International Journal of Mathematical Archive-3(12), 2012, 4879-4882
Available online through www.ijma.info ISSN 2229 – 5046

# ON SOME FIXED POINT THEOREMS IN 2–UNIFORM SPACES

*V. Srinivasa kumar & **T. V. L. Narayana

*Assistant Professor, Department of Mathematics, JNTUH College of Engineering, JNTU, Hyderabad-500085, A.P., India

**D.No-4-5-14, SNP Agraharam, 2nd line, Bapatla-522101, Guntur (Dt), A.P. India

(Received on: 23-11-12; Revised & Accepted on: 24-12-12)

### ABSTRACT

*In this paper, some fixed point theorems in 2–uniform spaces are established and contraction type mappings in 2 – uniform spaces are introduced.*

***AMS subject Classification:*** *37C25, 47H10*

***Key words:*** *Pseudo 2 – metric, Uniformity, 2 – uniform space, Contraction type mapping.*

## INTRODUCTION

In this paper we introduce contraction type mappings in 2 – uniform spaces and we present some fixed point theorems of operators in 2 – uniform spaces. These theorems generalize the results of many authors such as *Lal* and *Singh, Das* and *Sharma* etc.

In what follows $X$ and $\mathbb{R}$ stand for a non-empty set and the real line respectively and $X^3 = X \times X \times X$. If $A$ and $B$ are any two sets then by the symbol $A \leq B$ we mean that $A$ is contained in $B$.

## 1. PRELIMINARIES

**1.1 Definition:** A pseudo 2–metric for $X$ is a mapping $\rho : X^3 \to \mathbb{R}$ such that for all $a, b, c$ and $d$ in $X$ we have
$(i)$ $\rho(a,b,c) > 0$ and $\rho(a,b,c) = 0$ if at least two of $a, b, c$ are equal.
$(ii)$ $\rho(a,b,c) = \rho(b,c,a) = \rho(c,a,b) = ...$
$(iii)$ $\rho(a,b,c) \leq \rho(a,b,d) + \rho(a,d,c) + \rho(d,b,c)$

A set $X$ together with a pseudo 2 – metric $\rho$ is called a pseudo 2 – metric space $(X, \rho)$.

**1.2 Definition:** If $U$ is any subset of $X^3$ then $U^{-1} = \{(z,y,x) / (x,y,z) \in U\}$. We define the diagonal of $X^3$ to be the set $\Delta = \{(x,x,x) / x \in X\}$.

**1.3 Definition:** A 2 – uniformity for $X$ is a non-void family $\mathscr{U}$ of subsets of $X^3$ such that
$(i)$ each member of $\mathscr{U}$ contains $\Delta$
$(ii)$ if $U \in \mathscr{U}$ then $V \circ V \circ V \leq U$ for some $V$ in $\mathscr{U}$
$(iii)$ if $U$ and $V$ are two members of $U$ then $U \bigcap V \in \mathscr{U}$
$(iv)$ if $U \in \mathscr{U}$ and $U \leq V \leq X^3$ then $V \in \mathscr{U}$

By a 2 – uniform space we mean a non-empty set $X$ together with a 2 – uniformity $\mathscr{U}$ on $X$ and we denote it by $(X, \mathscr{U})$.

**Corresponding author:** *V. Srinivasa kumar
*Assistant Professor, Department of Mathematics, JNTUH College of Engineering, JNTU, Hyderabad-500085, A.P., India

**1.4 Definition:** If $(X,\mathscr{U})$ is 2 – uniform space then a subset $\mathscr{B}$ of $\mathscr{U}$ is called a basis for $(X,\mathscr{U})$

$(i)$ if each member of $\mathscr{B}$ contains the diagonal $\Delta$

$(ii)$ $U \in \mathscr{B}$ then $U^{-1}$ contains a member of $\mathscr{B}$

$(iii)$ if $U \in \mathscr{B}$ then $V \circ V \circ V \le U$ for some $V$ in $\mathscr{B}$

$(iv)$ the intersection of two members of $\mathscr{B}$ contains a member of $\mathscr{B}$

**1.5 Remark:** By $V \circ V \circ V$ we mean that the composition by treating $V$ as a relation in the order $V : X \to X \times X$, $V : X \times X \to X$ and $V : X \to X \times X$ respectively.

**1.6 Definition:** A 2 – uniform space $(X,\mathscr{U})$ is said to be sequentially complete if every cauchy sequence in $X$ converges to a point in $X$ .

**1.7 Definition:** If $(X,\rho)$ is a pseudo 2–metric space and if $r$ is a positive real number then we define $V_{(\rho,r)} = \left\{(x,y,z) \in X^3 / \rho(x,y,z) < r\right\}$.

**1.8 Notation:**

1. We denote $\mathcal{P}$ for the family of pseudo 2 – metrics on $X$ generating the uniformity.

2. $\mathcal{V}$ denotes family of all sets of the form $\bigcap_{i=1}^{n} V_{(\rho_i,r_i)}$ where $\rho_i \in \mathcal{P}$ and $r_i$ is a positive real number for $i = 1,2,3,...n$ ($n$ is not fixed).

3. If $V \in \mathcal{V}$ then $V = \bigcap_{i=1}^{n} V_{(\rho_i,r_i)}$. If $\alpha$ is positive then $\alpha V = \bigcap_{i=1}^{n} V_{(\rho_i,\alpha r_i)}$.

**1.9 Definition:** Let $\mathscr{B}$ be a basis for the 2 – uniform space $(X,\mathscr{U})$ and let $f$ be a mapping from $X$ into itsself.

(a) $f$ said to be a contraction with respect to $\mathscr{B}$ if $\left(f(x), f(y), z\right) \in U$ whenever $(x,y,z) \in U \in \mathscr{B}$.

(b) $f$ said to be an expansion with respect to $\mathscr{B}$ if $(x,y,z) \in U$ whenever $\left(f(x), f(y), z\right) \in U \in \mathscr{B}$.

## 2. SOME PRELIMINARY LEMMAS

**2.1 Lemma:** If $V \in \mathcal{V}$ and $\alpha, \beta$ are positive then $\alpha(\beta V) = (\alpha\beta)V$ .

**2.2 Lemma:** If $V \in \mathcal{V}$ and $\alpha, \beta$ are positive then $\alpha V \le \beta V$ whenever $\alpha \le \beta$ .

**2.3 Lemma:** Let $\rho$ be any pseudo 2 – metric on $X$ and $\alpha, \beta$ be any two positive real numbers. If $(x,y,z) \in \alpha V_{(\rho,r_1)} \circ \beta V_{(\rho,r_2)}$ then $\rho(x,y,z) < \alpha r_1 + \beta r_2$ .

**2.4 Lemma:** If $V \in \mathcal{V}$ and $\alpha, \beta$ are two positive real numbers then $\alpha V \circ \beta V \le (\alpha - \beta)V$ .

**2.5 Lemma:** Let $(x,y,z) \in X^3$. Then for every $V \in \mathcal{V}$ there exists a positive real number $\alpha$ such that $(x,y,z) \in \alpha V$ .

**2.6 Lemma:** If $V \in \mathcal{V}$ then there exists a pseudo 2 – metric $\rho$ on $X$ such that $V = V_{(\rho,1)}$ .

## 3. SOME FIXED POINT THEOREMS OF OPERATORS

In this section, we assume that $(X,\mathscr{U})$ is a 2 – uniform space which is also sequentially complete Hausdorff space.

**3.1 Theorem:** Let $\mathscr{A} = \left\{S_1, S_2, ..., S_p\right\}$ and $\mathscr{B} = \left\{T_1, T_2, ..., T_q\right\}$ be two sets of operators such that

$(a)$ each $S_i$ and $T_j$ map $X$ into itself

$(b)$ $S_i S_j = S_j S_i$ for $1 \le i, j \le p$ and $T_\alpha T_\beta = T_\beta T_\alpha$ for $1 \le \alpha, \beta \le q$

 

$(c)$ for all $x, y \in X$, for every $a \in X$ and each $\rho \in \mathcal{P}$ and for any five members $V_1, V_2, V_3, V_4, V_5$ in $\mathcal{V}$, $(S(x), T(y), a) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5$ where $S \in \mathcal{A}$ and $T \in \mathcal{B}$ and each $\alpha_i$ is a non negative real number independent of $x, y, a, V_1, V_2, V_3, V_4$ and $V_5$ such that

$$0 < \frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3}, \frac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_4} < 1, 1 - \alpha_2 - \alpha_3 \neq 0, 1 - \alpha_1 - \alpha_4 \neq 0.$$

If $\left(x, S(x), a\right) \in V_1, \left(y, T(x), a\right) \in V_2, \left(x, T(y), a\right) \in V_3, \left(y, S(x), a\right) \in V_4, \left(x, y, a\right) \in V_5$ then all $S_i (1 \leq i \leq p)$ and $T_j (1 \leq j \leq q)$ have a common unique fixed point.

**Proof:** Clearly $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$. Suppose that $k_1 = \dfrac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3}$ and $k_2 = \dfrac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_4}$. Let $V \in \mathcal{V}$ and $\rho \in \mathcal{P}$ suppose that $x, y, a$ are any three points of $X$.. Put $\rho(x, S(x), a) = r_1$, $\rho(y, T(y), a) = r_2, \rho(x, T(y), a) = r_3, \rho = (y, S(x), a) = r_4$ and $\rho(x, y, a) = r_5$ and take $\varepsilon > 0$, then $(x, S(x), a) \in (r_1 + \varepsilon) V$, $(y, T(y), a) \in (r_2 + \varepsilon) V$, $(x, S(x), a) \in (r_3 + \varepsilon) V$, $(y, T(y), a) \in (r_4 + \varepsilon) V$, $(x, y, a) \in (r_5 + \varepsilon) V$.

Then we have

$$(S(x), T(y), a) \in \alpha_1 (r_1 + \varepsilon) V \circ \alpha_2 (r_2 + \varepsilon) V \circ \alpha_3 (r_3 + \varepsilon) V \circ \alpha_3 (r_3 + \varepsilon) V \circ \alpha_4 (r_4 + \varepsilon) V \circ \alpha_5 (r_5 + \varepsilon)$$

$$\begin{aligned} \Rightarrow \rho(S(x), T(y), a) &\leq \alpha_1 (r_1 + \varepsilon) + \alpha_2 (r_2 + \varepsilon) + \alpha_3 (r_3 + \varepsilon) + \alpha_4 (r_4 + \varepsilon) + \alpha_5 (r_5 + \varepsilon) \\ &= \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4 + \alpha_5 r_5 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\rho(S(x), T(y), a) \leq \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4 + \alpha_5 r_5$.

Fix $x_0 \notin X$. Construct a sequence $\{x_n\}$ in $X$ such that $x_{2n+1} = S(x_{2n})$ and $x_{2n+2} = T(x_{2n+1})$ where $n = 0, 1, 2, 3, \ldots$.

Clearly $\{x_n\}$ is a Cauchy sequence in $X$ and hence there exists a point $u$ in $X$ such that $u = \lim_{n \to \infty} x_n$.

Then

$$\begin{aligned} \rho(u, S(u), a) &\leq \rho(u, S(u), x_{2n}) + \rho(u, x_{2n}, a) + \rho(x_{2n}, S(u), a) \\ &= \rho(u, S(u), x_{2n}) + \rho(u, x_{2n}, a) + \rho(T(x_{2n-1}), S(u), a) \\ &\leq \rho(u, S(u), x_{2n}) + \rho(u, x_{2n}, a) + \alpha_1 \rho(u, S(u), a) + \alpha_2 \rho(x_{2n-1}, T(x_{2n-1}), a) \\ &\quad + \alpha_3 \rho(u, x_{2n-1}, a) + \alpha_4 \rho(x_{2n-1}, S(u), a) + \alpha_5 \rho(x_{2n}, a) \end{aligned}$$

Letting $n \to \infty$, we get $(1 - \alpha_1 - \alpha_4) \rho(u, S(u), a) \leq 0 \Rightarrow \rho(u, S(u), a) = 0 \Rightarrow u = S(u)$

$\Rightarrow u$ is a fixed point of $S$.

Similarly $u$ is a fixed point of $T$. Furthermoer $u$ is unique common fixed point of $S$ and $T$.

**3.2 Theorem:** Suppose that $S: X \to X$ and $T: X \to X$ are two operators such that

(a) $ST = TS$ (b) For all $x, y, z_1, z_2$ in $X$, for each $\rho \in \mathcal{P}$ and for any six members $V_1, V_2, V_3, V_4, V_5, V_6$ in $\mathcal{V}$, $\left(S(x), T(y), a\right) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5 \circ \alpha_6 V_6$.

If $\left(x, S^k(z_1), a\right) \in V_1, \left(y, T^k(z_2), a\right) \in V_2, \left(x, S^k(z_2), a\right) \in V_3, \left(y, S^k(z_2), a\right) \in V_4$, $\left(S^k(z_1), T^k(z_2), a\right) \in V_5$ and $\left(x, y, a\right) \in V_6$ where $k$ is a positive integer and $\sum_{i=1}^{6} \alpha_i > 1$ then $S$ and $T$ have a unique common fixed point in $X$.

 

**3.3 Theorem:** Suppose that $S: X \to X$ and $T: X \to X$ are two operators such that
(a) $ST = TS$ (b) For all $x, y, z_1, z_2$ in $X$, for each $\rho \in \mathcal{P}$ and for any five members $V_1, V_2, V_3, V_4, V_5$ in $\mathcal{V}$, $\left(S(x), T(y), a\right) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5$.

If $\left(x, S^k(z_1), a\right) \in V_1, \left(y, T^k(z_2), a\right) \in V_2, \left(x, S^k(z_2), a\right) \in V_3, \left(y, S^k(z_2), a\right) \in V_4, \left(x, y, a\right) \in V_5$ where $k$ is a positive integer and $\sum_{i=1}^{5} \alpha_i > 1$ then $S$ and $T$ have a unique common fixed point in $X$.

**3.4 Theorem:** Suppose that $S: X \to X$ and $T: X \to X$ are two operators such that
(a) $ST = TS$ (b) For all $x, y, z_1, z_2, z_3, z_4$ in $X$, for each $\rho \in \mathcal{P}$ and for any four members $V_1, V_2, V_3, V_4$ in $\mathcal{V}$, $\left(S(x), T(y), a\right) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4$.

If $\left(x, S^k(z_1), a\right) \in V_1, \left(y, T^k(z_2), a\right) \in V_2, \left(S(z_1), S^k(z_3), a\right) \in V_3, \left(T(y), S^k(z_4), a\right) \in V_4$ where $k$ is a positive integer and $\sum_{i=1}^{4} \alpha_i > 1$ then $S$ and $T$ have a unique common fixed point in $X$.

## REFERENCES


**1.** Das, B.K., and Sharma A.K., *A Fixed point theorem*, Bull. Cal. Math. Soc., Vol. 72, p.p 263-266, 1980.

**2.** Kelley, J. L., General *Topology*, Springer – Verlag, New York, 1955.

**3.** Lal, S.N., and Singh, A.K., An *analogue of Banach's contraction principal for 2-metric spaces*, Bull. Austral. Math.Soc. Vol. 18, p. p 137-143, 1978.

**4.** Singh, U.N., and Singh, R.K., *Generalized fixed point theorems in metric spaces*, Jnanabha, Vol. 18, p. p 151-163.


**Source of support: Nil, Conflict of interest: None Declared**