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ON FUZZY SEMI-PRE-BOUNDARY

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ABSTRACT

In this paper we introduce the concept of fuzzy $\mathscr{C}$-semi-pre-boundary by using the arbitrary complement function $\mathscr{C}$ and by using fuzzy $\mathscr{C}$-semi-pre closure of a fuzzy topological space where $\mathscr{C}:[0, 1]\to[0, 1]$ is a function. Let λ be a fuzzy subset of a fuzzy topological space X and let $\mathscr{C}$ be a complement function. Then the fuzzy $\mathscr{C}$–semi-pre- boundary of λ is defined as $SPBd_{\mathscr{C}}\lambda = SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)$, where $SPCl_{\mathscr{C}}\lambda$ is the fuzzy $\mathscr{C}$-semi-pre-closure of λ and $\mathscr{C}\lambda(x) = \mathscr{C}(\lambda(x))$, $0 \le x \le 1$. In this paper we discuss the basic properties of fuzzy $\mathscr{C}$-semi-pre-boundary.

***MSC 2010**: 54A40, 3E72.*

***Key words:** Fuzzy $\mathscr{C}$-semi-pre-boundary, fuzzy $\mathscr{C}$-semi-pre closure, fuzzy $\mathscr{C}$-semi-pre closed sets and fuzzy topology.*

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1. INTRODUCTION

Athar and Ahmad [2] defined the notion of fuzzy semi boundary in fuzzy topological spaces and studied[1] the properties of fuzzy semi boundary. The authors introduced the concept of fuzzy $\mathscr{C}$-closed sets, fuzzy $\mathscr{C}$-semi closed sets, fuzzy $\mathscr{C}$-pre closed sets and fuzzy $\mathscr{C}$-semi-pre-closed sets in fuzzy topological spaces, where $\mathscr{C}$: $[0, 1]\to[0, 1]$ is an arbitrary complement function.

In this paper, we introduce the concept of fuzzy semi-pre-boundary by using the arbitrary complement function $\mathscr{C}$, instead of the usual fuzzy complement function, and by using fuzzy $\mathscr{C}$-semi-pre-closure instead of fuzzy semi-pre-closure.

Such a generalized fuzzy semi-pre-boundary is defined as $SPBd_{\mathscr{C}}\lambda=SPCl_{\mathscr{C}}\lambda\wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)$, called the fuzzy $\mathscr{C}$-semi-pre-boundary of λ, where $SPCl_{\mathscr{C}}\lambda$ is the intersection of all fuzzy $\mathscr{C}$-semi-pre- closed sets containing λ and $\mathscr{C}\lambda(x)=\mathscr{C}(\lambda(x))$, $0 \le x \le 1$.

For the basic concepts and notations, one can refer Chang [7]. The concepts that are needed in this paper are discussed in the second section. The third section is dealt with the concept of fuzzy $\mathscr{C}$- semi-pre-boundary.

2. PRELIMINARIES

Throughout this paper (X,τ) denotes a fuzzy topological space in the sense of Chang. Let $\mathscr{C}$: $[0, 1]\to[0, 1]$ be a complement function. If λ is a fuzzy subset of (X,τ) then the complement $\mathscr{C}\lambda$ of a fuzzy subset λ is defined by $\mathscr{C}\lambda(x) = \mathscr{C}(\lambda(x))$ for all $x\in X$. A complement function $\mathscr{C}$ is said to satisfy

(i) the boundary condition if $\mathscr{C}(0) = 1$ and $\mathscr{C}(1) = 0$,
(ii) monotonic condition if $x \le y \Rightarrow \mathscr{C}(x) \ge \mathscr{C}(y)$, for all $x, y \in [0, 1]$,
(iii) involutive condition if $\mathscr{C}(\mathscr{C}(x)) = x$, for all $x \in[0, 1]$.

The properties of fuzzy complement function $\mathscr{C}$ and $\mathscr{C}\lambda$ are given in George Klir [8] and Bageerathi *et al* [4]. The following lemma will be useful in sequel.

Lemma 2.1[4]. Let $\mathscr{C}$: $[0, 1] \to [0, 1]$ be a complement function that satisfies the monotonic and involutive conditions. Then for any family $\{\lambda_\alpha: \alpha\in\Delta\}$ of fuzzy subsets of X, we have
(i) $\mathscr{C}(\sup\{\lambda_\alpha(x): \alpha\in\Delta\}) = \inf\{\mathscr{C}(\lambda_\alpha(x)): \alpha\in\Delta\}= \inf\{(\mathscr{C}\lambda_\alpha(x)): \alpha\in\Delta\}$ and
(ii) $\mathscr{C}(\inf\{\lambda_\alpha(x): \alpha\in\Delta\}) = \sup\{\mathscr{C}(\lambda_\alpha(x)): \alpha\in\Delta\}= \sup\{(\mathscr{C}\lambda_\alpha(x)): \alpha\in\Delta\}$ for $x\in X$.

Definition 2.2 [4]. A fuzzy subset λ of X is fuzzy $\mathscr{C}$-closed in (X,τ) if $\mathscr{C}\lambda$ is fuzzy open in (X,τ). The fuzzy $\mathscr{C}$-closure of λ is defined as the intersection of all fuzzy $\mathscr{C}$-closed sets μ containing λ. The fuzzy $\mathscr{C}$-closure of λ is denoted by $Cl_{\mathscr{C}}\lambda$ that is equal to $\wedge\{\mu: \mu \ge \lambda, \mathscr{C}\mu\in\tau\}$.

Lemma 2.3[4]. If the complement function $\mathscr{C}$ satisfies the monotonic and involutive conditions, then for any fuzzy subset λ of X, $\mathscr{C}(Int\,\lambda) = Cl_{\mathscr{C}}(\mathscr{C}\lambda)$ and $\mathscr{C}(Cl_{\mathscr{C}}\lambda) = Int\,(\mathscr{C}\lambda)$.

Lemma 2.4[4]. Let (X,τ) be a fuzzy topological space. Let $\mathscr{C}$ be a complement function that satisfies the boundary, monotonic and involutive conditions. Then for any family $\{\lambda_\alpha : \alpha\in \Delta\}$ of fuzzy subsets of X. we have $\mathscr{C}(\vee\{\lambda_\alpha: \alpha\in\Delta\}) = \wedge\{\mathscr{C}\lambda_\alpha: \alpha\in\Delta\}$ and $\mathscr{C}(\wedge\{\lambda_\alpha: \alpha\in\Delta\}) = \vee\{\mathscr{C}\lambda_\alpha: \alpha\in\Delta\}$.

Definition 2.5 [Definition 2.15, [4]]. A fuzzy topological space (X,τ) is $\mathscr{C}$-product related to another fuzzy topological space (Y,σ) if for any fuzzy subset ν of X and ζ of Y, whenever $\mathscr{C}\lambda \ngeq \nu$ and $\mathscr{C}\mu \ngeq \zeta$ imply $\mathscr{C}\lambda \times 1 \vee 1 \times \mathscr{C}\mu \geq \nu \times \zeta$, where $\lambda \in \tau$ and $\mu \in\sigma$, there exist $\lambda_1 \in\tau$ and $\mu_1\in\sigma$ such that $\mathscr{C}\lambda_1 \geq \nu$ or $\mathscr{C}\mu_1 \geq \zeta$ and $\mathscr{C}\lambda_1 \times 1 \vee 1 \times \mathscr{C}\mu_1 = \mathscr{C}\lambda \times 1 \vee 1 \times \mathscr{C}\mu$.

Lemma 2.6 [Theorem 2.19, [4]]. Let (X,τ) and (Y,σ) be $\mathscr{C}$-product related fuzzy topological spaces. Then for a fuzzy subset λ of X and a fuzzy subset μ of Y, $Cl_{\mathscr{C}}(\lambda \times \mu) = Cl_{\mathscr{C}}\lambda \times Cl_{\mathscr{C}}\mu$.

Definition 2.7 [Definition 3.1, [6]]. Let (X,τ) be a fuzzy topological space and $\mathscr{C}$ be a complement function. Then λ is called fuzzy $\mathscr{C}$-semi-pre open if there exists a $\mathscr{C}$- pre open set μ such that $\mu \leq \lambda \leq Cl_{\mathscr{C}}\mu$.

Lemma 2.8[6]. Let (X, τ) be a fuzzy topological space and let $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive properties. Then a fuzzy set λ of a fuzzy topological space (X,τ) is

(i) fuzzy $\mathscr{C}$-semi-pre open if and only if $\lambda \leq Cl_{\mathscr{C}}Int\ Cl_{\mathscr{C}}(\lambda)$.
(ii) fuzzy $\mathscr{C}$- semi-pre closed in X if $Int\ Cl_{\mathscr{C}}Int\ (\lambda)) \leq \lambda$.
(iii) fuzzy $\mathscr{C}$- semi-pre closed if and only if $\mathscr{C}\lambda$ is fuzzy $\mathscr{C}$- semi-pre open.
(iv) the arbitrary union of fuzzy $\mathscr{C}$- semi-pre open sets is fuzzy $\mathscr{C}$- semi-pre open.

Lemma 2.9 [3]. If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the fuzzy subsets of X then

$$(\lambda_1 \wedge \lambda_2) \times (\lambda_3 \wedge \lambda_4) = (\lambda_1 \times \lambda_4) \wedge (\lambda_2 \times \lambda_3).$$

Lemma 2.10 [Lemma 5.1, [4]]. Suppose f is a function from X to Y. Then $f^{-1}(\mathscr{C}\mu) = \mathscr{C}(f^{-1}(\mu))$ for any fuzzy subset μ of Y.

Definition 2.11 [9]. If λ is a fuzzy subset of X and μ is a fuzzy subset of Y, then $\lambda \times \mu$ is a fuzzy subset of $X \times Y$, defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\}$ for each $(x, y)\in X \times Y$.

Lemma 2.12 [Lemma 2.1, [3]]. Let $f: X \to Y$ be a function. If $\{\lambda_\alpha\}$ a family of fuzzy subsets of Y, then

(i) $f^{-1}(\vee \lambda_\alpha) = \vee f^{-1}(\lambda_\alpha)$ and
(ii) $f^{-1}(\wedge \lambda_\alpha) = \wedge f^{-1}(\lambda_\alpha)$.

Lemma 2.13 [Lemma 2.2, [3]]. If λ is a fuzzy subset of X and μ is a fuzzy subset of Y, then $\mathscr{C}(\lambda\times\mu) = \mathscr{C}\lambda \times 1 \vee 1 \times \mathscr{C}\mu$.

3. FUZZY $\mathscr{C}$–SEMI-PRE-BOUNDARY

In this section, the concept of fuzzy $\mathscr{C}$- semi-pre-boundary is introduced and its properties are discussed.

Definition 3.1. Let λ be a fuzzy subset of a fuzzy topological space X and let $\mathscr{C}$ be a complement function. Then the fuzzy $\mathscr{C}$-semi-pre-boundary of λ is defined as $\mathcal{SP}Bd_{\mathscr{C}}\lambda = \mathcal{SP}Cl_{\mathscr{C}}\lambda\wedge\mathcal{SP}Cl_{\mathscr{C}}(\mathscr{C}\lambda)$.

Since the arbitrary intersection of fuzzy $\mathscr{C}$-semi-pre-closed sets is fuzzy $\mathscr{C}$-semi-pre closed, $\mathcal{SP}Bd_{\mathscr{C}}\lambda$ is fuzzy $\mathscr{C}$-semi-pre closed.

We identify $\mathcal{SP}Bd_{\mathscr{C}}\lambda$ with $\mathcal{SP}Bd(\lambda)$ when $\mathscr{C}(x) = 1-x$, the usual complement function.

Proposition 3.2. Let (X,τ) be a fuzzy topological space and $\mathscr{C}$ be a complement function that satisfies the involutive condition. Then for any fuzzy subset λ of X, $\mathcal{SP}Bd_{\mathscr{C}}\lambda = \mathcal{SP}Bd_{\mathscr{C}}(\mathscr{C}\lambda)$.

Proof. By using Definition 3.1, $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda = \mathscr{SP}Cl_{\mathscr{C}}\lambda \wedge \mathscr{SP}Cl_{\mathscr{C}}(\mathscr{C}\lambda)$. Since $\mathscr{C}$ satisfies the involutive condition $\mathscr{C}(\mathscr{C}\lambda) = \lambda$, that implies $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda = \mathscr{SP}Cl_{\mathscr{C}}(\mathscr{C}\lambda) \wedge \mathscr{SP}Cl_{\mathscr{C}}\mathscr{C}(\mathscr{C}\lambda)$.

Again by using Definition 3.1, $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda = \mathscr{SP}\mathrm{Bd}_{\mathscr{C}}(\mathscr{C}\lambda)$.

The following example shows that, the word "involutive" can not be dropped from the hypothesis of Proposition 4.2.

Example 3.3. Let X = {a, b, c} and $\tau = \{0, \{a_{.3}, b_{.7}\}, \{a_{.5}, b_{.2}, c_{.6}\}, \{a_{.5}, b_{.7}, c_{.6}\}, \{a_{.3}, b_{.2}\}, 1\}$.
Let $\mathscr{C}(x) = \dfrac{1-x}{1+x^2}$, $0 \le x \le 1$, be the complement function. We note that the complement function $\mathscr{C}$ does not satisfy the involutive condition. The family of all fuzzy $\mathscr{C}$-closed sets is $\mathscr{C}(\tau)=\{0,\{a_{.642}, b_{.201}, c_1\},\{a_{.4},b_{.769}, c_{.294}\},\{a_{.4}, b_{.201}, c_{.294}\},\{a_{.642}, b_{.769},c_1\}, 1\}$.

Let $\lambda = \{a_{.5}, b_{.8}, c_{.4}\}$. Then it can be calculated that $\mathscr{SP}Cl_{\mathscr{C}}\lambda = \{a_{.5}, b_{.8}, c_{.4}\}$.

Now $\mathscr{C}\lambda = \{a_{.4}, b_{.122}, c_{.57}\}$ and the value of $\mathscr{SP}Cl_{\mathscr{C}}\mathscr{C}\lambda = \{a_{.4}, b_{.122}, c_{.517}\}$. Hence $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda = \mathscr{SP}Cl_{\mathscr{C}}\lambda \wedge \mathscr{SP}Cl_{\mathscr{C}}(\mathscr{C}\lambda) = \{a_{.4}, b_{.122}, c_{.517}\}$. Also $\mathscr{C}(\mathscr{C}\lambda) = \{a_{.517}, b_{.865}, c_{.381}\}$, $\mathscr{SP}Cl_{\mathscr{C}}\mathscr{C}(\mathscr{C}\lambda) = \{a_{.517}, b_{.865}, c_{.381}\}$. $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\mathscr{C}\lambda = \mathscr{SP}Cl_{\mathscr{C}}\mathscr{C}\lambda \wedge \mathscr{SP}Cl_{\mathscr{C}}\mathscr{C}(\mathscr{C}\lambda) = \{a_{.4}, b_{.122}, c_{.381}\}$. This implies that $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda \neq \mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\mathscr{C}\lambda$.

Proposition 3.4. Let (X,τ) be a fuzzy topological space and $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive conditions. If λ is fuzzy $\mathscr{C}$-semi-pre closed, then $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda \le \lambda$.

Proof. Let λ be fuzzy $\mathscr{C}$-semi-pre- closed. By using Definition 3.1, $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda = \mathscr{SP}Cl_{\mathscr{C}}\lambda \wedge \mathscr{SP}Cl_{\mathscr{C}}(\mathscr{C}\lambda)$. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Proposition 5.6(ii) in [6], we have $\mathscr{SP}Cl_{\mathscr{C}}\lambda = \lambda$. Hence $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda \le \mathscr{SP}Cl_{\mathscr{C}}\lambda = \lambda$.

The following example shows that if the complement function $\mathscr{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 3.4 is false.

Example 3.5. Let X = {a, b, c} and $\tau = \{0, \{a_{.6}, b_{.9}\}, \{a_{.7}, b_{.3}\}, \{a_{.6}, b_{.3}\}, \{a_{.7}, b_{.9}\}, 1\}$.
Let $\mathscr{C}(x) = \dfrac{2x}{1+x}$, $0 \le x \le 1$, be a complement function. From this, we see that the complement function $\mathscr{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathscr{C}$-closed sets is given by $\mathscr{C}(\tau) = \{0, \{a_{.75}, b_{.947}\}, \{a_{.824}, b_{.462}\}, \{a_{.75},b_{.462}\}, \{a_{.824},b_{.947}\}, 1\}$. Let $\lambda = \{a_{.8}, b_{.3}\}$, it can be found that $Int\,\lambda = \{a_{.7}, b_{.3}\}$, $Cl_{\mathscr{C}}Int\,\lambda = \{a_{.824}, b_{.462}\}$ and $Int\,Cl_{\mathscr{C}}Int\,\lambda = \{a_{.7}, b_{.3}\}$. That implies $Int\,Cl_{\mathscr{C}}\lambda \le \lambda$. This shows that λ is fuzzy $\mathscr{C}$- pre closed. Further it can be calculated that $\mathscr{SP}Cl_{\mathscr{C}}\lambda = \{a_{.85}, b_{.632}\}$. Now $\mathscr{C}\lambda = \{a_{.889}, b_{.67}\}$ and $\mathscr{SP}Cl_{\mathscr{C}}\mathscr{C}\lambda = \{a_{.889}, b_{.67}\}$. Hence $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda = \mathscr{SP}Cl_{\mathscr{C}}\lambda \wedge \mathscr{SP}Cl_{\mathscr{C}}(\mathscr{C}\lambda) = \{a_{.85}, b_{.632}\}$. This implies that $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda \not\le \lambda$. This shows that the conclusion of Proposition 3.4 is false.

Proposition 3.6. Let (X,τ) be a fuzzy topological space and $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive conditions. If λ is fuzzy $\mathscr{C}$-semi-pre open then $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda \le \mathscr{C}\lambda$.

Proof. Let λ be fuzzy $\mathscr{C}$-semi-pre open. Since $\mathscr{C}$ satisfies the involutive condition, this implies that $\mathscr{C}(\mathscr{C}\lambda)$ is fuzzy $\mathscr{C}$-semi-pre open. By using Lemma 2.8, $\mathscr{C}\lambda$ is fuzzy $\mathscr{C}$-semi-pre closed. Since $\mathscr{C}$ satisfies the monotonic and the involutive conditions, by using Proposition 3.4, $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}(\mathscr{C}\lambda) \le \mathscr{C}\lambda$. Also by using Proposition 3.2, we get $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}(\lambda) \le \mathscr{C}\lambda$. This completes the proof.

Example 3.7. Let X = {a, b, c} and $\tau = \{0, \{a_{.3}, b_{.7}\}, \{a_{.5}, b_{.2}, c_{.6}\}, \{a_{.5}, b_{.7}, c_{.6}\}, \{a_{.3}, b_{.2}\}, 1\}$.
Let $\mathscr{C}(x) = \dfrac{1-x}{1+x^2}$, $0 \le x \le 1$, be the complement function. We note that the complement function $\mathscr{C}$ does not satisfy the involutive condition. The family of all fuzzy $\mathscr{C}$-closed sets is $\mathscr{C}(\tau)=\{0,\{a_{.642}, b_{.201}, c_1\},\{a_{.4},b_{.769}, c_{.294}\},\{a_{.4}, b_{.201}, c_{.294}\},\{a_{.642}, b_{.769},c_1\}, 1\}$.

Let $\lambda = \{a_{.4}, b_{.122}, c_{.57}\}$, the value of $\mathscr{SP}Cl_{\mathscr{C}}\lambda = \{a_{.4}, b_{.122}, c_{.517}\}$. Now $\mathscr{C}\lambda = \{a_{.517}, b_{.865}, c_{.381}\}$ and $\mathscr{SP}Cl_{\mathscr{C}}\lambda = \{a_{.517}, b_{.865}, c_{.381}\}$, it follows that $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda = \mathscr{SP}Cl_{\mathscr{C}}\lambda \wedge \mathscr{SP}Cl_{\mathscr{C}}(\mathscr{C}\lambda) = \{a_{.4}, b_{.122}, c_{.381}\}$. This shows that $\mathscr{SP}\mathrm{Bd}_{\mathscr{C}}\lambda \not\le \mathscr{C}\lambda$.

Therefore the conclusion of Proposition 3.6 is false.

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Proposition 3.8. Let (X, τ) be a fuzzy topological space and $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive conditions. If $\lambda \le \mu$ and μ is fuzzy $\mathscr{C}$-semi-pre closed then $SPBd_{\mathscr{C}}\lambda \le \mu$.

Proof. Let $\lambda \le \mu$ and μ be fuzzy $\mathscr{C}$-semi-pre closed. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Proposition 5.6(iv) in [6], we have $\lambda \le \mu$ implies $SPCl_{\mathscr{C}}\lambda \le SPCl_{\mathscr{C}}\mu$.

By using Definition 3.1, $SPBd_{\mathscr{C}}\lambda = SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)$. Since $SPCl_{\mathscr{C}}\lambda \le SPCl_{\mathscr{C}}\mu$, we have $SPBd_{\mathscr{C}}\lambda \le SPCl_{\mathscr{C}}\mu \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda) \le SPCl_{\mathscr{C}}\mu$. Again by using Proposition 5.6 (ii) in [6], we have $SPCl_{\mathscr{C}}\mu = \mu$.

This implies that $SPBd_{\mathscr{C}}\lambda \le \mu$.

The following example shows that if the complement function $\mathscr{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 3.8 is false.

Example 3.9. From Example 3.5, let X = {a, b} and τ = **{**0, {a.$_6$, b.$_9$}, {a.$_7$, b.$_3$}, {a.$_6$, b.$_3$},{a.$_7$, b.$_9$}, 1**}**.
Let λ = {a.$_7$, b.$_{45}$} and μ = {a.$_{76}$, b.$_5$}. Then it can be found that $Int\ \mu$ = {a.$_7$, b.$_3$}, $Cl_{\mathscr{C}}Int\ \mu$ = {a.$_{75}$, b.$_{462}$} and $Int\ Cl_{\mathscr{C}}Int\ \mu$ = {a.$_7$, b.$_3$}. That implies $Int\ Cl_{\mathscr{C}}Int\ \mu \le \mu$. This shows that μ is fuzzy $\mathscr{C}$-pre closed. It can be computed that $SPCl_{\mathscr{C}}\lambda$ = {a.$_8$, b.$_{47}$}.

Now $\mathscr{C}\lambda$ = {a.$_{824}$, b.$_{62}$} and $SPCl_{\mathscr{C}}\mathscr{C}\lambda$ = {a.$_{824}$, b.$_{47}$}. $SPBd_{\mathscr{C}}\lambda = SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)$ = { a.$_8$, b.$_{47}$}. This shows that $SPBd_{\mathscr{C}}\lambda \not\le \mu$.

Therefore the conclusion of Proposition 3.8 is false.

Proposition 3.10. Let (X,τ) be a fuzzy topological space and $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive conditions. If $\lambda \le \mu$ and μ is fuzzy $\mathscr{C}$-semi-pre open then $SPBd_{\mathscr{C}}\lambda \le \mathscr{C}\mu$.

Proof. Let $\lambda \le \mu$ and μ is fuzzy $\mathscr{C}$-semi-pre open. Since $\mathscr{C}$ satisfies the monotonic condition, by using Proposition 5.6(iv) in [6], we have $\mathscr{C}\mu \le \mathscr{C}\lambda$ that implies $SPCl_{\mathscr{C}}\mathscr{C}\mu \le SPCl_{\mathscr{C}}\mathscr{C}\lambda$. By using Definition 3.1, $SPBd_{\mathscr{C}}\lambda = SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}\mathscr{C}\lambda$. Taking complement on both sides, we get $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) = \mathscr{C}(SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda))$. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Lemma 2.1, we have $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) = \mathscr{C}(SPCl_{\mathscr{C}}\lambda) \vee \mathscr{C}(SPCl_{\mathscr{C}}(\mathscr{C}\lambda))$. Since $SPCl_{\mathscr{C}}\mathscr{C}\mu \le SPCl_{\mathscr{C}}\mathscr{C}\lambda$, $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) \ge \mathscr{C}(SPCl_{\mathscr{C}}\mathscr{C}\mu) \vee \mathscr{C}(SPCl_{\mathscr{C}}\lambda)$, by using Proposition 5.5(ii) in [6], $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) \ge SPInt_{\mathscr{C}}\mu \vee SPInt_{\mathscr{C}}\mathscr{C}\lambda \ge pInt_{\mathscr{C}}\mu$. Since μ is fuzzy $\mathscr{C}$–semi-pre open, $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) \ge \mu$. Since $\mathscr{C}$ satisfies the monotonic conditions, $SPBd_{\mathscr{C}}\lambda \le \mathscr{C}\mu$.

The following example shows that if the complement function $\mathscr{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 4.10 is false.

Example 3.11. From Example 3.5, let X = {a, b} and τ = **{**0, {a.$_6$, b.$_9$}, {a.$_7$, b.$_3$}, {a.$_6$, b.$_3$}, {a.$_7$, b.$_9$}, 1**}**. Let λ = {a.$_6$, b.$_3$} and μ = {a.$_{65}$, b.$_4$}. Then it can be evaluated that $Cl_{\mathscr{C}}\lambda$ = {a.$_{75}$, b.$_{462}$}, $Int\ Cl_{\mathscr{C}}\lambda$ = {a.$_6$, b.$_3$} and $Cl_{\mathscr{C}}Int\ \lambda$ = {a.$_{75}$, b.$_{462}$}. Thus we see that $\lambda \le Cl_{\mathscr{C}}(Int\lambda)$. By using Lemma 2.8, λ is fuzzy $\mathscr{C}$-semi-pre open. It can be computed that $SPCl_{\mathscr{C}}\lambda$ = {a.$_{85}$, b.$_{632}$}. Now $\mathscr{C}\lambda$ = {a.$_{75}$, b.$_{462}$} and $SPCl_{\mathscr{C}}\mathscr{C}\lambda$ = {a.$_{85}$, b.$_{632}$}. $SPBd_{\mathscr{C}}\lambda = SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)$ = {a.$_{85}$, b.$_{632}$}. This shows that $SPBd_{\mathscr{C}}\lambda \not\le \mathscr{C}\mu$.

Proposition 3.12. Let (X,τ) be a fuzzy topological space. Let $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, we have $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) = SPInt_{\mathscr{C}}\lambda \vee SPInt_{\mathscr{C}}(\mathscr{C}\lambda)$.

Proof. By using Definition 3.1, $SPBd_{\mathscr{C}}\lambda = SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)$. Taking complement on both sdes, we get $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) = \mathscr{C}(SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda))$. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Lemma 2.4(ii), $(SPBd_{\mathscr{C}}\lambda) = \mathscr{C}(SPCl_{\mathscr{C}}\lambda) \vee \mathscr{C}(SPCl_{\mathscr{C}}(\mathscr{C}\lambda))$. Also by using Proposition 5.6(ii) in [6], that implies $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) = SPInt_{\mathscr{C}}(\mathscr{C}\lambda) \vee SPInt_{\mathscr{C}}(\mathscr{C}(\mathscr{C}\lambda))$. Since $\mathscr{C}$ satisfies the involutive condition, $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) = SPInt_{\mathscr{C}}\lambda \vee SPInt_{\mathscr{C}}(\mathscr{C}\lambda)$.

The following example shows that if the monotonic and involutive conditions of the complement function $\mathscr{C}$ are dropped, then the conclusion of Proposition 3.12 is false.

Example 3.13. Let X = {a, b} and τ = **{**0, {a.$_3$, b.$_8$}, {a.$_2$, b.$_5$}, {a.$_7$, b.$_1$}, {a.$_3$, b.$_5$}, {a.$_3$, b.$_1$}, {a.$_2$, b.$_1$}, {a.$_7$, b.$_8$}, {a.$_7$, b.$_5$}, 1**}**. Let $\mathscr{C}(x) = \sqrt{x}$, $0 \le x \le 1$ be the complement function. From this example, we see that $\mathscr{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathscr{C}$-closed sets is $\mathscr{C}(\tau)$ = $\{0, \{a_{.548}, b_{.894}\}, \{a_{.447}, b_{.707}\}, \{a_{.837}, b_{.316}\}, \{_{.548}, b_{.707}\}, \{a_{.548}, b_{.316}\}, \{a_{.447}, b_{.316}\}, \{a_{.837}, b_{.894}\}, \{a_{.837}, b_{.707}\}, 1\}$.

Let $\lambda = \{a_{.6}, b_{.3}\}$. Then it can be evaluated that $SPInt_{\mathscr{C}}\lambda = \{a_{.3}, b_{.1}\}$, $\mathscr{C}\lambda = \{a_{.775}, b_{.548}\}$ and $SPInt_{\mathscr{C}}\mathscr{C}\lambda = \{a_{.775}, b_{.548}\}$. Thus we see that $SPInt_{\mathscr{C}}\lambda \vee SPInt_{\mathscr{C}}\mathscr{C}\lambda = \{a_{.775}, b_{.548}\}$. It can be computed that $SPCl_{\mathscr{C}}\lambda = \{a_{.6}, b_{.8}\}$. Now $\mathscr{C}\lambda = \{a_{.775}, b_{.548}\}$, $SPCl_{\mathscr{C}}\mathscr{C}\lambda = \{a_{.837}, b_{.707}\}$ and $SPBd_{\mathscr{C}}\lambda = SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda) = \{a_{.6}, b_{.707}\}$. Also $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) = \{a_{.775}, b_{.840}\}$. Thus we see that $\mathscr{C}(SPBd_{\mathscr{C}}\lambda) \neq SPInt_{\mathscr{C}}\lambda \vee SPInt_{\mathscr{C}}\mathscr{C}\lambda$. Therefore the conclusion of Proposition 3.12 is false.

Proposition 3.14. Let (X,τ) be a fuzzy topological space. Let $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, we have $SPBd_{\mathscr{C}}(\lambda) = SPCl_{\mathscr{C}}(\lambda) \wedge \mathscr{C}(SPInt_{\mathscr{C}}(\lambda))$.

Proof. By using Definition 3.1, we have $SPBd_{\mathscr{C}}(\lambda) = SPCl_{\mathscr{C}}(\lambda) \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)$. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Proposition 5.5(ii) in [6], we have $SPBd_{\mathscr{C}}(\lambda) = SPCl_{\mathscr{C}}(\lambda) \wedge \mathscr{C}(SPInt_{\mathscr{C}}(\lambda))$.

The next example shows that if the complement function $\mathscr{C}$ does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 3.14 is false.

Example 3.15. From Example 3.5, let X = {a, b} and $\tau = \{0, \{a_{.6}, b_{.9}\}, \{a_{.7}, b_{.3}\}, \{a_{.6}, b_{.3}\}, \{a_{.7}, b_{.9}\}, 1\}$. Let $\lambda = \{a_{.9}, b_{.5}\}$. Then it can be evaluated that $SPInt_{\mathscr{C}}\lambda = \{a_{.75}, b_{.462}\}$ and $\mathscr{C}(SPInt_{\mathscr{C}}\lambda) = \{a_{.857}, b_{.632}\}$ and it can be computed that $SPCl_{\mathscr{C}}\lambda = \{a_{.9}, b_{.5}\}$. Now $\mathscr{C}\lambda = \{a_{.947}, b_{.667}\}$, $SPCl_{\mathscr{C}}\mathscr{C}\lambda = \{a_{.947}, b_{.667}\}$ and $SPBd_{\mathscr{C}}\lambda = SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda) = \{a_{.9}, b_{.5}\}$. Also $SPCl_{\mathscr{C}}\lambda \wedge \mathscr{C}(SPInt_{\mathscr{C}}\lambda) = \{a_{.857}, b_{.5}\}$. Thus we see that $SPBd_{\mathscr{C}}\lambda \neq SPCl_{\mathscr{C}}\lambda \wedge \mathscr{C}(SPInt_{\mathscr{C}}\lambda)$. Therefore the conclusion of Proposition 3.14 is false.

Proposition 3.16. Let (X,τ) be a fuzzy topological space. Let $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any subset λ of X, $SPBd_{\mathscr{C}}(SPInt_{\mathscr{C}}(\lambda)) \leq SPBd_{\mathscr{C}}(\lambda)$.

Proof. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.14, we have $SPBd_{\mathscr{C}}(SPInt_{\mathscr{C}}(\lambda)) = SPCl_{\mathscr{C}}(SPInt_{\mathscr{C}}(\lambda)) \wedge \mathscr{C}(SPInt_{\mathscr{C}}(SPInt_{\mathscr{C}}(\lambda)))$. Since $SPInt_{\mathscr{C}}(\lambda)$ is fuzzy $\mathscr{C}$-semi-pre open, $SPBd_{\mathscr{C}}(SPInt_{\mathscr{C}}(\lambda)) = SPCl_{\mathscr{C}}(SPInt_{\mathscr{C}}(\lambda)) \wedge \mathscr{C}(SPInt_{\mathscr{C}}(\lambda))$. Since $SPInt_{\mathscr{C}}(\lambda) \leq \lambda$, by using Proposition 5.6(ii) in [6], $SPCl_{\mathscr{C}}(SPInt_{\mathscr{C}}(\lambda)) \leq SPCl_{\mathscr{C}}(\lambda)$. Thus $SPBd_{\mathscr{C}}(SPInt_{\mathscr{C}}(\lambda)) \leq SPCl_{\mathscr{C}}(\lambda) \wedge \mathscr{C}(SPInt_{\mathscr{C}}(\lambda))$. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Proposition 5.5 in [6], $SPBd_{\mathscr{C}}(SPInt_{\mathscr{C}}(\lambda)) \leq SPCl_{\mathscr{C}}(\lambda) \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)$. By using Definition 3.1, we have $SPBd_{\mathscr{C}}(SPInt_{\mathscr{C}}(\lambda)) \leq SPBd_{\mathscr{C}}(\lambda)$.

Proposition 3.17. Let (X,τ) be a fuzzy topological space. Let $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then $SPBd_{\mathscr{C}}(SPCl_{\mathscr{C}}(\lambda)) \leq SPBd_{\mathscr{C}}(\lambda)$.

Proof. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.14, $SPBd_{\mathscr{C}}(SPCl_{\mathscr{C}}(\lambda)) = SPCl_{\mathscr{C}}(SPCl_{\mathscr{C}}(\lambda)) \wedge \mathscr{C}(SPInt_{\mathscr{C}}(SPCl_{\mathscr{C}}(\lambda)))$. By using Proposition 5.6(iii) in [6], we have $SPCl_{\mathscr{C}}(SPCl_{\mathscr{C}}(\lambda)) = SPCl_{\mathscr{C}}(\lambda)$, that implies $SPBd_{\mathscr{C}}(SPCl_{\mathscr{C}}(\lambda)) = SPCl_{\mathscr{C}}(\lambda) \wedge \mathscr{C}(SPInt_{\mathscr{C}}(SPCl_{\mathscr{C}}(\lambda)))$. Since $\lambda \leq SPCl_{\mathscr{C}}(\lambda)$, that implies $SPInt_{\mathscr{C}}(\lambda) \leq SPInt_{\mathscr{C}}(Cl_{\mathscr{C}}(\lambda))$. Therefore, $SPBd_{\mathscr{C}}(SPCl_{\mathscr{C}}(\lambda)) \leq SPCl_{\mathscr{C}}(\lambda) \wedge \mathscr{C}(SPInt_{\mathscr{C}}(\lambda))$. By using Proposition 5.5 (ii) in [6], and by using Definition 3.1, we get $SPBd_{\mathscr{C}}(SPCl_{\mathscr{C}}(\lambda)) \leq SPBd_{\mathscr{C}}(\lambda)$.

Theorem 3.18. Let (X,τ) be a fuzzy topological space. Let $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then $SPBd_{\mathscr{C}}(\lambda \vee \mu) \leq SPBd_{\mathscr{C}}\lambda \vee SPBd_{\mathscr{C}}\mu$.

Proof. By using Definition 3.1, $SPBd_{\mathscr{C}}(\lambda \vee \mu) = SPCl_{\mathscr{C}}(\lambda \vee \mu) \wedge SPCl_{\mathscr{C}}(\mathscr{C}(\lambda \vee \mu))$. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Proposition 5.7(i) in [6], that implies $SPBd_{\mathscr{C}}(\lambda \vee \mu) = (SPCl_{\mathscr{C}}(\lambda) \vee SPCl_{\mathscr{C}}(\mu)) \wedge SPCl_{\mathscr{C}}(\mathscr{C}(\lambda \vee \mu))$. By using Lemma 2.4 and Proposition 5.7(ii) in [6], $SPBd_{\mathscr{C}}(\lambda \vee \mu) \leq (SPCl_{\mathscr{C}}(\lambda) \vee SPCl_{\mathscr{C}}(\mu)) \wedge (SPCl_{\mathscr{C}}(\mathscr{C}\lambda) \wedge SPCl_{\mathscr{C}}(\mathscr{C}\mu))$. That is, $SPBd_{\mathscr{C}}(\lambda \vee \mu) \leq (SPCl_{\mathscr{C}}(\lambda) \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)) \vee (SPCl_{\mathscr{C}}(\mu)) \wedge SPCl_{\mathscr{C}}(\mathscr{C}\mu))$. Again by using Definition 3.1, $SPBd_{\mathscr{C}}(\lambda \vee \mu) \leq SPBd_{\mathscr{C}}(\lambda) \vee SPBd_{\mathscr{C}}(\mu)$.

Theorem 3.19. Let (X,τ) be a fuzzy topological space. Suppose the complement function $\mathscr{C}$ satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space X, we have
$SPBd_{\mathscr{C}}(\lambda \wedge \mu) \leq (SPBd_{\mathscr{C}}(\lambda) \wedge SPCl_{\mathscr{C}}(\mu)) \vee (SPBd_{\mathscr{C}}(\mu) \wedge SPCl_{\mathscr{C}}(\lambda))$.

Proof. By using Definition 3.1, we have $SPBd_{\mathscr{C}}(\lambda \wedge \mu) = SPCl_{\mathscr{C}}(\lambda \wedge \mu) \wedge SPCl_{\mathscr{C}}(\mathscr{C}(\lambda \wedge \mu))$. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Proposition 5.7(i), Proposition 5.7 (ii) in [6] and by using Lemma

2.4(iv), we get $SPBd_{\mathscr{C}}(\lambda\wedge\mu) \leq (SPCl_{\mathscr{C}}(\lambda)\wedge SPCl_{\mathscr{C}}(\mu))\wedge(SPCl_{\mathscr{C}}(\mathscr{C}\lambda)\vee SPCl_{\mathscr{C}}(\mathscr{C}\mu))$ is equal to $[SPCl_{\mathscr{C}}(\lambda)\wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)]\wedge(SPCl_{\mathscr{C}}(\mu))\vee[SPCl_{\mathscr{C}}(\mu)\wedge SPCl_{\mathscr{C}}(\mathscr{C}\mu)]\wedge SPCl_{\mathscr{C}}(\lambda)$. Again by using Definition 3.1, we get $SPBd_{\mathscr{C}}(\lambda\wedge\mu) \leq (SPBd_{\mathscr{C}}(\lambda)\wedge SPCl_{\mathscr{C}}(\mu))\vee(SPBd_{\mathscr{C}}(\mu)\wedge SPCl_{\mathscr{C}}(\lambda))$.

Proposition 3.20. Let (X, τ) be a fuzzy topological space. Suppose the complement function $\mathscr{C}$ satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of a fuzzy topological space X, we have

(i) $SPBd_{\mathscr{C}}(SPBd_{\mathscr{C}}(\lambda)) \leq SPBd_{\mathscr{C}}(\lambda)$

(ii) $SPBd_{\mathscr{C}}SPBd_{\mathscr{C}}SPBd_{\mathscr{C}}\lambda \leq SPBd_{\mathscr{C}}SPBd_{\mathscr{C}}\lambda$.

Proof. By using Definition 3.1, $SPBd_{\mathscr{C}}\lambda = SPCl_{\mathscr{C}}\lambda \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda)$.

We have $SPBd_{\mathscr{C}}SPBd_{\mathscr{C}}\lambda = SPCl_{\mathscr{C}}(SPBd_{\mathscr{C}}\lambda) \wedge SPCl_{\mathscr{C}}[\mathscr{C}(SPBd_{\mathscr{C}}\lambda)] \leq SPCl_{\mathscr{C}}(SPBd_{\mathscr{C}}\lambda)$. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Proposition 5.6(ii) in [6], $SPCl_{\mathscr{C}}\lambda = \lambda$, where λ is fuzzy $\mathscr{C}$-pre closed.

Here $SPBd_{\mathscr{C}}\lambda$ is fuzzy $\mathscr{C}$-pre closed. So, $SPCl_{\mathscr{C}}(SPBd_{\mathscr{C}}\lambda) = SPBd_{\mathscr{C}}\lambda$. This implies that $SPBd_{\mathscr{C}}SPBd_{\mathscr{C}}\lambda \leq SPBd_{\mathscr{C}}\lambda$. This proves (i). (ii) Follows from (i).

Proposition 3.21. Let λ be a fuzzy $\mathscr{C}$-semi-pre closed subset of a fuzzy topological space X and μ be a fuzzy $\mathscr{C}$-semi-pre closed subset of a fuzzy topological space Y, then $\lambda \times \mu$ is a fuzzy $\mathscr{C}$-semi-pre closed set of the fuzzy product space $X \times Y$ where the complement function $\mathscr{C}$ satisfies the monotonic and involutive conditions.

Proof. Let λ be a fuzzy $\mathscr{C}$-semi-pre closed subset of a fuzzy topological space X. Then by applying Lemma 2.8, $\mathscr{C}\lambda$ is fuzzy $\mathscr{C}$-semi-pre open set in X. Also if $\mathscr{C}\lambda$ is fuzzy $\mathscr{C}$- semi-pre- open set in X, then $\mathscr{C}\lambda \times 1$ is fuzzy $\mathscr{C}$-semi-pre-open in $X \times Y$. Similarly let μ be a fuzzy $\mathscr{C}$-semi-pre closed subset of a fuzzy topological space X. Then by using Lemma 2.8, $\mathscr{C}\mu$ is fuzzy $\mathscr{C}$-semi-pre open set in Y. Also if $\mathscr{C}\mu$ is fuzzy $\mathscr{C}$-semi-pre open set in Y then $\mathscr{C}\mu \times 1$ is fuzzy $\mathscr{C}$-semi-pre open in $X \times Y$. Since the arbitrary union of fuzzy $\mathscr{C}$- semi-pre open sets is fuzzy $\mathscr{C}$- semi-pre open. So, $\mathscr{C}\lambda \times 1 \vee 1 \times \mathscr{C}\mu$ is fuzzy $\mathscr{C}$-semi-preopen in $X \times Y$. By using Lemma 2.13, $\mathscr{C}(\lambda \times \mu) = \mathscr{C}\lambda \times 1 \vee 1 \times \mathscr{C}\mu$, hence $\mathscr{C}(\lambda \times \mu)$ is fuzzy $\mathscr{C}$-semi-pre open. By using Lemma 2.8, $\lambda \times \mu$ is fuzzy $\mathscr{C}$-semi-pre closed of the fuzzy product space $X \times Y$.

Theorem 3.22. Let $\mathscr{C}$ be a complement function that satisfies the monotonic and involutive conditions. If λ is a fuzzy subset of a fuzzy topological space X and μ is a fuzzy subset of a fuzzy topological space Y, then

(i) $SPCl_{\mathscr{C}}\lambda \times SPCl_{\mathscr{C}}\mu \geq SPCl_{\mathscr{C}}(\lambda \times \mu)$

(ii) $SPInt_{\mathscr{C}}\lambda \times SPInt_{\mathscr{C}}\mu \leq SPInt_{\mathscr{C}}(\lambda \times \mu)$.

Proof. By using Definition 2.11, $(SPCl_{\mathscr{C}}\lambda \times SPCl_{\mathscr{C}}\mu)(x, y) = \min\{SPCl_{\mathscr{C}}\lambda(x), SPCl_{\mathscr{C}}\mu(y)\} \geq \min\{\lambda(x), \mu(y)\} = (\lambda\times\mu)(x, y)$. This shows that $SPCl_{\mathscr{C}}\lambda \times SPCl_{\mathscr{C}}\mu \geq (\lambda \times \mu)$.

By using Proposition 5.6 in [6], $SPCl_{\mathscr{C}}(\lambda \times \mu) \leq SPCl_{\mathscr{C}}(SPCl_{\mathscr{C}}\lambda\times SPCl_{\mathscr{C}}\mu) = SPCl_{\mathscr{C}}\lambda\times SPCl_{\mathscr{C}}\mu$. By using Definition 2.11, $(SPInt_{\mathscr{C}}\lambda \times SPInt_{\mathscr{C}}\mu)(x, y) = \min\{SPInt_{\mathscr{C}}\lambda(x), SPInt_{\mathscr{C}}\mu(y)\} \leq \min\{\lambda(x), \mu(y)\} = (\lambda\times\mu)(x, y)$. This shows that $SPInt_{\mathscr{C}}\lambda \times SPInt_{\mathscr{C}}\mu \leq (\lambda \times \mu)$. By using Proposition 5.2 in [6], $SPInt_{\mathscr{C}}(SPInt_{\mathscr{C}}\lambda \times SPInt_{\mathscr{C}}\mu) \leq SPInt_{\mathscr{C}}(\lambda \times \mu)$, that implies $SPInt_{\mathscr{C}}\lambda \times SPInt_{\mathscr{C}}\mu \leq SPInt_{\mathscr{C}}(\lambda \times \mu)$.

Theorem 3.23. Let X and Y be $\mathscr{C}$-product related fuzzy topological spaces. Then for a fuzzy subset λ of X and a fuzzy subset μ of Y,

(i) $SPCl_{\mathscr{C}}(\lambda \times \mu) = SPCl_{\mathscr{C}}\lambda \times SPCl_{\mathscr{C}}\mu$

(ii) $SPInt_{\mathscr{C}}(\lambda \times \mu) = SPInt_{\mathscr{C}}\lambda \times SPInt_{\mathscr{C}}\mu$.

Proof. By using Theorem 3.22, it is sufficient to show that $SPCl_{\mathscr{C}}(\lambda\times\mu) \geq SPCl_{\mathscr{C}}\lambda\times SPCl_{\mathscr{C}}\mu$. By using Definition 5.4 in [6], we have $SPCl_{\mathscr{C}}(\lambda \times \mu) = \inf\{\mathscr{C}(\lambda_\alpha \times \mu_\beta): \mathscr{C}(\lambda_\alpha \times \mu_\beta) \geq \lambda \times \mu$ where λ_α and μ_β are fuzzy $\mathscr{C}$-semi-pre open$\}$. By using Lemma 2.11,

we have
$$
\begin{aligned}
SPCl_{\mathscr{C}}(\lambda \times \mu) &= \inf\{\mathscr{C}\lambda_\alpha \times 1 \vee 1 \times \mathscr{C}\lambda_\beta : \mathscr{C}\lambda_\alpha \times 1 \vee 1 \times \mathscr{C}\mu_\beta \geq \lambda \times \mu\}\\
&= \inf\{\mathscr{C}\lambda_\alpha \times 1 \vee 1 \times \mathscr{C}\mu_\beta : \mathscr{C}\lambda_\alpha \geq \lambda \text{ or } \mathscr{C}\mu_\beta \geq \mu\}\\
&= \min(\inf\{\mathscr{C}\lambda_\alpha \times 1 \vee 1 \times \mathscr{C}\mu_\beta : \mathscr{C}\lambda_\alpha \geq \lambda\}, \inf\{\mathscr{C}\lambda_\alpha \times 1\vee 1 \times \mathscr{C}\mu_\beta : \mathscr{C}\mu_\beta \geq \}).
\end{aligned}
$$

$$
\begin{aligned}
\text{Now } \inf\{\mathscr{C}\lambda_\alpha \times 1\vee 1 \times \mathscr{C}\mu_\beta : \mathscr{C}\lambda_\alpha \geq \lambda\} &\geq \inf\{\mathscr{C}\lambda_\alpha \times 1 : \mathscr{C}\lambda_\alpha \geq \lambda\}\\
&= \inf\{\mathscr{C}\lambda_\alpha : \mathscr{C}\lambda_\alpha \geq \lambda\} \times 1
\end{aligned}
$$

$$= (SPCl_{\mathscr{C}}\lambda) \times 1.$$

Also $\inf\{\, \mathscr{C}\lambda_\alpha \times 1 \vee 1 \times \mathscr{C}\mu_\beta : \mathscr{C}\mu_\beta \geq \mu \,\} \geq \inf\{\, 1 \times \mathscr{C}\mu_\beta : \mathscr{C}\mu_\beta \geq \mu \,\}$

$$= 1 \times \inf\{\, \mathscr{C}\mu_\beta : \mathscr{C}\mu_\beta \geq \mu \,\}$$
$$= 1 \times SPCl_{\mathscr{C}}\mu.$$

The above discussions imply that

$$SPCl_{\mathscr{C}}(\lambda \times \mu) \geq \min(SPCl_{\mathscr{C}}\lambda \times 1,\ 1 \times SPCl_{\mathscr{C}}\mu)$$
$$= SPCl_{\mathscr{C}}\lambda \times SPCl_{\mathscr{C}}\mu.$$

(ii) follows from (i) and using Proposition 5.5 in [6].

Theorem 3.24. Let X_i, i = 1, 2 ….n, be a family of $\mathscr{C}$-product related fuzzy topological spaces. If λ_i is a fuzzy subset of X_i, and the complement function $\mathscr{C}$ satisfies the monotonic and involutive conditions, then

$$SPBd_{\mathscr{C}}\Big(\prod_{i=1}^{n} \lambda_i\Big) = [SPBd_{\mathscr{C}}\lambda_1 \times SPCl_{\mathscr{C}}\lambda_2 \times \ldots \times SPCl_{\mathscr{C}}\lambda_n] \vee [SPCl_{\mathscr{C}}\lambda_1 \times SPBd_{\mathscr{C}}\lambda_2 \times \ldots \times SPCl_{\mathscr{C}}\lambda_n] \vee \ldots \vee [SPCl_{\mathscr{C}}\lambda_1 \times SPCl_{\mathscr{C}}\lambda_2 \times \ldots \times SPBd_{\mathscr{C}}\lambda_n].$$

Proof. It suffices to prove this for n = 2. By using Proposition 3.14,

we have $SPBd_{\mathscr{C}}(\lambda_1 \times \lambda_2) = SPCl_{\mathscr{C}}(\lambda_1 \times \lambda_2) \wedge \mathscr{C}(SPInt_{\mathscr{C}}(\lambda_1 \times \lambda_2))$

$$= (SPCl_{\mathscr{C}}\lambda_1 \times SPCl_{\mathscr{C}}\lambda_2) \wedge \mathscr{C}(SPInt_{\mathscr{C}}\lambda_1 \times SPInt_{\mathscr{C}}\lambda_2) \quad \text{[by using Theorem 3.23]}$$
$$= (SPCl_{\mathscr{C}}\lambda_1 \times SPCl_{\mathscr{C}}\lambda_2) \wedge \mathscr{C}[(SPInt_{\mathscr{C}}\lambda_1 \wedge SPCl_{\mathscr{C}}\lambda_1) \times (SPInt_{\mathscr{C}}\lambda_2 \wedge SPCl_{\mathscr{C}}\lambda_2)]$$
$$= (SPCl_{\mathscr{C}}\lambda_1 \times SPCl_{\mathscr{C}}\lambda_2) \wedge [\mathscr{C}(SPInt_{\mathscr{C}}\lambda_1 \wedge SPCl_{\mathscr{C}}\lambda_1) \times 1 \vee 1 \times \mathscr{C}(SPInt_{\mathscr{C}}\lambda_2 \wedge SPCl_{\mathscr{C}}\lambda_1)].$$

[By Lemma 2.13]. Since $\mathscr{C}$ satisfies the monotonic and involutive conditions, by using Proposition 5.5(i), Proposition 5.5(i) in [6] and also by using Lemma 2.11,

$$SPBd_{\mathscr{C}}(\lambda_1 \times \lambda_2) = (SPCl_{\mathscr{C}}\lambda_1 \times SPCl_{\mathscr{C}}\lambda_2) \wedge [(SPCl_{\mathscr{C}}\mathscr{C}\lambda_1 \vee SPInt_{\mathscr{C}}\mathscr{C}\lambda_1) \times 1 \vee 1 \times (SPCl_{\mathscr{C}}\mathscr{C}\lambda_2 \vee SPInt_{\mathscr{C}}\mathscr{C}\lambda_2)]$$
$$= (SPCl_{\mathscr{C}}\lambda_1 \times SPCl_{\mathscr{C}}\lambda_2) \wedge [SPCl_{\mathscr{C}}\mathscr{C}\lambda_1 \times 1 \vee 1 \times SPCl_{\mathscr{C}}\mathscr{C}\lambda_2]$$
$$= [(SPCl_{\mathscr{C}}\lambda_1 \times SPCl_{\mathscr{C}}\lambda_2) \wedge (SPCl_{\mathscr{C}}(\mathscr{C}\lambda_1) \times 1)] \vee [(SPCl_{\mathscr{C}}(\lambda_1) \times SPCl_{\mathscr{C}}\lambda_2) \wedge (1 \times SPCl_{\mathscr{C}}(\mathscr{C}\lambda_2))].$$

Again by using Lemma 2.9, we get

$$SPBd_{\mathscr{C}}(\lambda_1 \times \lambda_2) = [(SPCl_{\mathscr{C}}\lambda_1 \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda_1)) \times (1 \wedge SPCl_{\mathscr{C}}\lambda_2)] \vee [(SPCl_{\mathscr{C}}\lambda_2 \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda_2)) \times (1 \wedge SPCl_{\mathscr{C}}\lambda_1)]$$
$$= [(SPCl_{\mathscr{C}}(\lambda_1) \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda_1)) \times SPCl_{\mathscr{C}}(\lambda_2)] \vee [(SPCl_{\mathscr{C}}(\lambda_2) \wedge SPCl_{\mathscr{C}}(\mathscr{C}\lambda_2)) \times SPCl_{\mathscr{C}}(\lambda_1)]$$

$$SPBd_{\mathscr{C}}(\lambda_1 \times \lambda_2) = [SPBd_{\mathscr{C}}(\lambda_1) \times SPCl_{\mathscr{C}}(\lambda_2)] \vee [SPCl_{\mathscr{C}}(\lambda_1) \times SPBd_{\mathscr{C}}(\lambda_2)].$$

Theorem 3.25. Let f: $X \to Y$ be a fuzzy continuous function. Suppose the complement function $\mathscr{C}$ satisfies the monotonic and involutive conditions. Then

$$SPBd_{\mathscr{C}}(f^{-1}(\mu)) \leq f^{-1}(SPBd_{\mathscr{C}}(\mu)), \text{ for any fuzzy subset } \mu \text{ in } Y.$$

Proof. Let f be a fuzzy continuous function and μ be a fuzzy subset in Y. By using Definition 3.1, we have $SPBd_{\mathscr{C}}(f^{-1}(\mu)) = SPCl_{\mathscr{C}}(f^{-1}(\mu)) \wedge SPCl_{\mathscr{C}}(\mathscr{C}(f^{-1}(\mu))$. By using Lemma 2.10, $SPBd_{\mathscr{C}}(f^{-1}(\mu)) = SPCl_{\mathscr{C}}(f^{-1}(\mu)) \wedge SPCl_{\mathscr{C}}(f^{-1}(\mathscr{C}\mu))$.

Since f is fuzzy continuous and $f^{-1}(\mu) \leq f^{-1}(SPCl_{\mathscr{C}}(\mu))$, it follows that $SPCl_{\mathscr{C}}(f^{-1}(\mu)) \leq f^{-1}(SPCl_{\mathscr{C}}(\mu))$. This together with the above imply that $SPBd_{\mathscr{C}}(f^{-1}(\mu)) \leq f^{-1}(SPCl_{\mathscr{C}}(\mu)) \wedge f^{-1}(SPCl_{\mathscr{C}}(\mathscr{C}\mu))$. By using Lemma 2.12, $SPBd_{\mathscr{C}}(f^{-1}(\mu)) \leq f^{-1}(SPCl_{\mathscr{C}}(\mu) \wedge SPCl_{\mathscr{C}}(\mathscr{C}\mu))$. That is $SPBd_{\mathscr{C}}(f^{-1}(\mu)) \leq f^{-1}(SPBd_{\mathscr{C}}(\mu))$.

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