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ASYMPTOTIC BEHAVIOR OF SOME MATHEMATICAL MODELS

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ABSTRACT

The objective of this paper is to investigate the asymptotic behavior of the solutions of an economic model

$$\begin{cases} x_{n+1} = x_n + \alpha x_n \left(a - c_1 - 2bx_n - by_n\right), \quad n = 0,1,2,.... \\ y_{n+1} = \dfrac{1}{2b}\left(a - c_2 - bx_n\right) \end{cases} \tag{1}$$

where $\alpha \in [0; 1]$*, a, b,* c_1*,* c_2 *are positive constants, and of a known population model*

$$x_{n+1} = \alpha + \beta . x_{n-1} e^{-x_n}, n = 0,1,2... \tag{2}$$

where α *is the immigration rate and* β *is the population growth rate.*

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1 INTRODUCTION

Difference equations manifest themselves as mathematical models describing real life situation in probability theory, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc,... Nonlinear difference equations of order greater than one are of paramount importance in applications where the (n+1)st generation (or state) of the system depends on the previous *n* generations (or state). Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics.

Stability criteria and periodic behavior are derived for difference equations by many authors. In this paper, we study the asymptotic behavior of all positive solutions of the following economic system:

$$\begin{cases} x_{n+1} = x_n + \alpha x_n \left(a - c_1 - 2bx_n - by_n\right), \; n = 0,1,2,.... \\ y_{n+1} = \dfrac{1}{2b}\left(a - c_2 - bx_n\right) \end{cases} \tag{1}$$

where $\alpha \in [0; 1]$, a; b; c_1; c_2 are positive constants, and of a known pop-ulation model

$$x_{n+1} = \alpha + \beta . x_{n-1} e^{-x_n}, n = 0,1,2... \tag{2}$$

where α is the immigration rate and β is the population growth rate.

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2. ASYMPTOTIC APPROXIMATION OF THE ECONOMIC SYSTEM (1)

In [5] H. El. Metwally investigated some qualitative behavior of the solutions of the following economic model

$$\begin{cases} x_{n+1} = x_n + \alpha x_n \left(a - c_1 - 2bx_n - by_n\right), \quad n = 0,1,2,.... \\ y_{n+1} = \dfrac{1}{2b}\left(a - c_2 - bx_n\right) \end{cases} \tag{1}$$

where $\alpha \in [0; 1]$, a, b, c_1, c_2 are positive constants.

The author obtained some results about the boundedness, the periodicity and the global attractivity of the solutions of this higher order difference equa-tion. In this section, we study the asymptotic behavior of all positive solutions of the following system (1).

Set β = a-c_1 and $a = \dfrac{a - c_2}{2b}$, we can write the system (1) in the form

$$\begin{cases} x_{n+1} = x_n + \alpha x_n \left(\beta - 2bx_n - by_n\right), \quad n = 0,1,2,.... \\ y_{n+1} = a - \dfrac{1}{2}x_{n.} \end{cases} \tag{1}$$

Finally, the system (1) can be written in only equation

$$x_{n+1} = \left(1 + \alpha\beta - \alpha ab\right)x_n - 2\alpha bx_n^2 + \frac{\alpha b}{2}x_n x_{n-1}. \tag{3}$$

The equilibrium point equation is

$$\overline{x} = \left(1 + \alpha\beta - \alpha ab\right)\overline{x} - \frac{7\alpha b}{2}\overline{x}$$

Thus, we get

$$\overline{x_1} = 0 \quad or \quad \overline{x_2} = \frac{2\left(\beta - ab\right)}{3b} > 0.$$

We set $f\left(x_n, x_{n-1}\right) = \left(1 + \alpha\beta - \alpha ab\right)x_n - 2\alpha abx_n^2 + \dfrac{\alpha b}{2}x_n.x_{n-1}$

$$f'_{x_n}\Big|_{\overline{x_2}} = \frac{3 - 4\alpha\left(\beta - ab\right)}{3}$$

$$f'_{x_{n-1}}\Big|_{\overline{x_2}} = \frac{\alpha\left(\beta - ab\right)}{3}$$

Note that the linearized equation of Eq (3) about the positive equilibrium point. We now investigate the asymptotical stability of Eq. (1) $\overline{x_2} = \dfrac{2\left(\beta - ab\right)}{3b}$ is

$$y_{n+1} = \frac{3 - 4\alpha\left(\beta - ab\right)}{3}y_n + \frac{\alpha\left(\beta - ab\right)}{3}y_{n-1}. \tag{4}$$

The characteristic polynomial associated with Eq (4) is

$$p(t) = t^2 + \frac{4\alpha\left(\beta - ab\right) - 3}{3}t - \frac{\alpha\left(\beta - ab\right)}{3} = 0. \tag{5}$$

Since, $p(0) = -\dfrac{\alpha(\beta - ab)}{3} < 0, p(1) = \alpha(\beta - ab) > 0$

$$p'(t) = 2t + \frac{4\alpha(\beta - ab) - 3}{3} > 0 \ \ for\ t \in (0,1)$$

It follows that for $0 \le \alpha \le \min\left(1; \dfrac{3}{4(\beta - ab)}\right)$, there is a unique positive root $t_0 \in (0,1)$ such that $p(t_0) = 0\ and\ 0 < t_0^2 < t_0 < 1$ such that $p(t_0^2) < p(t_0) = 0$. It means that

$$p(t_0) = t_0^2 + \frac{4\alpha(\beta - ab) - 3}{3} t_0 - \frac{\alpha(\beta - ab)}{3} = 0 \tag{6}$$

This fact motivated us to believe that there are solutions of Equation (5) which have the following asymptotics

$$x_n = \overline{x} + a_1 t_0^n + 0(t_0^n) \tag{7}$$

where $a_1 \in R$ and t_0 is the above mentioned root of Eq (6) we solve the open problem, showing that such a solution exists, developing Berg's ideas in [1-4] which are based on the asymptotics. The asymptotics for solutions of difference equation have been investigated by L. Berg and S. Stevi'c, see, for example [7-9], [13,14] and the reference therein. The problem is solved by constructing appropriate sequences y_n and z_n

$$y_n \le x_n \le z_n \tag{8}$$

for suffiently large n. In [1-4] some methods can be found for the construction of these bounds, see, also [13, 14].

From (7) we expect that for k $\ge$ 2 such solutions have the first three members in their asymptotics in the following form

$$\varphi_n = \overline{x} + a_1 t_0^n + b_1 t_0^{2n} \tag{9}$$

This is proved by developing Berg's ideas in [1-4] which are based on asymptotics. We need the following result in the proof of main theorems. The proof of the following theorem can be found in [13].

Theorem 2.1. Let f : $I^{k+2} \to I$ *be a continuous and nondecreasing function in each argument on the interval I* $\subset$ *R, and let* $\{y_n\}$ *and* $\{z_n\}$ *be sequences with* $y_n \le z_n \quad for\ n \ge n_0$ *and such that*

$$y_{n-k} \le f\left(n, y_{n-k+1}, \ldots, y_{n+1}\right), f\left(n, z_{n-k+1}, \ldots, z_{n+1}\right) \le z_{n-k}\ \ fo\ \ n > n_0 + k - 1 \tag{10}$$

Then there is a solution of the following difference equation

$$x_{n-k} = f\left(n, x_{n-k+1}, \ldots, x_{n+1}\right) \tag{11}$$

with property (8) for n $\ge n_0$

Theorem 2.2. *For a; b; c_1; c_2 are positive constants and* $\alpha \in \left[0, \min\left(1, \dfrac{3}{4(\beta - ab)}\right)\right]$ *there is a nonoscillatory solution of Eq (3) converging to the positive equilibirium point* $\overline{x} = \dfrac{2(\beta - ab)}{3b}$ *as* $n \to \infty$

Proof: First, Eq (3) can be written in the form

$$x_{n-1} = \frac{2}{ab} x_{n+1}.x_n^{-1} + 4x_n + 2a - \frac{2\beta}{b} - \frac{2}{\alpha b}$$

$$F\left(x_{n-1}, x_n, x_{n+1}\right) = \frac{2}{ab} x_{n+1}.x_n^{-1} + 4x_n + 2a - \frac{2\beta}{b} - \frac{2}{\alpha b} - x_{n-1} \tag{12}$$

We expect the solutions of Eq (3) have the asymptotic appropriation (9).

$$F = F\left(\varphi_{n-1}, \varphi_n, \varphi_{n+1}\right) = c\left(\bar{x} + a_1 t^{n+1} + b_1 t^{2n+2}\right)\left(\bar{x} + a_1 t^n + b_1 t^{2n}\right)^{-1} + 4\left(\bar{x} + a_1 t^n + b_1 t^{2n}\right) + d - \bar{x} - a_1 t^{n+1} - b_1 t^{2n+2}$$

$$with\ c = \frac{2}{\alpha b}\ and\ d = 2a - \frac{2\beta}{b} - \frac{2}{\alpha b}.$$

$$F = c\left(1 + \frac{a_1}{\bar{x}} t^{n+1} + \frac{b_1}{\bar{x}} t^{2n+2}\right)\left(1 + \frac{a_1}{\bar{x}} t^n + \frac{b_1}{\bar{x}} t^{2n}\right)^{-1} + 4\bar{x} + 4a_1 t^n + 4b_1 t^{2n} + d - \bar{x} - a_1 t^{n-1} - b_1 t^{2n-2}$$

$$F = \left(\frac{2}{\alpha b} + 3\frac{2\left(\beta - ab\right)}{3b} + 2a - \frac{2\beta}{b} - \frac{2}{ab}\right) + \left(\frac{ca_1 t^2}{\bar{x}} + \frac{4a_1\bar{x} - ca_1}{\bar{x}} t - a_1\right) t^{n-1}$$
$$+ \left[\frac{cb_1}{\bar{x}} t^4 - \frac{ca_1^2}{\bar{x}^2} t^3 + \left(4b_1 + \frac{ca_1^2}{\bar{x}^2} - \frac{cb_1}{\bar{x}}\right) t^2 - b_1\right] t^{2n-2} + o\left(t^{2n}\right)$$

By easy calculation we have

$$\frac{2}{ab} + \frac{2\left(\beta - ab\right)}{b} + 2a - \frac{2\beta}{b} - \frac{2}{\alpha b} = 0$$

and we can write the function F in following form

$$F = \left[t^2 + \frac{\alpha\beta}{2}\left(\frac{8\alpha\left(\beta - ab\right) - 6}{3\alpha b}\right) t - \frac{\alpha\left(\beta - ab\right)}{3}\right]\frac{a_1 t^n}{\bar{x} t} +$$
$$+ \left[ct^4 + \left(4\bar{x} - c\right) t^2 - \bar{x}\right]\frac{b_1 t^{2n}}{\bar{x} t^2} + \left(ca_1^2 - ca_1^2 t\right)\frac{t^{2n+2}}{\bar{x}^2} + o\left(t^{2n}\right) \tag{13}$$

As mentioned earlier exists $t_0 \in (0; 1)$ such that $p(t_0) = 0$ and $0 < t_0^2 < t_0 < 1,\ p\left(t_0^2\right) < 0$ Posing $t = t_0$, we obtain

$$F = \left[t_0^4 + \frac{4\bar{x} - c}{c} t_0^2 - \frac{\bar{x}}{c}\right]\frac{cb_1 t_0^{2n}}{\bar{x} t_0^2} + \left(1 - t_0\right)\frac{ca_1^2 t_o^{2n+2}}{\bar{x}^2} + o\left(t_0^{2n}\right)$$

$$F = p\left(t_0^2\right).\frac{cb_1 t_0^{2n}}{\bar{x} t_0^2} + \left(1 - t_0\right)\frac{ca_1^2 t_o^{2n+2}}{\bar{x}^2} + o\left(t_0^{2n}\right)$$

Therefore, we have

$$F = \left[p\left(t_0^2\right).\frac{cb_1 t_0^{2n}}{\bar{x} t_0^2} + \left(1 - t_0\right)\frac{ca_1^2 t_o^{2n+2}}{\bar{x}^2}\right] t_0^{2n} + o\left(t_0^{2n}\right)$$

$$F = \left(Bb_1 + C\right) t_0^{2n} + o\left(t_0^{2n}\right)$$

Setting $H_{t_0}(q) = Bq + C = 0 \rightarrow q_0 = -\dfrac{C}{B}$

$$H'_{t_0}(q) = B = \frac{p\left(t_0^2\right).c}{\overline{x}t_0^2} < 0$$

We obtain that there are $q_1 < q_0$ and $q_2 > q_0$ such that $H_{t_0}(q_1) > 0; H_{t_0}(q_2) < 0$; $q_1 < q_0 < q_2$.

We assume that $a_1 \neq 0,\ if\ \widehat{\varphi_n} = \overline{x} + a_1 t_0^n + q_0 t_0^{2n}$, we obtain

$$F\left(\widehat{\varphi}_{n-1}, \widehat{\varphi}_n, \widehat{\varphi}_{n+1}\right) \sim \left[q_0 B + C\right] t_0^{2n} + o\left(t_0^{2n}\right)$$

With the notation

$$y_n = \overline{x} + a_1 t_0^n + q_1 t_0^{2n}, z_n = \overline{x} + a_1 t_0^n + q_2 t_0^{2n}$$

We get

$$F\left(y_{n-1}, y_n, y_{n+1}\right) \sim \left[q_1 B + C\right] t_0^{2n},$$
$$F\left(z_{n-2}, z_{n-1}, z_n\right) \sim \left[q_2 B + C\right] t_0^{2n}$$

These relations show that inequalities (10) are satisfied for sufficiently large n, where $f = F + x_{n-1}$ and F is given by (12). Because the function $f\left(x_{n-1}, x_n, x_{n+1}\right)$ is continuous and nondecreasing on $\left[\overline{x}, +\infty\right)^3 \rightarrow \left[\overline{x}, +\infty\right)$. We easily have $f\left(\overline{x}, \overline{x}, \overline{x}\right) = \overline{x}$ We can apply the Theorem (2.1) with $I = \left[\overline{x}, \infty\right)$ and see that there is an $n_0 \geq 0$ and a solution of equation (3) with the asymptotics $x_n = \widehat{\varphi_n} + o(t_0^{2n})$, for $n_0 \geq 0$ where $b = q_0$ in $\widehat{\varphi_n}$. In particular, the solution converges monotonically to the positive equilibrium $\overline{x} = \dfrac{2\left(\beta - ab\right)}{3b}\ for\ n \geq n_0$ Hence, the solution x_{n+n_0+2} is also such a solution when n $\geq$-2.

Theorem 2.3. For a; b; c_1; c_2 are positive constants and $\alpha \in \left[0, \min\left(1, \dfrac{3}{4\left(\beta - ab\right)}\right)\right]$ there is a nonoscillatory solution of system equation

$$\begin{cases} x_{n+1} = x_n + \alpha x_n\left(a - c_1 - 2bx_n - by_n\right), \quad n = 0,1,2,.... \\ y_{n+1} = \dfrac{1}{2b}\left(a - c_2 - bx_n\right) \end{cases} \tag{1}$$

converging to the positive equilibrium point $\overline{x}, \overline{y}$ where

$$\overline{x} = \frac{2\left(\beta - ab\right)}{3b}$$
$$\overline{y} = \frac{1}{2b}\left(a - c_2 - \frac{2\left(\beta - ab\right)}{b}\right)$$

and $x_n = \widehat{\varphi}_n + o\left(t_0^{2n}\right)$

$$y_n = \frac{1}{2b}\left[a - c_2 - b(\widehat{\varphi}_n + o\left(t_0^{2n}\right))\right], n \geq -2$$

3. ASYMPTOTIC APPROXIMATION OF POPULATION MODEL (2)

In [5,6] the authors studied the asymptotic behavior of some know population models. They established that every solution of the bounded below by positive constants. They also provided sufficient conditions for the global asymptotically stable of all solution of that higher order difference equations.

The study of a nonoscillatory solution of difference equation converging to the positive equilibrium point $\bar{x}$ is extremely useful in the behavior of mathematical models of various biological systems and other application. This is due to the fact that difference equation are appropriate models for discribing situations where the variable is assumed to take only a discrete set of values and they arise frequently in the study of biological models, in the formation and analysis of discrete - time systems, the numerical intergation of differential equation by finite - difference schemes, the study of deterministic chaos, etc. For example El - Metwally [5] investigated the asymptotic behavior of the population model

$$x_{n+1} = \alpha + \beta . x_{n-1} e^{-x_n}, n = 0,1,2,... \tag{14}$$

where α is the immigration rate and β is the population growth rate.

In final section, we study the nonoscillatory solution of the equation (14) converging to the positive equilibrium point.

The equilibrium point equation is $\bar{x} = \alpha + \beta . \bar{x} . e^{-\bar{x}}$

$$\bar{x} - \beta \frac{\bar{x}}{e^{\bar{x}}} = \alpha$$

We pose

$$f\left(x_n, x_{n-1}\right) = \alpha + \beta . x_{n-1} e^{-x_n}$$

$$f'_{x_n}\Big|_{\bar{x}} = -\beta . \bar{x} e^{-\bar{x}} = \alpha - \bar{x}$$

$$f'_{x_{n-1}}\Big|_{\bar{x}} = -\beta . e^{-\bar{x}} = \frac{\bar{x} - \alpha}{\bar{x}}$$

Note that the linearized equation of Eq (14) about the positive equilibrium point

$$y_{n+1} = \left(\alpha - \bar{x}\right) y_n + \frac{\bar{x} - \alpha}{\bar{x}} y_{n-1}, n = 0,1,2,... \tag{15}$$

The characteristic polynomial associated with Eq (15) is

$$p(t) = \bar{x} t^2 + \bar{x}\left(\bar{x} - \alpha\right) t + \alpha - \bar{x} = 0$$

$$Since,\ p(0) = \alpha - \bar{x} < 0, p(1) = \bar{x} + \bar{x}\left(\bar{x} - \alpha\right) + \alpha - \bar{x} = \bar{x}\left(\bar{x} - \alpha\right) + \alpha > 0$$

$$p'(t) = 2\bar{x} t + \bar{x}\left(\bar{x} - \alpha\right) > 0 \ \ for \ \ t \in (0,1)$$

There is a unique positive root $t_0 \in (0,1)$ such that

$p\left(t_0\right) = 0 \ and \ 0 < t_0^2 < t_0 < 1 \ \ such\ that \ \ p\left(t_0^2\right) < p\left(t_0\right) = 0$. It means that

$$p\left(t_0\right) = \bar{x} t_0^2 + \bar{x}\left(\bar{x} - \alpha\right) t_0 + \alpha - \bar{x} = 0$$

This fact motivated us to believe that there are solutions of Equation (5) which have the following asymptotics

$$x_n = \bar{x} + a_1 t_0^n + o\left(t_0^n\right) \tag{16}$$

from (16) we expect that for k $\geq$ 2 such solutions have the first three members in their asymptotics in the following form

$$\varphi_n = \bar{x} + a_1 t_0^n + b_1 t_0^{2n} \tag{17}$$

Theorem 3.1: For α, β are positive constants there is a nonoscillatory solution of Equation (14) converging to the positive equilibrium point that satisfies

$$\bar{x}\left(1-\frac{\beta}{e^{\bar{x}}}\right)=\alpha \ \ as \ \ n \to \infty$$

Proof: From Eq (14) we can write in the form

$$F\left(x_{n-1}, x_n, x_{n+1}\right)=\frac{1}{\beta}.e^{x_n}\left(x_{n+1}-\alpha\right)-x_{n-1}=g\left(x_n, x_{n+1}\right)-x_{n-1} \tag{18}$$

$$g'_{x_{n+1}}=\frac{1}{\beta}e^{x_n}>0$$

$$g'_{x_{n+1}}=\frac{1}{\beta}e^{x_n}\left(x_{n+1}-\alpha\right)>0$$

We expect the solutions of Eq (14) have the asymptotic appropriation (16)'

$$F=F\left(\varphi_{n-1}, \varphi_n, \varphi_{n+1}\right)=\frac{1}{\beta}e^{\bar{x}+at^n+bt^{2n}}\left(\bar{x}+at^{n+1}+bt^{2n+2}-\alpha\right)-\left(\bar{x}+at^{n-1}+bt^{2n-2}\right)$$
$$=0 \ \ for \ t\in\left(0,\infty\right)$$

$$e^{\bar{x}+at^n+bt^{2n}}=\beta\left(\bar{x}+at^{n-1}+bt^{2n-2}\right)\left(\bar{x}+at^{n+1}+bt^{2n+2}-\alpha\right)^{-1}$$

$$\bar{x}+at^n+bt^{2n}=\ln\left[\beta\left(1+\frac{at^{n-1}}{\bar{x}}+\frac{bt^{2n-2}}{\bar{x}}\right)\left(1+\frac{at^{n+1}}{\bar{x}-\alpha}+\frac{bt^{2n+2}}{\bar{x}-\alpha}\right)^{-1}\right]$$

$$=\ln\frac{\beta\bar{x}}{\bar{x}-\alpha}\left(1+\frac{at^{n-1}}{\bar{x}}+\frac{bt^{2n-2}}{\bar{x}}\right).\left[1-\frac{at^{n+1}}{\bar{x}-\alpha}-\frac{bt^{2n+2}}{\bar{x}-\alpha}+\left(\frac{at^{n+1}+bt^{2n+2}}{\bar{x}-\alpha}\right)^2-...\right]$$

$$\ln\bar{x}+\frac{at^{n-1}+bt^{2n-2}}{\bar{x}}-\frac{a^2t^{2n-2}+2abt^{3n-3}+...}{2\bar{x}^2}+...$$

$$=\bar{x}+at^n+bt^{2n}+\ln\frac{\bar{x}-\alpha}{\beta}+\frac{at^{n+1}+bt^{2n+2}}{\bar{x}-\alpha}-\frac{a^2t^{2n+2}}{2\left(\bar{x}-\alpha\right)}+..... \tag{19}$$

From (19) we have

$$\ln\bar{x}=\bar{x}+\ln\frac{\bar{x}-\alpha}{\beta} \ \ or \ \ \left(\bar{x}-\alpha\right)e^{\bar{x}}=\beta\bar{x}$$

This is trust and from (19) we obtain

$$\frac{a}{\bar{x}}t^{n-1}=at^n+\frac{at^{n+1}}{\bar{x}-\alpha}$$

$$\left(\frac{a}{t\bar{x}}-a-\frac{at}{\bar{x}-\alpha}\right)t^n=\frac{\left[\bar{x}t^2+t\bar{x}\left(\bar{x}-\alpha\right)+\left(\alpha-\bar{x}\right)\right]}{t\bar{x}\left(\bar{x}-\alpha\right)}at^n=\frac{p\left(t\right)at^n}{t\bar{x}\left(\bar{x}-\alpha\right)}$$

As mentioned earlier exists $t_0\in\left(0,1\right)$ such that $p(t_0) = 0$ and $0<t_0^2<t_0<1, p\left(t_0^2\right)<0$. Posting $t = t_0$, we get

$$\left(\frac{a}{t_0\bar{x}}-a-\frac{at_0}{\bar{x}-\alpha}\right)t_0^n=\frac{p(t_0)at_0^n}{t_0\bar{x}(\bar{x}-\alpha)}=0$$

From (19) it follows $\dfrac{b}{\bar{x}}t_0^{2n-2}-\dfrac{a^2}{2\bar{x}^2}t_0^{2n-2}=bt_0^{2n}+\dfrac{bt_0^{2n+2}}{\bar{x}-\alpha}-\dfrac{a^2t_0^{2n+2}}{2(\bar{x}-\alpha)^2}$

$$F=\left\{\left(\frac{b}{\bar{x}}-\frac{a^2}{2\bar{x}^2}\right)\frac{1}{t_0^2}-b-\left[\frac{b}{\bar{x}-\alpha}-\frac{a^2}{2(\bar{x}-\alpha)^2}\right]t_0^{2n}\right\}t_0^{2n}+o\left(t_0^{2n}\right)$$

$$=\frac{2b(\bar{x}-\alpha)\bar{x}p(t_0^2)-a^2t_0^4\left[\bar{x}^2t_0^4-(\bar{x}-\alpha)^2\right]}{2\bar{x}^2(\bar{x}-\alpha)^2t_0^2}.t_0^{2n}+o\left(t_o^{2n}\right)$$

Setting F = (Bb+C)$t_0^{2n}+o\left(t_0^{2n}\right)$

$$H_{t_o}(q)=Bq+C=0\rightarrow q_0=-\frac{C}{B}$$

$$H_{t_0}^{'}(q)=B=\frac{2b(\bar{x}-\alpha)\bar{x}p(t_0^2)}{2\bar{x}^2(\bar{x}-\alpha)^2t_0^2}<0$$

We obtain that there are $q_1 < q_0$ and $q_2 > q_0$ such that $H_{to}(q_1) > 0$, $H_{t_0}(q_2)<0, q_1<q_0<q_2$. We assume that $a\neq 0,\ if\ \hat{\varphi}=\bar{x}+at_0^n+q_0t_0^{2n}$, we obtain

$$F\left(\hat{\varphi}_{n-1},\hat{\varphi}_n,\hat{\varphi}_{n+1}\right)\sim\left[q_0B+C\right]t_0^{2n}+o\left(t_0^{2n}\right)$$

With the notation

$$y_n=\bar{x}+at_0^n+q_1t_0^{2n},\ z_n=\bar{x}+at_0^n+q_2t_0^{2n}$$

We get

$$F\left(y_{n-1},y_n,y_{n+1}\right)\sim\left[q_1B+C\right]t_0^{2n};$$

$$F\left(z_{n-1},z_n,z_{n+1}\right)\sim\left[q_2B+C\right]t_0^{2n}$$

These relations show that inequalities (10) are satisfied for sufficiently large n, where $g=F+x_{n-1}$ and F is given by (18). Because the function $g\left(x_{n-1},x_n,x_{n+1}\right)$ is continuous and nondecreasing on $\left[\bar{x},+\infty\right)^3\rightarrow\left[\bar{x},+\infty\right)$. We easily have $g\left(\bar{x},\bar{x},\bar{x}\right)=\bar{x}$. We can apply the Theorem 2.1 with I = $\left[\bar{x},+\infty\right)$ and see that there is an $n_0\geq 0$ and a solution of equation (14) with theasymptotics $x_n=\hat{\varphi}_n+o\left(t_0^{2n}\right),\ for\ n\geq 0\ where\ b=q_0\ in\ \hat{\varphi}_n$. In particular, the solution converges monotonically to the positive equilibrium point for $n_0\geq 0$. The proof is complete.

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