ON SOME FIXED POINT THEOREMS IN 2–UNIFORM SPACES

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ABSTRACT

In this paper, some fixed point theorems in 2–uniform spaces are established and contraction type mappings in 2–uniform spaces are introduced.

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Key words: Pseudo 2–metric, Uniformity, 2–uniform space, Contraction type mapping.

INTRODUCTION

In this paper we introduce contraction type mappings in 2–uniform spaces and we present some fixed point theorems of operators in 2–uniform spaces. These theorems generalize the results of many authors such as Lal and Singh, Das and Sharma etc.

In what follows $X$ and $\mathbb{R}$ stand for a non-empty set and the real line respectively and $X^3 = X \times X \times X$. If $A$ and $B$ are any two sets then by the symbol $A \leq B$ we mean that $A$ is contained in $B$.

1. PRELIMINARIES

1.1 Definition: A pseudo 2–metric for $X$ is a mapping $\rho : X^3 \to \mathbb{R}$ such that for all $a, b, c$ and $d$ in $X$ we have

(i) $\rho(a, b, c) > 0$ and $\rho(a, b, c) = 0$ if at least two of $a, b, c$ are equal.

(ii) $\rho(a, b, c) = \rho(b, c, a) = \rho(c, a, b) = ...$

(iii) $\rho(a, b, c) \leq \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c)$

A set $X$ together with a pseudo 2–metric $\rho$ is called a pseudo 2–metric space $(X, \rho)$.

1.2 Definition: If $U$ is any subset of $X^3$ then $U^{-1} = \{(z, y, x) / (x, y, z) \in U\}$. We define the diagonal of $X^3$ to be the set $\Delta = \{(x, x, x) / x \in X\}$.

1.3 Definition: A 2–uniformity for $X$ is a non-void family $\%$ of subsets of $X^3$ such that

(i) each member of $\%$ contains $\Delta$

(ii) if $U \in \%$ then $V \circ V \circ V \leq U$ for some $V$ in $\%$

(iii) if $U$ and $V$ are two members of $U$ then $U \bigcap V \in \%$

(iv) if $U \in \%$ and $U \leq V \leq X^3$ then $V \in \%$

By a 2–uniform space we mean a non-empty set $X$ together with a 2–uniformity $\%$ on $X$ and we denote it by $(X, \%)$.

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1.4 Definition: If \((X, \mathcal{U})\) is 2–uniform space then a subset \(\mathcal{B}\) of \(\mathcal{U}\) is called a basis for \((X, \mathcal{U})\)
(i) if each member of \(\mathcal{B}\) contains the diagonal \(\Delta\)
(ii) \(U \in \mathcal{B}\) then \(U^{-1}\) contains a member of \(\mathcal{B}\)
(iii) if \(U \in \mathcal{B}\) then \(V \circ V \circ V \leq U\) for some \(V\) in \(\mathcal{B}\)
(iv) the intersection of two members of \(\mathcal{B}\) contains a member of \(\mathcal{B}\)

1.5 Remark: By \(V \circ V \circ V\) we mean that the composition by treating \(V\) as a relation in the order \(V : X \times X \to X\)
and \(V : X \to X \times X\) respectively.

1.6 Definition: A 2–uniform space \((X, \mathcal{U})\) is said to be sequentially complete if every cauchy sequence in \(X\)
converges to a point in \(X\).

1.7 Definition: If \((X, \rho)\) is a pseudo 2–metric space and if \(r\) is a positive real number then we define
\[V_{(\rho,r)} = \left\{(x,y,z) \in X^3 \mid \rho(x,y,z) < r\right\} .\]

1.8 Notation:
1. We denote \(\mathcal{P}\) for the family of pseudo 2–metrics on \(X\) generating the uniformity.
2. \(V\) denotes family of all sets of the form \(\bigcap_{i=1}^{n} V_{(\rho_i, r_i)}\) where \(\rho_i \in \mathcal{P}\) and \(r_i\) is a positive real number for \(i = 1, 2, 3, \ldots n\) (\(n\) is not fixed).
3. If \(V \in \mathcal{V}\) then \(V = \bigcap_{i=1}^{n} V_{(\rho_i, r_i)}\). If \(\alpha\) is positive then \(\alpha V = \bigcap_{i=1}^{n} V_{(\alpha \rho_i, \alpha r_i)}\).

1.9 Definition: Let \(\mathcal{B}\) be a basis for the 2–uniform space \((X, \mathcal{U})\) and let \(f\) be a mapping from \(X\) into itself.
(a) \(f\) is said to be a contraction with respect to \(\mathcal{B}\) if \(f(x, y, z) \in U\) whenever \((x, y, z) \in U \in \mathcal{B}\).
(b) \(f\) is said to be an expansion with respect to \(\mathcal{B}\) if \((x, y, z) \in U\) whenever \((f(x), f(y), z) \in U \in \mathcal{B}\).

2. SOME PRELIMINARY LEMMAS

2.1 Lemma: If \(V \in \mathcal{V}\) and \(\alpha, \beta\) are positive then \(\alpha V = (\alpha \beta)V\).

2.2 Lemma: If \(V \in \mathcal{V}\) and \(\alpha, \beta\) are positive then \(\alpha V \leq \beta V\) whenever \(\alpha \leq \beta\).

2.3 Lemma: Let \(\rho\) be any pseudo 2–metric on \(X\) and \(\alpha, \beta\) be any two positive real numbers. If \((x, y, z) \in \alpha V_{(\rho, r_1)} \circ \beta V_{(\rho, r_2)}\) then \(\rho(x, y, z) < \alpha r_1 + \beta r_2\).

2.4 Lemma: If \(V \in \mathcal{V}\) and \(\alpha, \beta\) are two positive real numbers then \(\alpha V \circ \beta V \leq (\alpha + \beta) V\).

2.5 Lemma: Let \((x, y, z) \in X^3\). Then for every \(V \in \mathcal{V}\) there exists a positive real number \(\alpha\) such that \((x, y, z) \in a V\).

2.6 Lemma: If \(V \in \mathcal{V}\) then there exists a pseudo 2–metric \(\rho\) on \(X\) such that \(V = V_{(\rho, 1)}\).

3. SOME FIXED POINT THEOREMS OF OPERATORS

In this section, we assume that \((X, \mathcal{U})\) is a 2–uniform space which is also sequentially complete Hausdorff space.

3.1 Theorem: Let \(\mathcal{A} = \{S_1, S_2, \ldots, S_p\}\) and \(\mathcal{B} = \{T_1, T_2, \ldots, T_q\}\) be two sets of operators such that
(a) each \(S_i\) and \(T_j\) map \(X\) into itself
(b) \(S_i S_j = S_j S_i\) for \(1 \leq i, j \leq p\) and \(T_\alpha T_\beta = T_\beta T_\alpha\) for \(1 \leq \alpha, \beta \leq q\)
(c) for all \( x, y \in X \), for every \( a \in X \) and each \( \rho \in P \) and for any five members \( V_1, V_2, V_3, V_4, V_5 \) in \( V \),

\[
(S(x), T(y), a) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5 \quad \text{where} \quad S \in \mathcal{A} \quad \text{and} \quad T \in \mathcal{B} \quad \text{and each} \quad \alpha_i \quad \text{is a non negative real number independent of} \quad x, y, a, V_1, V_2, V_3, V_4, V_5 \quad \text{such that}
\]

\[
0 < \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3}, \quad \frac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_4 - \alpha_5} < 1, 1 - \alpha_2 - \alpha_3 \neq 0, 1 - \alpha_4 - \alpha_5 \neq 0.
\]

If \((x, S(x), a) \in V_1,(y, T(x), a) \in V_2,(x, T(y), a) \in V_3,(y, S(x), a) \in V_4,(x, y, a) \in V_5 \) then all \( S_i (1 \leq i \leq p) \) and \( T_j (1 \leq j \leq q) \) have a common unique fixed point.

**Proof:** Clearly \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1 \). Suppose that \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1 \). Let \( V \in V \) and \( \rho \in P \) suppose that \( x, y, a \) are any three points of \( X \). Put \( \rho(x, S(x), a) = r_1 \), \( \rho(y, T(y), a) = r_2, \rho(x, T(y), a) = r_3, \rho = (y, S(x), a) = r_4 \) and \( \rho(x, y, a) = r_5 \) and take \( \varepsilon > 0 \), then \((x, S(x), a) \in (r_1 + \varepsilon) V, (y, T(y), a) \in (r_2 + \varepsilon) V, (x, S(x), a) \in (r_3 + \varepsilon) V, (y, T(y), a) \in (r_4 + \varepsilon) V, (x, y, a) \in (r_5 + \varepsilon) V) \).

Then we have

\[
(S(x), T(y), a) \in \alpha_1 (r_1 + \varepsilon) V \circ \alpha_2 (r_2 + \varepsilon) V \circ \alpha_3 (r_3 + \varepsilon) V \circ \alpha_4 (r_4 + \varepsilon) V \circ \alpha_5 (r_5 + \varepsilon) V
\]

\[\Rightarrow \rho(S(x), T(y), a) \leq \alpha_1 (r_1 + \varepsilon) + \alpha_2 (r_2 + \varepsilon) + \alpha_3 (r_3 + \varepsilon) + \alpha_4 (r_4 + \varepsilon) + \alpha_5 (r_5 + \varepsilon)
\]

\[= \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4 + \alpha_5 r_5 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \varepsilon
\]

Since \( \varepsilon > 0 \) is arbitrary, we have \( \rho(S(x), T(y), a) \leq \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4 + \alpha_5 r_5 \).

Fix \( x_0 \in X \). Construct a sequence \( \{x_n\} \) in \( X \) such that \( x_{2n+1} = S(x_{2n}) \) and \( x_{2n+2} = T(x_{2n+1}) \) where \( n = 0, 1, 2, 3, \ldots \).

Clearly \( \{x_n\} \) is a Cauchy sequence in \( X \) and hence there exists a point \( u \) in \( X \) such that \( u = \lim_{n \to \infty} x_n \).

Then

\[
\rho(u, S(u), a) \leq \rho(u, S(u), x_{2n}) + \rho(u, x_{2n}, a) + \rho(x_{2n}, S(u), a)
\]

\[= \rho(u, S(u), x_{2n}) + \rho(u, x_{2n}, a) + \rho(T(x_{2n-1}), S(u), a)
\]

\[\leq \rho(u, S(u), x_{2n}) + \rho(u, x_{2n}, a) + \alpha_i \rho(u, S(u), a) + \alpha_2 \rho(x_{2n-1}, T(x_{2n-1}), a)
\]

\[+ \alpha_4 \rho(u, x_{2n-1}, a) + \alpha_5 \rho(x_{2n-1}, S(u), a) + \alpha_6 \rho(x_{2n}, a)
\]

Letting \( n \to \infty \), we get \((1 - \alpha_i - \alpha_4) \rho(u, S(u), a) \leq 0 \Rightarrow \rho(u, S(u), a) = 0 \Rightarrow u = S(u) \Rightarrow u \) is a fixed point of \( S \).

Similarly \( u \) is a fixed point of \( T \). Furthermore \( u \) is unique common fixed point of \( S \) and \( T \).

**3.2 Theorem:** Suppose that \( S : X \to X \) and \( T : X \to X \) are two operators such that

(a) \( ST = TS \)  
(b) For all \( x, y, z_1, z_2 \in X \), for each \( \rho \in P \) and for any six members \( V_1, V_2, V_3, V_4, V_5, V_6 \) in \( V \),

\[
(S(x), T(y), a) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5 \circ \alpha_6 V_6.
\]

If \((x, S^k(z_1), a) \in V_1,(y, T^k(z_2), a) \in V_2,(x, S^k(z_2), a) \in V_3,(y, S^k(z_1), a) \in V_4,(S^k(z_1), T^k(z_2), a) \in V_5\), and \((x, y, a) \in V_6 \) where \( k \) is a positive integer and \( \sum_{j=1}^{6} \alpha_j > 1 \) then \( S \) and \( T \) have a unique common fixed point in \( X \).
3.3 Theorem: Suppose that \( S : X \rightarrow X \) and \( T : X \rightarrow X \) are two operators such that
(a) \( ST = TS \)
(b) For all \( x, y, z_1, z_2 \) in \( X \), for each \( \rho \in \mathcal{P} \) and for any five members \( V_1, V_2, V_3, V_4, V_5 \) in \( \mathcal{V} \),
\[
(S(x), T(y), a) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5.
\]
If \( (x, S^k(z_1), a) \in V_1, (y, T^k(z_2), a) \in V_2, (x, S^k(z_2), a) \in V_3, (y, S^k(z_2), a) \in V_4, (x, y, a) \in V_5 \) where \( k \) is a positive integer and \( \sum_{i=1}^{5} \alpha_i > 1 \) then \( S \) and \( T \) have a unique common fixed point in \( X \).

3.4 Theorem: Suppose that \( S : X \rightarrow X \) and \( T : X \rightarrow X \) are two operators such that
(a) \( ST = TS \)
(b) For all \( x, y, z_1, z_2, z_3, z_4 \) in \( X \), for each \( \rho \in \mathcal{P} \) and for any four members \( V_1, V_2, V_3, V_4 \) in \( \mathcal{V} \),
\[
(S(x), T(y), a) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4.
\]
If \( (x, S^k(z_1), a) \in V_1, (y, T^k(z_2), a) \in V_2, (S(z_1), S^k(z_1), a) \in V_3, (T(y), S^k(z_4), a) \in V_4 \) where \( k \) is a positive integer and \( \sum_{i=1}^{4} \alpha_i > 1 \) then \( S \) and \( T \) have a unique common fixed point in \( X \).

REFERENCES

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