{x / x \in R\}$ is a g**-open cover which has no finite sub cover whose g**-closures cover $X$. Therefore $R$ is not g**-QFHC.

**Remark 4.9:** In theorem (4.7) the condition $I$ is g**-codense is necessary as seen in the following example.

**Example 4.10:** In example (4.8), $(R, \tau)$ is g**-QCHC modulo $I$ but not g**-QCHC.

The following theorem contains a number of characterizations of g**-QFHC modulo $I$ spaces.

**Theorem 4.11:** For a topological space $(X, \tau)$ and an ideal $I$ on $X$, the following statements are equivalent.

(1) $(X, \tau)$ is G**-QFHC modulo $I$.

(2) For each family $\{\alpha \in \Omega\}$ of g**-closed sets having empty intersection there exists a finite sub family $\{A_{a_1}, A_{a_2}, \ldots, A_{a_n}\}$ such that $\bigcap_{i=1}^{n} g**int(A_{a_i}) \in I$.

(3) For each family $\{\alpha \in \Omega\}$ of g**-closed sets such that $g**int(A_{a})/\alpha \in \Omega$ with FIP modulo $I$ one has, $\bigcap_{\alpha \in A'} A \neq \varphi$.

**Proof:** (1) $\Rightarrow$ (2) Let $A' = \{A_{a} / \alpha \in \Omega\}$ be a family of g**-closed sets such that $\bigcap_{\alpha \in \Omega} A_{a} = \varphi$. Then $X = \bigcup_{\alpha \in \Omega} (G_{a})$ where $G_{a} = X - A_{a}$ is g**-open. By (1), there exists $\{G_{a_1}, G_{a_2}, \ldots, G_{a_n}\}$ such that $X - \bigcap_{i=1}^{n} g**cl(G_{a_i}) \in I$. (ie) $\bigcap_{i=1}^{n} [g**cl(G_{a_i})]^{c} \in I$. But $[g**cl(G_{a_i})]^{c} = X - g**cl(G_{a_i})$ which is equal to $g**int(X - G_{a_i}) = g**int(A_{a_i})$. $\therefore \bigcap_{i=1}^{n} g**int(A_{a_i}) \in I$.

(2) $\Rightarrow$ (3) : Let $A'$ be a family of g**-closed sets such that $\{g**int(A_{a}) / \alpha \in \Omega\}$ has FIP modulo $I$.

Suppose $\bigcap_{\alpha \in \Omega} A_{a} = \varphi$. Then by (2) there exists a finite sub family $\{A_{a_1}, A_{a_2}, \ldots, A_{a_n}\}$ such that $\bigcap_{i=1}^{n} g**int(A_{a_i}) \in I$ which is a contradiction. $\therefore \bigcap_{\alpha \in \Omega} A_{a} \neq \varphi$.

(3) $\Rightarrow$ (1) : Let $\{U_{a} / \alpha \in \Omega\}$ be a g**-open cover for $X$. To prove that there exists a finite sub family $\{U_{a_i} / i = 1, 2, \ldots, n\}$ such that $X - \bigcup_{i=1}^{n} g**cl(U_{a_i}) \in I$. If not there is no finite sub family with this property.

Now $\{X - U_{a} / \alpha \in \Omega\}$ is a family of g**-closed sets. For any finite sub family $U_{a_1}, U_{a_2}, \ldots, U_{a_n}$, $X - \bigcup_{i=1}^{n} g**cl(U_{a_i}) \notin I$. (ie) $\bigcap_{i=1}^{n} g**int(X - U_{a_i}) \notin I$. By (3), $\bigcap_{\alpha \in \Omega} (X - U_{a}) \neq \varphi$. $\therefore \bigcup_{\alpha \in \Omega} (U_{a}) \neq X$ which is a contradiction. Therefore $(X, \tau)$ is g**-QFHC.
Theorem 4.12: For a topological space \((X, \tau)\) and an ideal \(I\) on \(X\), the following statements are equivalent.

1. \((X, \tau)\) is \(g^{**}\)-QCHC modulo \(I\).
2. For each family \(A' = \{A_\alpha / \alpha \in \Omega\}\) of \(g^{**}\)-closed sets having empty intersection there exists a countable subfamily \(\{A_{\alpha_1}, A_{\alpha_2}, \ldots, \ldots\}\) such that \(\bigcap_{i=1}^{\infty} g^{**}\text{int}(A_{\alpha_i}) \in I\).
3. For each family \(A' = \{A_\alpha / \alpha \in \Omega\}\) of \(g^{**}\)-closed sets such that \(\{g^{**}\text{int}(A_\alpha) / \alpha \in \Omega\}\) with CIP modulo \(I\) one has \(\bigcap_{\alpha \in \Lambda} A \neq \emptyset\).

Proof: Similar to the above proof.

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Source of support: Nil, Conflict of interest: None Declared