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MAXIMUM TERMS OF COMPOSITE ENTIRE FUNCTIONS AND THEIR COMPARATIVE GROWTH RATES

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ABSTRACT

In the paper we investigate some comparative growth properties related to the maximum terms of composite entire functions.

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1. INTRODUCTION, DEFINITIONS AND NOTATIONS.

Let f be an entire function defined in the open complex plane ℂ. The maximum term $\mu(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by

$$\mu(r, f) = \max_{n \geq 0}\left(|a_n| r^n\right).$$

To start our paper we just recall the following definitions:

Definition 1. *The order* ρ_f *and lower order* λ_f *of an entire function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r},$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ *for k=1,2,3,… and* $\log^{[0]} x = x$.

Definition 2. *The hyper order* $\overline{\rho}_f$ *hyper lower order* $\overline{\lambda}_f$ *of an entire function f are defined as follows*

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r, f)}{\log r} \quad \text{and} \quad \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r, f)}{\log r}.$$

Since for $0 \leq r < R,\ \mu(r, f) \leq M(r, f) \leq \frac{R}{R-r}\mu(R, f)\ \{cf. [4]\}$

it is easy to see that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f)}{\log r}$$

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and

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]}\mu(r,f)}{\log r} \quad \text{and} \quad \overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]}\mu(r,f)}{\log r}.$$

Definition 3. ([3]) *Let f be an entire function of order zero. Then the quantities $\rho^*{}_f$, $\lambda^*{}_f$ and $\overline{\rho}^*{}_f$, $\overline{\lambda}^*{}_f$ are defined in the following way :*

$$\rho^*{}_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}, \quad \lambda^*{}_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}$$

and

$$\overline{\rho}^*{}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}, \quad \overline{\lambda}^*{}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}.$$

Definition 4. *The type σ_f of an entire function f is defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \; 0 < \rho_f < \infty .$$

In the paper we would like to establish some new results based on the comparative growth properties of maximum terms of composite entire functions. We do not explain the standard notations and definitions in the theory of entire functions as those are available in [5].

2. LEMMAS.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. ([1]) Let f and g be any two entire functions. Then for all sufficiently large values of r,

$$M\left(\frac{1}{8}M\left(\frac{r}{2},g\right) - |g(0)|, f\right) \le M(r, f\circ g) \le M(M(r,g), f).$$

Lemma 2. ([2]) *Let f be an entire function of finite lower order. If there exist entire functions $a_i \; (i = 1,2,\ldots,n; n \le \infty)$ satisfying $T(r,a_i) = o\{T(r,f)\}$ and $\sum\limits_{i=1}^{n} \delta(a_i; f) = 1$, the*

$$\lim_{r\to\infty} \frac{T(r,f)}{\log M(r,f)} = \frac{1}{\pi}.$$

3. THEOREMS.

In this section we present the main results of the paper.

Theorem 1. *Let f, g and k be any three entire functions with $(i)\, 0 < \lambda_k < \rho_k < \infty$, $(ii)\, 0 < \rho_g < \infty$, $(iii)\, \lambda^*{}_f < \rho^*{}_f < \infty$ and $(iv)\, 0 < \sigma_g < \infty$. Also let there exist entire functions $a_i \; (i = 1,2,\ldots,n; n \le \infty)$ satisfying $T(r,a_i) = o\{T(r,g)\}$ and $\sum\limits_{i=1}^{n} \delta(a_i; g) = 1$. Then*

$$\left(\frac{1}{4}\right)^{\rho_g} \frac{\pi\sigma_g\lambda_k}{\rho^*{}_f . \rho_g} \le \limsup_{r\to\infty} \frac{\log^{[2]}\mu(r, k\circ g)}{\log^{[2]}\mu\left(\exp\left(r^{\rho_g}\right), f\circ g\right)} \le \frac{\pi\sigma_g\rho_k}{\lambda^*{}_f . \lambda_g}.$$

Proof. Putting $R = 2r$ in the inequality

$$\mu(r, f) \le M(r, f) \le \frac{R}{R-r}\mu(R, f) \ \{cf.[4]\}$$

we get that

$$\mu(r, f) \le M(r, f) \le 2\mu(2r, f) \ .$$

Now in view of the first part of Lemma 1 and the inequality $M(r, f) \le 2\mu(2r, f)$, we obtain for all sufficiently large values of *r*,

$$\begin{aligned}\log^{[2]} \mu(r, k \circ g) &\ge \log^{[2]} \frac{1}{2} M\left(\frac{r}{2}, k \circ g\right) \\ &\ge \log^{[2]} M\left(\frac{1}{16} M\left(\frac{r}{4}, g\right), k\right) + O(1) \\ &\ge (\lambda_k - \varepsilon) \log M\left(\frac{r}{4}, g\right). \end{aligned} \tag{1}$$

Again in view of the second part of Lemma 1 and for all sufficiently large values of r,

$$\begin{aligned}\log^{[2]} \mu(\exp(r^{\rho_g}), f \circ g) &\le \log^{[2]} M(\exp(r^{\rho_g}), f \circ g) \\ &\le \log^{[2]} M(M(\exp(r^{\rho_g}), g), f) \\ &\le (\rho^*{}_f + \varepsilon).(\rho_g + \varepsilon).r^{\rho_g} \ . \end{aligned} \tag{2}$$

Now using Lemma 2 and from (1) and (2) we get for all sufficiently large values of r,

$$\begin{aligned}\frac{\log^{[2]} \mu(r, k \circ g)}{\log^{[2]} \mu(\exp(r^{\rho_g}), f \circ g)} &\ge \frac{(\lambda_k - \varepsilon) \log M\left(\frac{r}{4}, g\right)}{(\rho^*{}_f + \varepsilon).(\rho_g + \varepsilon).r^{\rho_g}} \\ &\ge \frac{(\lambda_k - \varepsilon)}{(\rho^*{}_f + \varepsilon).(\rho_g + \varepsilon)} . \frac{\log M\left(\frac{r}{4}, g\right)}{T\left(\frac{r}{4}, g\right)} . \frac{T\left(\frac{r}{4}, g\right)}{\left(\frac{r}{4}\right)^{\rho_g}} . \left(\frac{1}{4}\right)^{\rho_g} .\end{aligned}$$

Since $\varepsilon(>0)$ is arbitrary,

$$\limsup_{r \to \infty} \frac{\log^{[2]} \mu(r, k \circ g)}{\log^{[2]} \mu(\exp(r^{\rho_g}), f \circ g)} \ge \left(\frac{1}{4}\right)^{\rho_g} \frac{\pi \sigma_g \lambda_k}{\rho^*{}_f . \rho_g} . \tag{3}$$

Again by the second part of Lemma 1 and the inequality $\mu(r, f) \le M(r, f)$, we get for all sufficiently large values of *r*,

$$\begin{aligned}\log^{[2]} \mu(r, k \circ g) &\le \log^{[2]} M(r, k \circ g) \\ &\le \log^{[2]} M(M(r, g), k) \\ &\le (\rho_k + \varepsilon) \log M(r, g). \end{aligned} \tag{4}$$

Also in view of the first part of Lemma 1 and for all sufficiently large values of r we obtain that

$$\log^{[2]}\mu\left(\exp\left(r^{\rho_g}\right), f\circ g\right)\geq \log^{[2]}\frac{1}{2}M\left(\frac{\exp\left(r^{\rho_g}\right)}{2}, f\circ g\right)$$

$$\geq \log^{[2]} M\left(\frac{1}{16}M\left(\frac{\exp\left(r^{\rho_g}\right)}{4}, g\right), f\right)$$

$$\geq \left(\lambda^*_f-\varepsilon\right)\log^{[2]} M\left(\frac{\exp\left(r^{\rho_g}\right)}{4}, g\right)$$

$$\geq \left(\lambda^*_f-\varepsilon\right)\left(\lambda_g-\varepsilon\right).r^{\rho_g}. \tag{5}$$

Now from (4) and (5) we get for all sufficiently large values of r,

$$\frac{\log^{[2]}\mu\left(r,k\circ g\right)}{\log^{[2]}\mu\left(\exp\left(r^{\rho_g}\right), f\circ g\right)}\leq \frac{\left(\rho_k+\varepsilon\right)\log M\left(r,g\right)}{\left(\lambda^*_f-\varepsilon\right)\left(\lambda_g-\varepsilon\right).r^{\rho_g}}.$$

As $\varepsilon(>0)$ is arbitrary,

$$\limsup_{r\to\infty}\frac{\log^{[2]}\mu\left(r,k\circ g\right)}{\log^{[2]}\mu\left(\exp\left(r^{\rho_g}\right), f\circ g\right)}\leq \frac{\rho_k.\pi.\sigma_g}{\lambda^*_f.\lambda_g}. \tag{6}$$

Thus the theorem follows from (3) and (6).

Theorem 2. *Let* f, g *and* k *be any three entire functions with* $(i)\, 0<\lambda_k<\rho_k<\infty$, $(ii)\; 0<\rho_g<\infty$, *and* $(iii)\; \lambda^*_f<\rho^*_f<\infty$ $(iv)\; 0<\sigma_g<\infty$. *Also let there exist entire functions* $a_i\,(i=1,2,\ldots,n; n\leq\infty)$ *satisfying* $T\left(r,a_i\right)=o\{T(r,g)\}$ *and* $\sum_{i=1}^{n}\delta\left(a_i;g\right)=1$. *Then*

$$\left(\frac{1}{4}\right)^{\rho_g}\frac{\pi\sigma_g\lambda_k}{\rho^*_f.\rho_k}\leq \limsup_{r\to\infty}\frac{\log^{[2]}\mu\left(r,k\circ g\right)}{\log^{[2]}\mu\left(\exp\left(r^{\rho_g}\right), f\circ k\right)}\leq \frac{\pi\sigma_g\rho_k}{\lambda^*_f.\lambda_k}.$$

Proof. In view of the following inequality and putting $R=2r$

$$\mu(r,f)\leq M(r,f)\leq \frac{R}{R-r}\mu(R,f)\ \{cf.[4]\}$$

we get that

$$\mu(r,f)\leq M(r,f)\leq 2\mu(2r,f).$$

Now in view of the first part of Lemma 1 and the inequality $M(r,f)\leq 2\mu(2r,f)$, we obtain for all sufficiently large values of *r*,

$$\log^{[2]}\mu\left(r,k\circ g\right)\geq \log^{[2]}\frac{1}{2}M\left(\frac{r}{2},k\circ g\right)$$

$$\geq \log^{[2]} M\left(\frac{1}{16}M\left(\frac{r}{4},g\right),k\right)+O(1)$$

$$\geq (\lambda_k - \varepsilon)\log M\left(\frac{r}{4}, g\right). \tag{7}$$

Again in view of the second part of Lemma 1 and for all sufficiently large values of r,

$$\begin{aligned}\log^{[2]} \mu\left(\exp\left(r^{\rho_g}\right), f \circ k\right) &\leq \log^{[2]} M\left(\exp\left(r^{\rho_g}\right), f \circ k\right)\\ &\leq \log^{[2]} M\left(M\left(\exp\left(r^{\rho_g}\right), k\right), f\right)\\ &\leq \left(\rho^*{}_f + \varepsilon\right).\left(\rho_k + \varepsilon\right).r^{\rho_g} .\end{aligned} \tag{8}$$

Now using Lemma 2 and from (7) and (8) we get for all sufficiently large values of r,

$$\begin{aligned}\frac{\log^{[2]} \mu(r, k \circ g)}{\log^{[2]} \mu\left(\exp\left(r^{\rho_g}\right), f \circ k\right)} &\geq \frac{(\lambda_k - \varepsilon)\log M\left(\frac{r}{4}, g\right)}{\left(\rho^*{}_f + \varepsilon\right).\left(\rho_k + \varepsilon\right).r^{\rho_g}}\\ &\geq \frac{(\lambda_k - \varepsilon)}{\left(\rho^*{}_f + \varepsilon\right).\left(\rho_k + \varepsilon\right)}.\frac{\log M\left(\frac{r}{4}, g\right)}{T\left(\frac{r}{4}, g\right)}.\frac{T\left(\frac{r}{4}, g\right)}{\left(\frac{r}{4}\right)^{\rho_g}}.\left(\frac{1}{4}\right)^{\rho_g}.\end{aligned}$$

Since $\varepsilon(>0)$ is arbitrary,

$$\limsup_{r \to \infty} \frac{\log^{[2]} \mu(r, k \circ g)}{\log^{[2]} \mu\left(\exp\left(r^{\rho_g}\right), f \circ k\right)} \geq \left(\frac{1}{4}\right)^{\rho_g} \frac{\pi \sigma_g \lambda_k}{\rho^*{}_f . \rho_k}. \tag{9}$$

Again by the second part of Lemma 1 and the inequality $\mu(r, f) \leq M(r, f)$, we get for all sufficiently large values of r,

$$\begin{aligned}\log^{[2]} \mu(r, k \circ g) &\leq \log^{[2]} M(r, k \circ g)\\ &\leq \log^{[2]} M(M(r, g), k)\\ &\leq (\rho_k + \varepsilon)\log M(r, g).\end{aligned} \tag{10}$$

Also in view of the first part of Lemma 1 and for all sufficiently large values of r we obtain that

$$\begin{aligned}\log^{[2]} \mu\left(\exp\left(r^{\rho_g}\right), f \circ k\right) &\geq \log^{[2]} \frac{1}{2} M\left(\frac{\exp\left(r^{\rho_g}\right)}{2}, f \circ k\right)\\ &\geq \log^{[2]} M\left(\frac{1}{16} M\left(\frac{\exp\left(r^{\rho_g}\right)}{4}, k\right), f\right)\\ &\geq \left(\lambda^*{}_f - \varepsilon\right)\log^{[2]} M\left(\frac{\exp\left(r^{\rho_g}\right)}{4}, k\right)\\ &\geq \left(\lambda^*{}_f - \varepsilon\right) \cdot (\lambda_k - \varepsilon) \cdot r^{\rho_g} .\end{aligned} \tag{11}$$

Now from (10) and (11) we get for all sufficiently large values of r,

$$\frac{\log^{[2]}\mu(r,k\circ g)}{\log^{[2]}\mu\left(\exp\left(r^{\rho_g}\right),f\circ k\right)}\leq\frac{(\rho_k+\varepsilon)\log M(r,g)}{\left(\lambda^*_f-\varepsilon\right)(\lambda_k-\varepsilon).r^{\rho_g}}.$$

As $\varepsilon(>0)$ is arbitrary,

$$\limsup_{r\to\infty}\frac{\log^{[2]}\mu(r,k\circ g)}{\log^{[2]}\mu\left(\exp\left(r^{\rho_g}\right),f\circ k\right)}\leq\frac{\rho_k.\pi.\sigma_g}{\lambda^*_f.\lambda_k}. \tag{12}$$

Thus the theorem follows from (9) and (12).

Theorem 3. *Let* f, g *and* k *be any three entire functions with* $(i)\ 0<\overline{\lambda}_k<\overline{\rho}_k<\infty$, $(ii)\ 0<\lambda_g<\rho_g<\infty$, $(iii)\ \overline{\lambda}^*_f<\overline{\rho}^*_f<\infty$ *and* $(iv)\ 0<\sigma_g<\infty$ *.Also let there exist entire functions* $a_i\,(i=1,2,\ldots,n;n\leq\infty)$ *satisfying* $T\left(r,a_i\right)=o\{T(r,g)\}$ *and* $\sum_{i=1}^{n}\delta\left(a_i;g\right)=1.$ *Then*

$$\left(\frac{1}{4}\right)^{\rho_g}\frac{\pi\sigma_g\overline{\lambda}_k}{\overline{\rho}^*_f.\rho_g}\leq\limsup_{r\to\infty}\frac{\log^{[3]}\mu(r,k\circ g)}{\log^{[3]}\mu\left(\exp\left(r^{\rho_g}\right),f\circ g\right)}\leq\frac{\pi\sigma_g\overline{\rho}_k}{\overline{\lambda}^*_f.\lambda_g}.$$

Proof. Considering the following inequality and taking $R=2r$

$$\mu(r,f)\leq M(r,f)\leq\frac{R}{R-r}\mu(R,f)\ \{cf.[4]\}$$

we obtain that

$$\mu(r,f)\leq M(r,f)\leq 2\mu(2r,f).$$

Now in view of the first part of Lemma 1 and the inequality $M(r,f)\leq 2\mu(2r,f)$, we obtain for all sufficiently large values of *r*,

$$\begin{aligned}\log^{[3]}\mu(r,k\circ g)&\geq\log^{[3]}\frac{1}{2}M\left(\frac{r}{2},k\circ g\right)\\&\geq\log^{[3]}M\left(\frac{1}{16}M\left(\frac{r}{4},g\right),k\right)+O(1)\\&\geq\left(\overline{\lambda}_k-\varepsilon\right)\log M\left(\frac{r}{4},g\right).\end{aligned} \tag{13}$$

Again in view of the second part of Lemma 1 and for all sufficiently large values of r,

$$\begin{aligned}\log^{[3]}\mu\left(\exp\left(r^{\rho_g}\right),f\circ g\right)&\leq\log^{[3]}M\left(\exp\left(r^{\rho_g}\right),f\circ g\right)\\&\leq\log^{[3]}M\left(M\left(\exp\left(r^{\rho_g}\right),g\right),f\right)\\&\leq\left(\overline{\rho}^*_f+\varepsilon\right)\left(\rho_g+\varepsilon\right).r^{\rho_g}.\end{aligned} \tag{14}$$

Now using Lemma 2 and from (13) and (14) we get for all sufficiently large values of r,

$$\frac{\log^{[3]}\mu(r,k\circ g)}{\log^{[3]}\mu\left(\exp\left(r^{\rho_g}\right),f\circ g\right)}\geq\frac{\left(\overline{\lambda}_k-\varepsilon\right)\log M\left(\frac{r}{4},g\right)}{\left(\overline{\rho}^*_f+\varepsilon\right)\left(\rho_g+\varepsilon\right).r^{\rho_g}}$$

$$\geq \frac{\left(\overline{\lambda}_k - \varepsilon\right)}{\left(\overline{\rho}^*{}_f + \varepsilon\right)\left(\rho_g + \varepsilon\right)} \cdot \frac{\log M\left(\frac{r}{4}, g\right)}{T\left(\frac{r}{4}, g\right)} \cdot \frac{T\left(\frac{r}{4}, g\right)}{\left(\frac{r}{4}\right)^{\rho_g}} \cdot \left(\frac{1}{4}\right)^{\rho_g}.$$

Since $\varepsilon(>0)$ is arbitrary,

$$\limsup_{r\to\infty} \frac{\log^{[3]} \mu(r, k\circ g)}{\log^{[3]} \mu\left(\exp\left(r^{\rho_g}\right), f\circ g\right)} \geq \left(\frac{1}{4}\right)^{\rho_g} \frac{\pi\sigma_g \overline{\lambda}_k}{\overline{\rho}^*{}_f \cdot \rho_g}. \tag{15}$$

Again by the second part of Lemma 1 and the inequality $\mu(r, f) \leq M(r, f)$, we get for all sufficiently large values of r,

$$\begin{aligned} \log^{[3]} \mu(r, k\circ g) &\leq \log^{[3]} M(r, k\circ g) \\ &\leq \log^{[3]} M(M(r,g), k) \\ &\leq \left(\overline{\rho}_k + \varepsilon\right)\log M(r,g). \end{aligned} \tag{16}$$

Also in view of the first part of Lemma 1 and for all sufficiently large values of r we obtain that

$$\begin{aligned} \log^{[3]} \mu\left(\exp\left(r^{\rho_g}\right), f\circ g\right) &\geq \log^{[3]} \frac{1}{2} M\left(\frac{\exp\left(r^{\rho_g}\right)}{2}, f\circ g\right) \\ &\geq \log^{[3]} M\left(\frac{1}{16} M\left(\frac{\exp\left(r^{\rho_g}\right)}{4}, g\right), f\right) \\ &\geq \left(\overline{\lambda}^*{}_f - \varepsilon\right)\log^{[2]} M\left(\frac{\exp\left(r^{\rho_g}\right)}{4}, g\right) \\ &\geq \left(\lambda^*{}_f - \varepsilon\right)\cdot\left(\lambda_g - \varepsilon\right)\cdot r^{\rho g}. \end{aligned} \tag{17}$$

Now from (16) and (17) we get for all sufficiently large values of r,

$$\frac{\log^{[3]} \mu(r, k\circ g)}{\log^{[3]} \mu\left(\exp\left(r^{\rho_g}\right), f\circ g\right)} \leq \frac{\left(\overline{\rho}_k + \varepsilon\right)\log M(r,g)}{\left(\overline{\lambda}^*{}_f - \varepsilon\right)\left(\lambda_g - \varepsilon\right) r^{\rho_g}}.$$

As $\varepsilon(>0)$ is arbitrary,

$$\limsup_{r\to\infty} \frac{\log^{[3]} \mu(r, k\circ g)}{\log^{[3]} \mu\left(\exp\left(r^{\rho_g}\right), f\circ g\right)} \leq \frac{\overline{\rho}_k . \pi . \sigma_g}{\overline{\lambda}^*{}_f . \lambda_g}. \tag{18}$$

Thus the theorem follows from (15) and (18).

Theorem 4. *Let* f, g *and* k *be any three entire functions with* $(i)\, 0 < \lambda_k < \rho_k < \infty$, $(ii)\, 0 < \overline{\lambda}_k < \overline{\rho}_k < \infty$, $(iii)\, 0 < \rho_g < \infty$, $(iv)\, 0 < \overline{\lambda}^*{}_f < \overline{\rho}^*{}_f < \infty$ *and* $(v)\, 0 < \sigma_g < \infty$. *Also let there exist entire functions* $a_i\, (i = 1,2,...,n; n \leq \infty)$ *satisfying* $T(r, a_i) = o\{T(r,g)\}$ *and* $\sum_{i=1}^{n} \delta(a_i; g) = 1$. *Then*

$$\left(\frac{1}{4}\right)^{\rho_g}\frac{\pi\sigma_g\overline{\lambda}_k}{\overline{\rho}^*_f.\rho_k}\le\limsup_{r\to\infty}\frac{\log^{[3]}\mu(r,k\circ g)}{\log^{[3]}\mu\left(\exp\left(r^{\rho_g}\right),f\circ k\right)}\le\frac{\pi\sigma_g\overline{\rho}_k}{\overline{\lambda}^*_f.\lambda_k}.$$

Proof. Putting $R=2r$ in the inequality

$$\mu(r,f)\le M(r,f)\le\frac{R}{R-r}\mu(R,f)\ \{cf.[4]\}$$

We obtain that

$$\mu(r,f)\le M(r,f)\le 2\mu(2r,f).$$

Now in view of the first part of Lemma 1 and the inequality $M(r,f)\le 2\mu(2r,f)$, we obtain for all sufficiently large values of r,

$$\begin{aligned}\log^{[3]}\mu(r,k\circ g)&\ge\log^{[3]}\frac{1}{2}M\left(\frac{r}{2},k\circ g\right)\\&\ge\log^{[3]}M\left(\frac{1}{16}M\left(\frac{r}{4},g\right),k\right)+O(1)\\&\ge\left(\overline{\lambda}_k-\varepsilon\right)\log M\left(\frac{r}{4},g\right).\end{aligned}\tag{19}$$

Again in view of the second part of Lemma 1 and for all sufficiently large values of r,

$$\begin{aligned}\log^{[3]}\mu\left(\exp\left(r^{\rho_g}\right),f\circ k\right)&\le\log^{[3]}M\left(\exp\left(r^{\rho_g}\right),f\circ k\right)\\&\le\log^{[3]}M\left(M\left(\exp\left(r^{\rho_g}\right),k\right),f\right)\\&\le\left(\overline{\rho}^*_f+\varepsilon\right)\left(\rho_k+\varepsilon\right).r^{\rho_g}.\end{aligned}\tag{20}$$

Now using Lemma 2 and from (19) and (20) we get for all sufficiently large values of r,

$$\begin{aligned}\frac{\log^{[3]}\mu(r,k\circ g)}{\log^{[3]}\mu\left(\exp\left(r^{\rho_g}\right),f\circ k\right)}&\ge\frac{\left(\overline{\lambda}_k-\varepsilon\right)\log M\left(\frac{r}{4},g\right)}{\left(\overline{\rho}^*_f+\varepsilon\right)\left(\rho_k+\varepsilon\right).r^{\rho_g}}\\&\ge\frac{\left(\overline{\lambda}_k-\varepsilon\right)}{\left(\overline{\rho}^*_f+\varepsilon\right)\left(\rho_k+\varepsilon\right)}.\frac{\log M\left(\frac{r}{4},g\right)}{T\left(\frac{r}{4},g\right)}.\frac{T\left(\frac{r}{4},g\right)}{\left(\frac{r}{4}\right)^{\rho_g}}.\left(\frac{1}{4}\right)^{\rho_g}.\end{aligned}$$

Since $\varepsilon(>0)$ is arbitrary,

$$\limsup_{r\to\infty}\frac{\log^{[3]}\mu(r,k\circ g)}{\log^{[3]}\mu\left(\exp\left(r^{\rho_g}\right),f\circ k\right)}\ge\left(\frac{1}{4}\right)^{\rho_g}\frac{\pi\sigma_g\overline{\lambda}_k}{\overline{\rho}^*_f.\rho_k}.\tag{21}$$

Again by the second part of Lemma 1 and the inequality $\mu(r,f)\le M(r,f)$, we get for all sufficiently large values of r,

$$\begin{aligned}\log^{[3]}\mu(r,k\circ g)&\le\log^{[3]}M(r,k\circ g)\\&\le\log^{[3]}M(M(r,g),k)\\&\le\left(\overline{\rho}_k+\varepsilon\right)\log M(r,g).\end{aligned}\tag{22}$$

Also in view of the first part of Lemma 1 and for all sufficiently large values of r we obtain that

$$\log^{[3]}\mu\left(\exp\left(r^{\rho_g}\right), f\circ k\right) \geq \log^{[3]}\frac{1}{2}M\left(\frac{\exp\left(r^{\rho_g}\right)}{2}, f\circ k\right)$$

$$\geq \log^{[3]} M\left(\frac{1}{16}M\left(\frac{\exp\left(r^{\rho_g}\right)}{4}, k\right), f\right)$$

$$\geq \left(\overline{\lambda}^{*}_{f} - \varepsilon\right)\log^{[2]} M\left(\frac{\exp\left(r^{\rho_g}\right)}{4}, k\right)$$

$$\geq \left(\lambda^{*}_{f} - \varepsilon\right)\cdot\left(\lambda_k - \varepsilon\right)\cdot r^{\rho g}\,. \tag{23}$$

Now from (22) and (23) we get for all sufficiently large values of r,

$$\frac{\log^{[3]}\mu(r, k\circ g)}{\log^{[3]}\mu\left(\exp\left(r^{\rho_g}\right), f\circ k\right)} \leq \frac{\left(\overline{\rho}_k + \varepsilon\right)\log M(r, g)}{\left(\overline{\lambda}^{*}_{f} - \varepsilon\right)\left(\lambda_k - \varepsilon\right).r^{\rho_g}}.$$

As $\varepsilon(>0)$ is arbitrary,

$$\limsup_{r\to\infty}\frac{\log^{[3]}\mu(r, k\circ g)}{\log^{[3]}\mu\left(\exp\left(r^{\rho_g}\right), f\circ k\right)} \leq \frac{\overline{\rho}_k.\pi.\sigma_g}{\overline{\lambda}^{*}_{f}.\lambda_k}. \tag{24}$$

Thus the theorem follows from (21) and (24).

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