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DYSFUNCTION IN SMOOTH PURSUIT EYE MOVEMENT- A CASE STUDY OF UNDAMPED FREE OSCILLATIONS

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ABSTRACT

This paper presents a simple deterministic model of dysfunctions of eye- tracking. The model is formulated as a second order nonlinear ordinary differential equation, incorporating non Hookesien cubic restoring force. The equation is solved analytically by employing a perturbation technique with the nonlinear restoring force coefficient as the perturbation parameter. Jump phenomena in angular displacement, angular velocity were discussed for wide spectrum of parameter values. Identified the equilibrium points of the model equation and stability of equilibrium points is also discussed.

Key words: *Pursuit Eye Tracking, Nonlinear Oscillations, Jump Phenomena.*

1. INTRODUCTION

Dysfunctions [5, 10] in smooth pursuit eye movement are frequently encountered in schizophrenia patients [3], and also in some individuals with disorders of their central nervous system may be due to generic reasons [4]. The person suffering with such dysfunction would have to rotate the eye to track the signal which is coming from the periodically moving target [1],[2]. When the target motion is periodic the eye ball oscillations observed as a case study of undamped free oscillations [6]. This situation is modeled mathematically using a second order nonlinear ordinary differential equation of duffing type with a cubic non -Hooksian restoring force subject to non homogeneous initial conditions. An approximate solution of the modeled equation is obtained by employing a perturbation technique [9]. The perturbation parameter (ε) is characteristic of the nonlinearity of the restoring force. The angular displacement versus dimension less time profiles, the angular velocity versus dimensionless time profiles and phase plane portraits are illustrated for wide spectra of the perturbation parameter (ε), the initial angular displacement (a) and the initial angular velocity (b). A look at these illustrations shows the resonating character of both the angular displacement and angular velocity with the increase of initial data and increasing the coefficient of nonlinear restoring force, and spiral type of variations in the phase planes in the case of soft spring are discussed. The phenomenon of jump in dysfunction dynamic parameters angular displacement, angular velocity was discussed for wide spectrum of parameters and establishing the nonlinearity in the jump of dysfunction dynamic parameters using Lagrangian interpolation. Identified the equilibrium points of the model equation and stability of equilibrium points is also discussed.

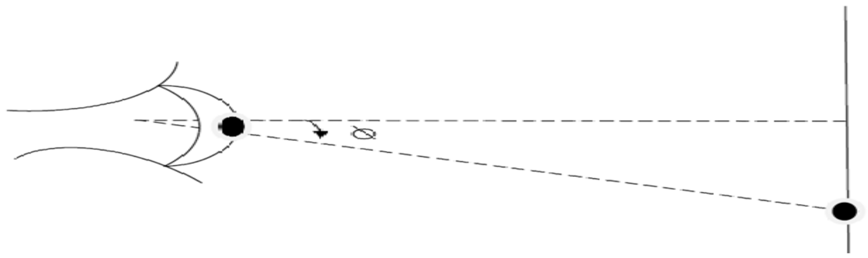

Fig. 1: Schematic sketch of the eye ball movement under investigation

2. NOTATION ADOPTED

Φ : The angle between normal to the screen and the line connecting T he target's position at time t, and the eye

I : Moment of inertia of the eye about the axis i.e. normal to the screen

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α	:	The damping coefficient
k	:	Hooksian restoring constant
L	:	A nonlinear non hooksian restoring coefficient
A	:	peak to peak amplitude of the target moving periodically
w_d	:	The frequency of the target

3. MATHEMATICAL MODEL EQUATION

The deterministic eye dynamics in the presence of a target which is moving periodically, a nonlinear differential equation, is given by [perspectives in biological dynamics and theoretical medicine]

$$I\frac{d^2\phi}{dt^2}+\alpha\frac{d\phi}{dt}+k\phi-L\phi^3=A\cos(\omega_d t) \tag{1}$$

$$\phi(0)=\alpha,\ \phi^{\bullet}(0)=\beta \tag{2}$$

where α and β are the initial values of the angular displacement and initial angular velocity of the eye ball respectively.

In terms of the following non dimensional parameters,

$$t=\frac{\tau}{\omega_0},\ \frac{\phi}{\phi_0}=\psi,\frac{k}{I}=\omega_0^2,\frac{\omega_d}{\omega_0}=\Omega,\frac{\alpha}{I\omega_0}=2\delta,\frac{\phi_0^2 L}{I\omega_0^2}=\varepsilon,\frac{A}{I\omega_0^2\phi_0}=\Gamma$$

$$\phi(t)=\phi_0\psi(\frac{\tau}{\omega_0}),\quad \phi(0)=\phi_0\psi(0)$$

$$\alpha=\phi_0\psi_0,\quad \frac{\alpha}{\phi_0}=a$$

$$\phi^{\bullet}(0)=\omega_0\phi_0 \qquad \psi^{\bullet}(0),\beta=\omega_0\phi_0 b \tag{3}$$

By using (3) equation (1) can be reduced to

$$\varphi_0 I\frac{d^2\psi}{dt^2}+\alpha\varphi_0\frac{d\psi}{dt}+k\varphi_0\psi-L\varphi_0^{\ 3}\psi^3 = A\cos(\omega_d t) \tag{4}$$

$$\omega_0^{\ 2}\varphi_0 I\frac{d^2\psi}{d\tau^2}+\alpha\varphi_0\omega_0\frac{d\psi}{d\tau}+k\varphi_0\psi-L\varphi_0^{\ 3}\psi^3 = A\cos(\omega_d\frac{\tau}{\omega_0}) \tag{5}$$

$$\frac{d^2\psi}{d\tau^2}+2\delta\frac{d\psi}{d\tau}+\psi-\varepsilon\psi^3=\Gamma\cos(\Omega\tau) \tag{6}$$

with initial conditions $\psi(0)=a,\psi^{*}(0)=b$ (7)

The coefficient ε signifies non Hooksian character of the nonlinear restoring force. This formulation signifies undamped free oscillations of dysfunctions in eye movement take δ =0, r=0

The equation (4) reduces to $\frac{d^2\psi}{d\tau^2}+\psi-\varepsilon\psi^3=0$ (8)

with initial conditions $\psi(0)=a,\ \psi(0)$=b (9)

The spring is soft (or) hard accordingly ε is negative and positive respectively

4. ANALYTIC SOLUTION:

Let the $\psi(\tau)=\psi^{(0)}(\tau)+\varepsilon\psi^{(1)}(\tau)+\varepsilon^2\psi^{(2)}(\tau)+. \ .$ (10)

Substituting (8) in the equation (6) and collecting the like powers of ε on both sides of equality we get the equations in the successive stages of approximation.

5. THE BASIC (OR) ZERO TH ORDER APPROXIMATION

In this approximation the equation to be solved is

$$\frac{d^2\psi^{(0)}}{d\tau^2}+\psi^{(0)}=0 \tag{11}$$

With the initial conditions $\psi^{(0)}(0)=a, \quad \psi^{(0)\bullet}(0)=b$ (12)

This yields the solution is $\psi^{(0)}(\tau) = a\cos(\tau)+b\sin(\tau)$ (13)

6. THE FIRST ORDER APPROXIMATION

The equation for $\psi^{(1)}(\tau)$ is $\dfrac{d^2\psi^{(1)}}{d\tau^2}+\psi^{(1)}=(a\cos\tau+b\sin\tau)^3$ (14)

With initial conditions $\psi^{(1)}(0)=0, \quad \psi^{\bullet(1)}(0)=0$ (15)

The solution of equation (14) satisfying the initial conditions in (15) is

$$\begin{aligned}\psi^{(1)}(\tau)=&\frac{(a^3-3a^2b)}{32}\cos\tau+\frac{9(b^3+5a^2b)}{32}\sin\tau-\frac{(a^3-3a^2b)}{32}\cos 3\tau\\ &-\frac{(3a^2b-b^3)}{32}\sin 3\tau-\frac{(3b^3+3a^2b)}{8}\tau\cos\tau+\frac{(3a^3+3ab^2)}{8}\tau\sin\tau\end{aligned} \tag{16}$$

Hence up to this order of approximation the angular displacement is given by

$$\psi(\tau)=\psi^{(0)}(t)+\varepsilon\psi^{(1)}(\tau) \tag{17}$$

$$\psi(\tau)=(a\cos\tau+b\sin\tau)+\frac{\varepsilon}{32}\left[\begin{aligned}&(a^3-3a^2b)\cos\tau+9(b^3+5a^2b)\sin\tau-(a^3-3a^2b)\cos 3\tau\\ &-(3a^2b-b^3)\sin 3\tau-4(3b^3+3a^2b)\tau\cos\tau+4(3a^3+3ab^2)\tau\sin\tau\end{aligned}\right] \tag{18}$$

And the angular velocity is given by

$$\psi^{\bullet}(\tau)=(-a\sin\tau+b\cos\tau)+\frac{\varepsilon}{32}\left[\begin{aligned}&-(a^3-3a^2b)\sin\tau+9(b^3+5a^2b)\cos\tau+3(a^3-3a^2b)\sin 3\tau\\ &-3(3a^2b-b^3)\cos 3\tau-4(3b^3+3a^2b)(\cos\tau-\tau\sin\tau)\\ &+4(3a^3+3ab^2)\tau(\cos\tau+\sin\tau)\end{aligned}\right] \tag{19}$$

7. THE ANGULAR DISPLACEMENT VERSES TIME PROFILES FOR DIFFERENT VALUES OF PARAMETERS ARE GIVEN BELOW

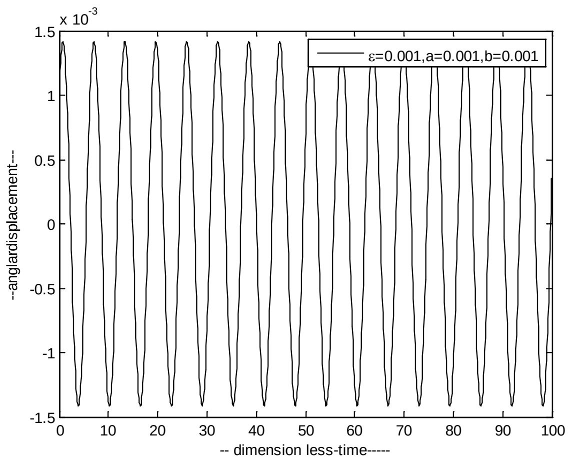

Fig. 2: variation of angular displacement versus time for the nonlinear restoring coefficient ε=0.001, initial angular displacement a=0.001 and initial angular velocity b=0.001, For this parameter values the amplitude of angular displacement is not changing as time increases.

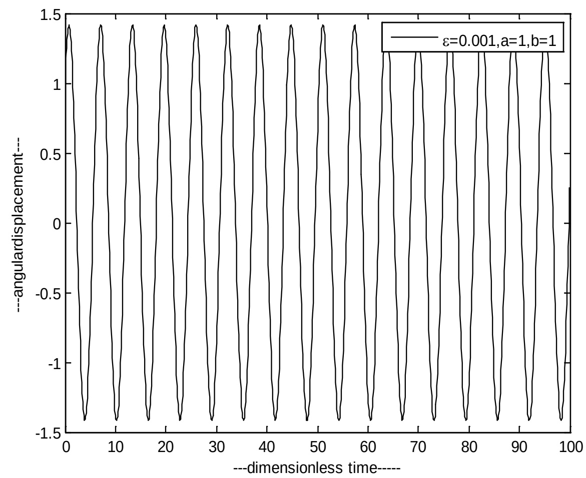

Fig. 3: variation of angular displacement versus time for the nonlinear restoring coefficient ε=0.001, initial angular displacement a=1 and initial angular velocity b=1. For this parameter values the amplitude of angular displacement is not changing as time increases.

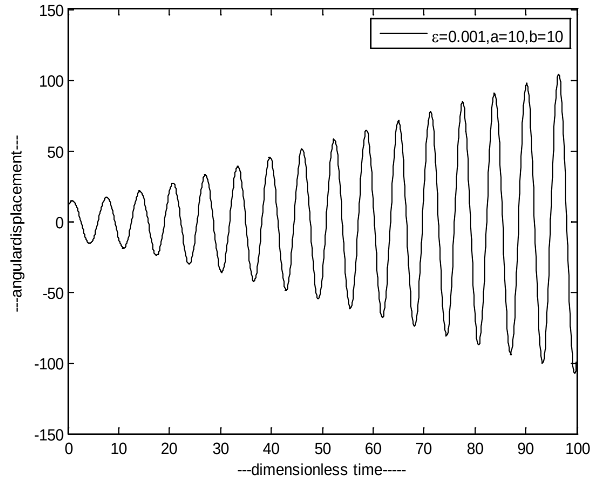

Fig. 4: variation of angular displacement versus time for the nonlinear restoring coefficient ε=0.001, initial angular displacement a=10 and initial angular velocity b=10. For this parameter values the amplitude of angular displacement is increasing as time increases.

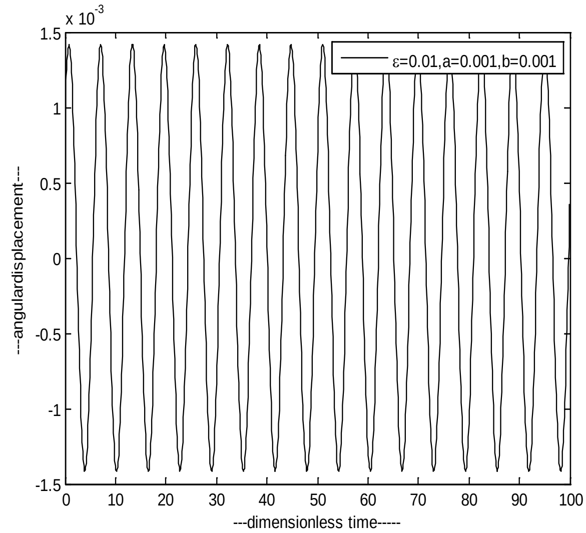

Fig. 5: variation of angular displacement versus time for the nonlinear restoring coefficient ε=0.01, initial angular displacement a=0.001 and initial angular velocity b=0.001, For this parameter values the amplitude of angular displacement is not changing as time increases.

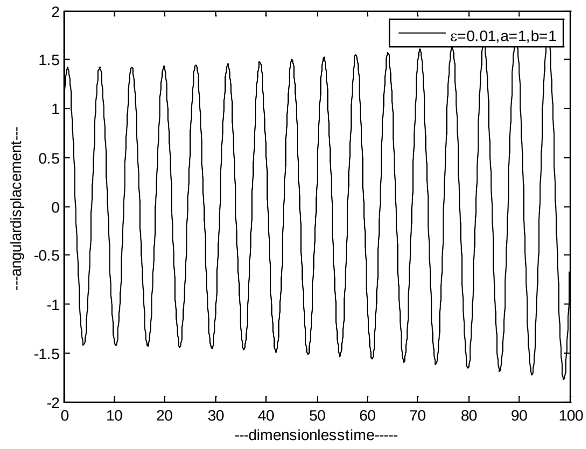

Fig. 6: variation of angular displacement versus time for the nonlinear restoring coefficient ε=0.01, initial angular displacement a=1 and initial angular velocity b=1. For this parameter values the amplitude of angular displacement is increasing as time increases.

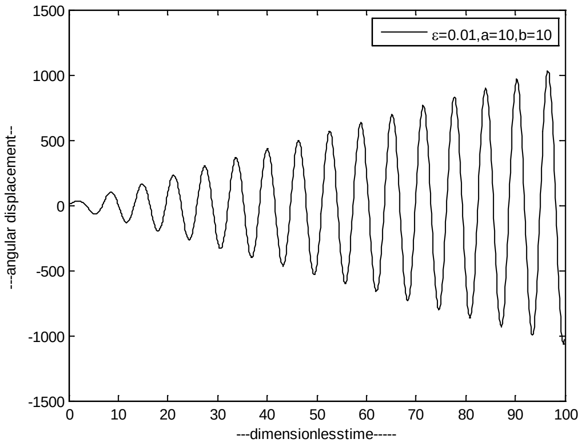

Fig. 7: variation of angular displacement versus time for the nonlinear restoring coefficient ε=0.01, initial angular displacement a=10 and initial angular velocity b=10. For this parameter values the amplitude of angular displacement is increasing rapidly as time increases.

8. PHASE PLANE PORTRAITS

The angular velocity versus angular displacement profiles (analytical phase plane portraits) are given for different values of parameter

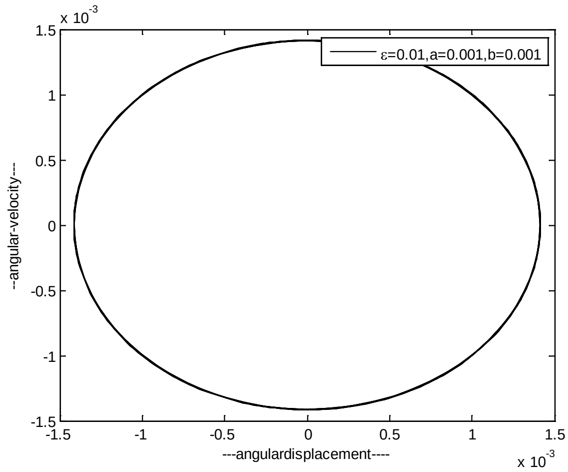

Fig. 8: variation of angular velocity versus angular displacement for the nonlinear restoring coefficient ε=0.01, initial angular displacement a=0.001 and initial angular velocity b=0.001

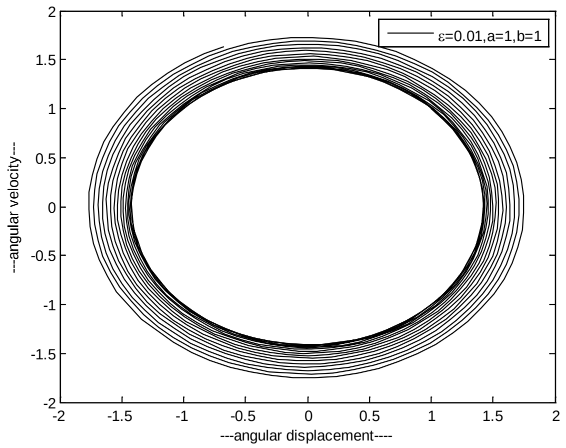

Fig. 9: variation of angular velocity versus angular displacement for the nonlinear restoring coefficient ε=0.01, initial angular displacement a=1and initial angular velocity b=1

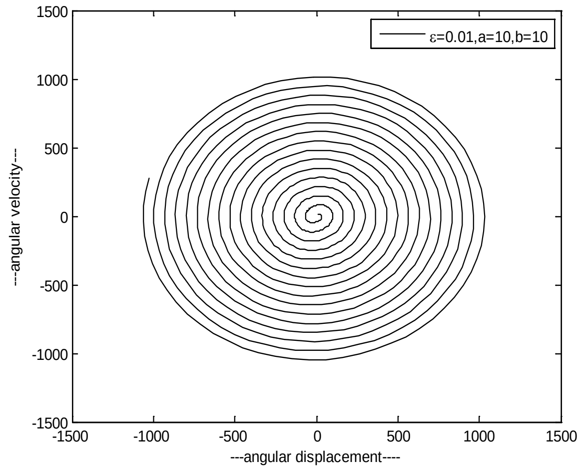

Fig. 10: variation of angular velocity versus angular displacement for the nonlinear restoring coefficient ε=0.01, initial angular displacement a=10and initial angular velocity b=10.

9. JUMP PHENOMINA IN PEAK TO PEAK AMPLITUDE OF DYSFUNCTION DYNAMIC PARAMETERS -ANGULAR DISPLACEMENT AND ANGULAR VELOCITY

The dysfunction dynamical parameters angular displacement and angular velocity are simulated in the time interval [0,100] using the analytical solution of the model (18), (19) and using the software MATLAB. For different values of the parameters in the model such as initial angular displacement, initial angular velocity and coefficient of nonlinear restoring force are taken at different levels and using the analytical solution of the model. Jump in peak to peak amplitude of The dysfunction dynamical parameters angular displacement and angular velocity calculated as shown in the Fig (11).

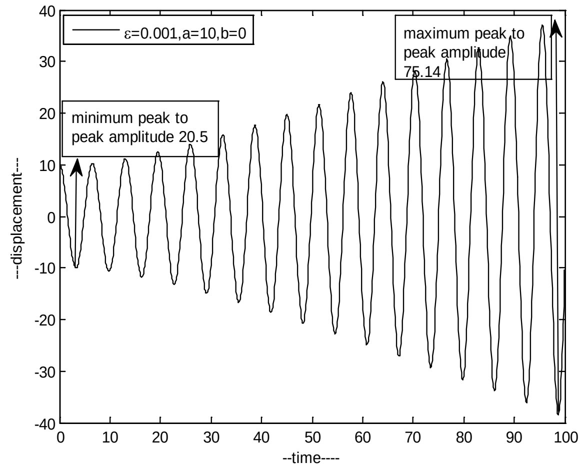

Fig. 11: The figure shows the jump phenomena of dysfunction dynamic parameter i.e (the angular displacement)

10. JUMP PHENOMINA IN PEAK TO PEAK AMPLITUDE OF DYSFUNCTION DYNAMIC PARAMETERS -ANGULAR DISPLACEMENT:

The Nonlinear restoring coefficient ε is taken in three levels(0.001,0.01,0.1,1) together with no initial angular displacement and varying the initial angular velocity(b),Then The jump in peak to peak amplitude in angular displacement is function of initial angular velocity (b). The graph of Lagrangian interpolated polynomial is given for each case, by taking the values of parameter (b) taken on x-axis and the corresponding jump in peak to peak amplitude taken on Y-axis.

Table: 1

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.001	0	0.1	0.2	0.2	0
0.001	0	0.5	1	1	0
0.001	0	1	2	2	0
0.001	0	5	10.064	13.8	3.736
0.001	0	10	20.5	75.14	54.64
0.001	0	15	32	235	203
0.001	0	25	63	1106	1043
0.001	0	50	310	8818	8508

Table: 1 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular displacement when ε=0.001, initial angular displacement (a=0) and with varying angular velocity

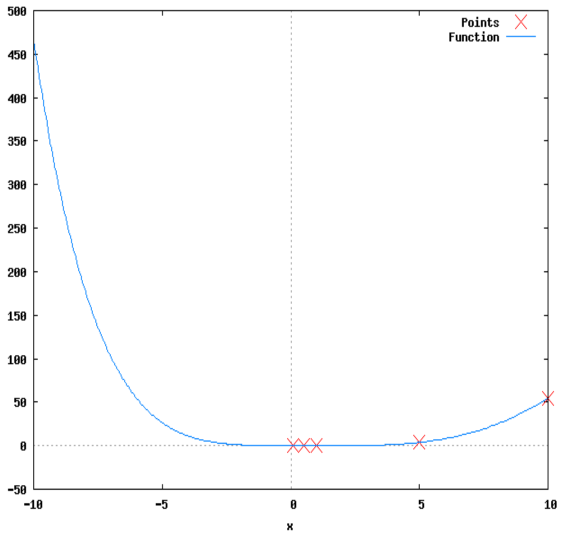

Fig. 12: When nonlinear restoring coefficient ε =0.001, with no initial angular displacement and initial angular displacement (b) is changing . The jump is linear up to b=5 and then it is nonlinear.

Table: 2

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.01	0	0.1	0.2	0.2	0
0.01	0	0.5	1	1	0
0.01	0	1	2	2	0
0.01	0	5	10.746	90	79.254
0.01	0	10	32	720	688
0.01	0	15	84	2500	2416
0.01	0	25	366	11826	11460
0.01	0	50	2824	88240	85416

Table 2: Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular displacement when ε=0.01 , initial angular displacement (a=0) and with varying angular velocity.

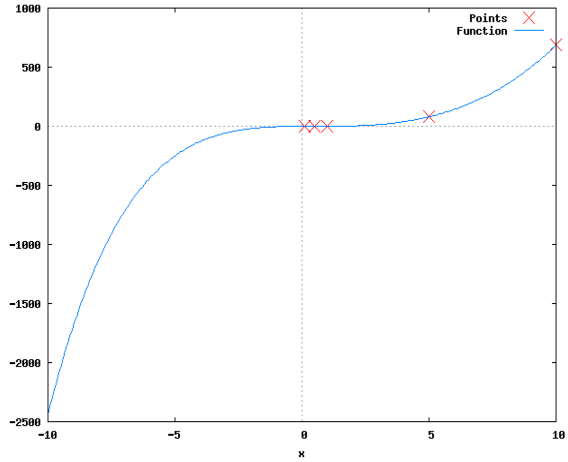

Fig. 13: When nonlinear restoring coefficient ε =0.01, with no initial angular displacement initial angular displacement (b) is changing. The jump is linear up to b < 5 and then it is nonlinear.

Table: 3

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.1	0	0.1	0.2	0.2	0
0.1	0	0.5	1	1.3744	0.3744
0.1	0	1	2.05	7.514	5.464
0.1	0	5	31	921.8	890.8
0.1	0	10	234	7058	6824
0.1	0	15	761.2	24200	23438.8
0.1	0	25	3546	114300	110754
0.1	0	50	29060	942400	913340

Table: 3 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular displacement when ε=0.1, initial angular displacement (a=0) and with varying angular velocity.

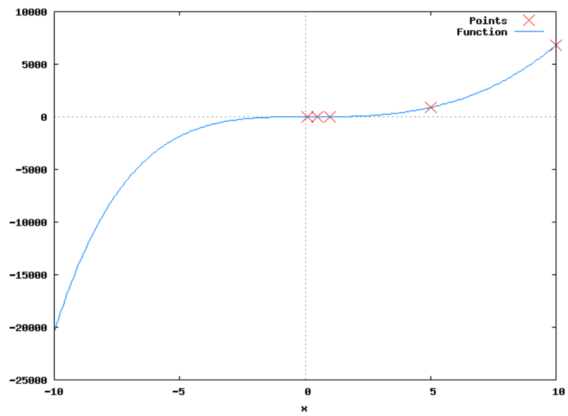

Fig. 14: When nonlinear restoring coefficient ε =0.1, with no initial angular displacement initial angular displacement (b) is changing. The jump is linear up to when b < 2 and then it is nonlinear.

Table: 4

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
1	0	0.1	0.2004	0.2138	0.0134
1	0	0.5	1.0746	9.486	8.4114
1	0	1	3.094	76.74	73.646
1	0	5	294.6	9224	8929.4
1	0	10	2336	74580	72244
1	0	15	3924	242200	238276
1	0	25	36780	1183000	1146220
1	0	50	294000	9424000	9130000

Table: 4 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular displacement when ε=1, initial angular displacement (a=0) and with varying angular velocity.

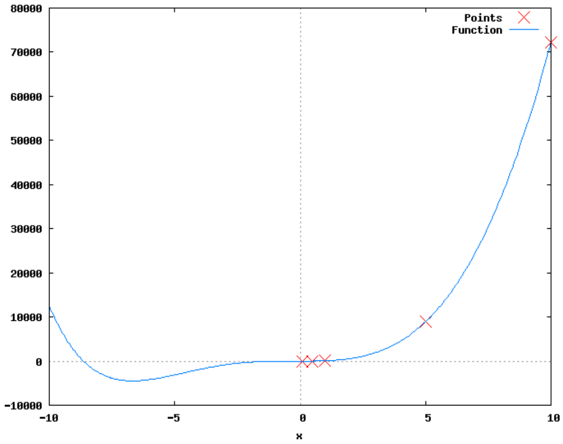

Fig. 15: When nonlinear restoring coefficient ε =1, with no initial angular displacement initial angular displacement (b) is changing. The jump is linear up to b < 1 and then it is nonlinear.

11. JUMP PHENOMENA WHEN NO TAKE OFF ANGULAR VELOCITY

The Nonlinear restoring coefficient ε is taken in three levels(0.001,0.01,0.1,1) together with no initial angular velocity(b), and changing the initial angular displacement(a) then jump in peak to peak amplitude in angular displacement is function of initial angular displacement (a). The graph of Lagrangian interpolated polynomial is given for each case, by taking The values of parameter (a) taken on x-axis and the corresponding jump in peak to peak amplitude taken on Y-axis.

Table: 5

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.001	0.1	0	0.2	0.2	0
0.001	0.5	0	1	1	0
0.001	1	0	2	2	0
0.001	5	0	10	13.7	3.7
0.001	10	0	20	76.74	56.74
0.001	15	0	31.08	251.8	220.72
0.001	25	0	49.62	1160.2	1110.58
0.001	50	0	146.1	9268	9121.9

Table: 5 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular displacement when ε=0.001, initial angular velocity (b=0) and with varying angular displacement.

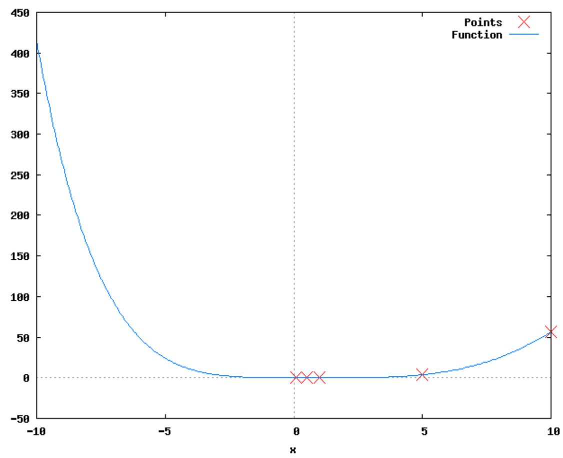

Fig. 16: When nonlinear restoring coefficient ε =0.001, with no initial angular velocity and with varying angular displacement (a). The jump is linear up to when b=5 and then it is nonlinear.

Table: 6

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.01	0.1	0	0.2	0.2	0
0.01	0.5	0	1	1	0
0.01	1	0	2	2.216	0.126
0.01	5	0	11.952	93.18	81.228
0.01	10	0	60.46	741.8	680.74
0.01	15	0	114.74	2502	2387.26
0.01	25	0	547.8	11588	11040.2
0.01	50	0	1531.4	92720	91188.6

Table: 6 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular displacement when ε=0.01, initial angular velocity (b=0) and with varying angular displacement.

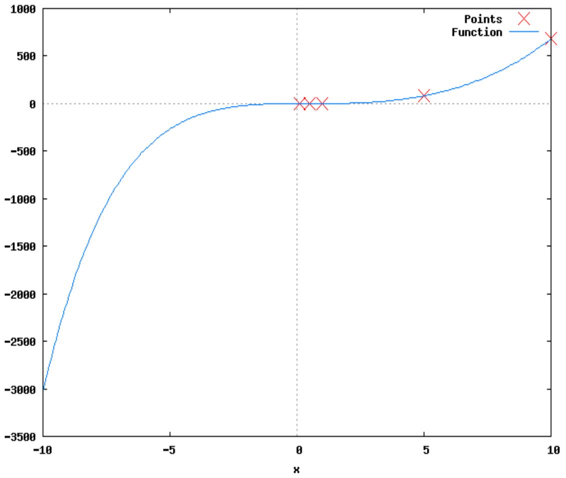

Fig. 17: When nonlinear restoring coefficient ε =0.01, with no initial angular velocity and with varying angular displacement (a). The jump is linear up to b<5 and then it is nonlinear.

Table: 7

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.1	0.1	0	0.2	0.2	0
0.1	0.5	0	1	1.3574	0.3574
0.1	1	0	2	7.674	5.674
0.1	5	0	42.7	926.8	884.1
0.1	10	0	354.6	7418	7063.4
0.1	15	0	1169.8	25040	23870.2
0.1	25	0	5218	115900	110682
0.1	50	0	44540	927200	882660

Table: 7: Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular displacement when ε=0.1, initial angular velocity (b=0) and with varying angular displacement.

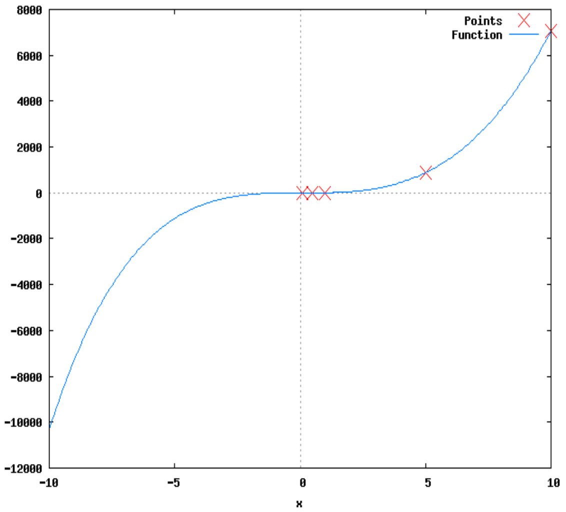

Fig. 18: When nonlinear restoring coefficient ε =0.1, with no initial angular velocity and with varying angular displacement (a). The jump is linear up to b<1 and then it is nonlinear.

Table: 8

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
1	0.1	0	0.2	0.2	0
1	0.5	0	1.1984	9.318	8.1196
1	1	0	3.84	74.1	70.26
1	5	0	444.6	9272	8827.4
1	10	0	3562	74180	70618
1	15	0	12028	250400	238372
1	25	0	55700	1159000	1103300
1	50	0	445600	9272000	8826400

Table: 8 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular displacement when ε=1, initial angular velocity (b=0) and with varying angular displacement.

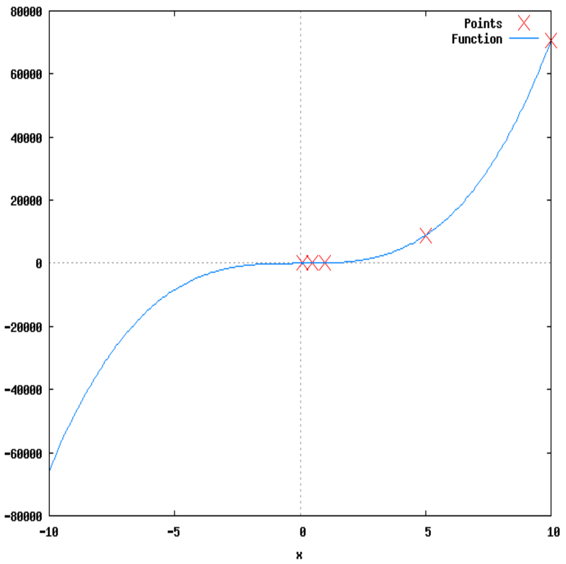

Fig. 19: When nonlinear restoring coefficient ε =1, with no initial angular velocity and with varying angular displacement (a). The jump is linear up to when b=0.5 and then it is nonlinear.

12. .JUMP PHENOMINA IN PEAK TO PEAK AMPLITUDE OF DYSFUNCTION DYNAMIC PARAMETER - ANGULAR VELOCITY

Table: 9

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.001	0	0.1	0.2	0.2	0
0.001	0	0.5	1	1	0
0.001	0	1	2	2	0
0.001	0	5	10	13.548	3.548
0.001	0	10	20.12	76.5	56.38
0.001	0	15	30.9	251.6	220.7
0.001	0	25	63.78	1157.8	1094.02
0.001	0	50	449.6	9274	8824.4

Table: 9 Table gives the values of minimum and maximum values of peak to peak amplitude is and corresponding jump in peak to peak amplitude in angular velocity when ε=0.001, initial angular displacement (a=0) and with varying angular displacement(a).

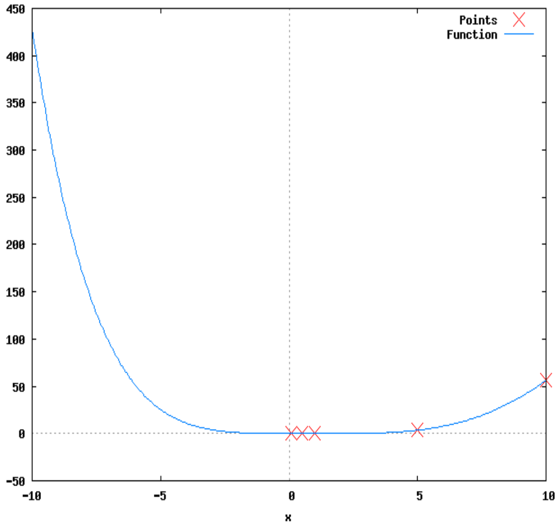

Fig. 20 When nonlinear restoring coefficient ε =0.001, with no initial angular displacement and with varying angular velocity (b). The jump is linear up to b=5 and then it is nonlinear.

Table: 10

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.01	0	0.1	0.2	0.2	0
0.01	0	0.5	1	1	0
0.01	0	1	1.998	2.128	0.13
0.01	0	5	10.408	93.1	82.692
0.01	0	10	35.7	741.4	705.7
0.01	0	15	120.6	2504	2383.4
0.01	0	25	578.2	11596	11017.8
0.01	0	50	4664	92780	88116

Table: 10 Table gives the values of minimum and maximum values of peak to peak amplitude is and corresponding jump in peak to peak amplitude in angular velocity when ε=0.01 and initial angular displacement (a=0) and with varying angular velocity (b).

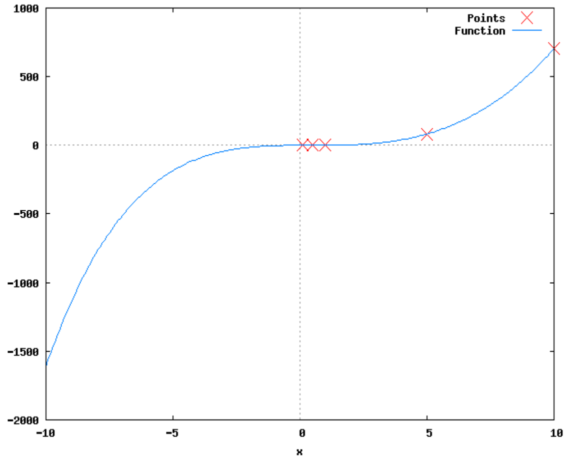

Fig. 21 When nonlinear restoring coefficient ε =0.01, with no initial angular displacement and with varying angular velocity (b). The jump is linear up to b<5 and then it is nonlinear.

Table: 11

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.1	0	0.1	0.2	0.2	0
0.1	0	0.5	1	1.3528	0.3528
0.1	0	1	2.012	7.65	5.638
0.1	0	5	44.96	927.4	882.44
0.1	0	10	370.8	7422	7051.2
0.1	0	15	1265.8	25040	23774.2
0.1	0	25	5910	115960	110050
0.1	0	50	47260	927800	880540

Table: 11 Table gives the values of minimum and maximum values of peak to peak amplitude is and corresponding jump in peak to peak amplitude in angular velocity when ε=0.1, initial angular displacement (a=0) and with varying angular velocity(b).

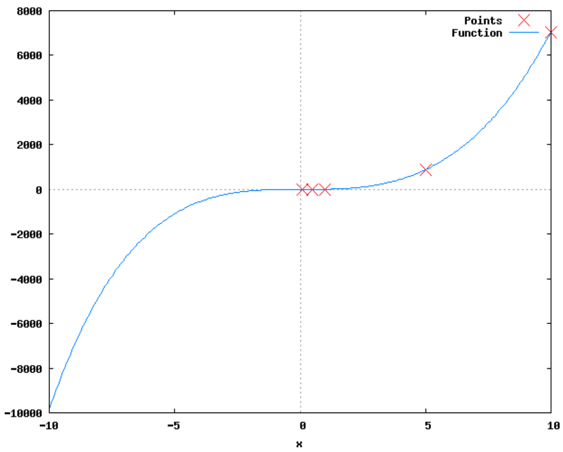

Fig. 22 When nonlinear restoring coefficient ε =0.1, with no initial angular displacement and with varying angular velocity (b). The jump is linear up to b=1 and then it is nonlinear.

Table: 13

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
1	0	0.1	0.19972	0.2126	0.01288
1	0	0.5	1.0296	9.31	8.2804
1	0	1	3.57	74.14	70.57
1	0	5	469.2	9278	8808.8
1	0	10	3786	74220	70434
1	0	15	12788	250400	237612
1	0	25	59100	1159600	1100500
1	0	50	474000	9278000	8804000

Table: 12 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular velocity when ε=1, initial angular displacement (a=0) and with varying angular velocity(b).

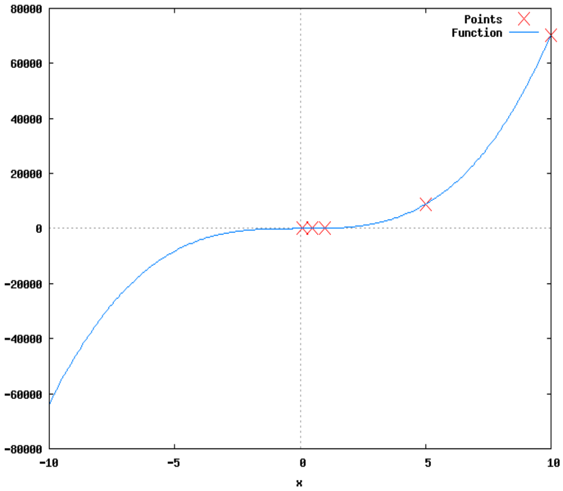

Fig. 23 When nonlinear restoring coefficient ε =1, with no initial angular displacement and with varying angular velocity (b). The jump is linear up to b=0.5 and then it is nonlinear.

Table: 13

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.001	0.1	0	0.2	0.2	0
0.001	0.5	0	1	1	0
0.001	1	0	2	2	0
0.001	5	0	9.938	13.4	3.462
0.001	10	0	20	75.14	55.14
0.001	15	0	30.76	247.6	216.84
0.001	25	0	76.14	1141	1064.86
0.001	50	0	602.4	9132	8529.6

Table: 13 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular velocity when ε=0.001, initial angular velocity (b=0) and with varying angular displacement(a).

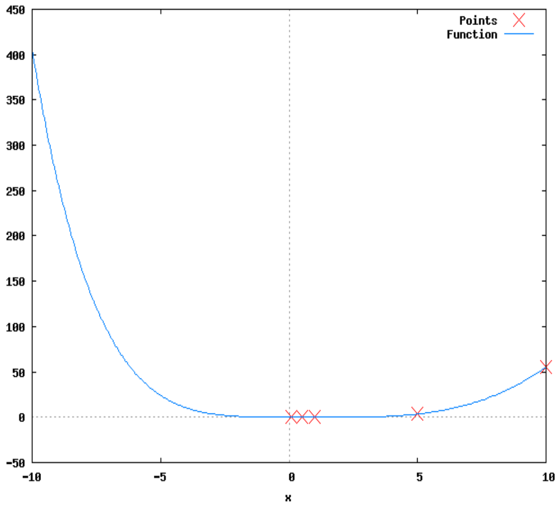

Fig. 24 When nonlinear restoring coefficient ε =0.001, with no initial angular velocity(b) and with varying angular displacement (a). The jump is linear up to b< 2 and then it is nonlinear.

Table: 14

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.01	0.1	0	0.2	0.2	0
0.01	0.5	0	0.9992	1.0034	0.0042
0.01	1	0	2	2.128	0.128
0.01	5	0	10.384	91.6	81.216
0.01	10	0	46.96	730.4	683.44
0.01	15	0	162.28	2466	2303.72
0.01	25	0	420.4	11414	10993.6
0.01	50	0	3578	91320	87742

Table: 14 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular velocity when ε=0.01, initial angular velocity (b=0) and with varying angular displacement(a).

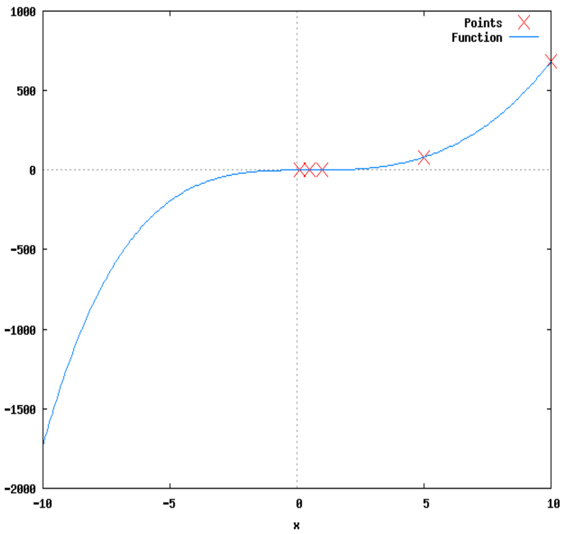

Fig. 25 When nonlinear restoring coefficient ε =0.01, with no initial angular velocity (b) and with varying angular displacement (a). The jump is linear up b=0.5and then it is nonlinear.

Table: 15

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
0.1	0.1	0	0.2	0.2	0
0.1	0.5	0	0.9938	1.34	0.3462
0.1	1	0	2	7.514	5.514
0.1	5	0	32.30	913.2	880.9
0.1	10	0	279.8	7306	7026.2
0.1	15	0	956.2	24660	23703.8
0.1	25	0	4512	114160	109648
0.1	50	0	36320	913200	876880

Table: 15 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular velocity when ε=0.1, initial angular velocity (b=0) and with varying angular displacement(a).

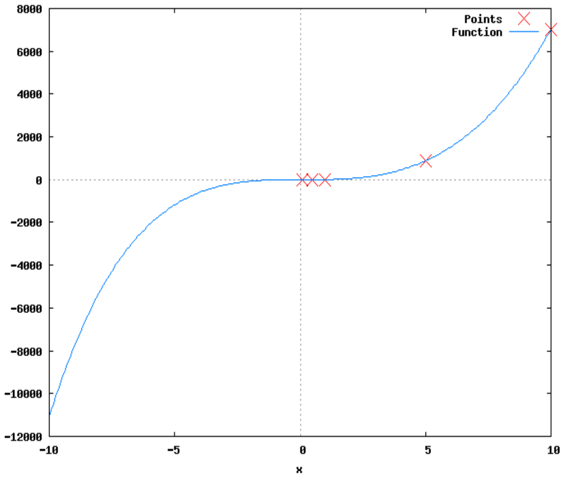

Fig. 26 When nonlinear restoring coefficient ε =0.1, with no initial angular velocity (b) and with varying angular displacement (a). The jump is linear up to b=0.1 and then it is nonlinear.

Table: 16

ε	a	b	Minimum value of peak to peak amplitude	Maximum value of peak to peak amplitude	Jump in peak to peak amplitude
1	0.1	0	0.2	0.2	0
1	0.5	0	0.9486	9.16	8.2114
1	1	0	4.696	73.04	68.344
1	5	0	359.2	9132	8772.8
1	10	0	2900	73060	70160
1	15	0	9786	246600	236814
1	25	0	45360	1141600	1096240
1	50	0	363000	9132000	8769000

Table: 16 Table gives the values of minimum and maximum values of peak to peak amplitude and corresponding jump in peak to peak amplitude in angular velocity when ε=1, initial angular velocity (b=0) and with varying angular displacement(a).

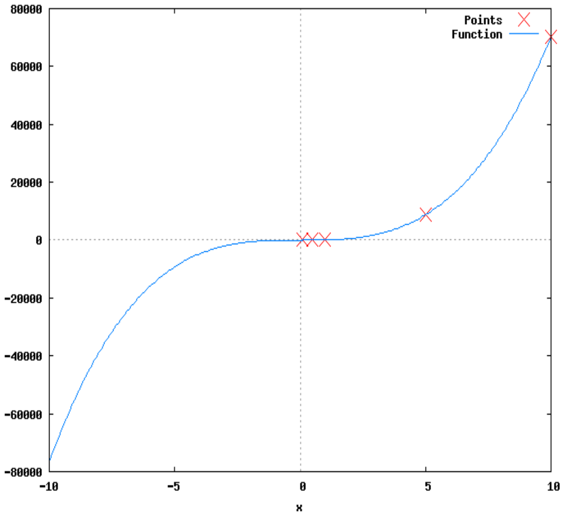

Fig. 27: When nonlinear restoring coefficient ε =0.1, with no initial angular velocity (b) and with varying angular displacement (a). The jump is linear up to when b=5 and then it is nonlinear.

13. STABILITY ANALYSIS

The differential equation

$$\frac{d^2\psi}{d\tau^2}+\psi-\varepsilon\psi^3=0 \tag{20}$$

The above differential equation divided into two first order differential equations

$$\frac{d\psi}{d\tau}=y \tag{21}$$

$$\frac{dy}{d\tau}=\varepsilon\psi^3-\psi \tag{22}$$

The system has the following four equilibrium states (i)-(iii) resulting from

$$\frac{d\psi}{d\tau}=y=0;$$

$$\frac{dy}{d\tau}=\varepsilon\psi^3-\psi=0 \tag{23}$$

E_1: state in which both angular displacement angular velocity are zero

$$\bar{\psi}=0;\ \bar{y}=0 \tag{24}$$

E_2: The state in which only the angular displacement is not equal to zero

And angular velocity is equal to zero $\bar{\psi}=\pm(1/\sqrt{\varepsilon})$; $\bar{y}=0$ (25)

14. STABILITY OF THE EQUILIBRIUM STATES

15. Stability of the Equilibrium State E_1:

$$\bar{\psi}=0;\ \bar{y}=0.$$

We consider slight deviations $u_1(\tau)$ and $u_2(\tau)$ over the steady state $(\bar{\psi}, \bar{y})$

$$\psi=\bar{\psi}+u_1(\tau) \tag{26}$$

$$y=\bar{y}+u_2(\tau) \tag{27}$$

Where $u_1(\tau)$ and $u_2(\tau)$ are small so that terms other than the first order can be neglected.

By substituting (2.26) and (2.27) in (2.21) and (2.22) we get

$$\frac{du_1}{d\tau}=\bar{y}+u_2 \tag{28}$$

$$\frac{du_2}{d\tau}=\varepsilon[(\bar{\psi})^3+3(\bar{\psi})^2u_1+3(\bar{\psi})u_1^2+u_1^3]-(\bar{\psi})-u_1 \tag{29}$$

By neglecting products and second and higher powers of u_1 and u_2, we get

$$\frac{du_1}{d\tau}=u_2;\quad \frac{du_2}{d\tau}=-u_1 \tag{30);(31}$$

Whose roots are $\pm i$, both the roots are complex. Hence the steady state is **unstable.** Further from (30) and (31) we get

$$u_1=u_{10}\cos\tau+u_{20}\sin\tau; \tag{32}$$

$$u_2 = -u_{10}\sin\tau + u_{20}\cos\tau \tag{33}$$

Where u_{10}, u_{20} are initial values of u_1, u_2 respectively and the solution curves are shown in Figures 28 to 31 and the conclusions are presented below.

16. Stability of the equilibrium states E_2, E_3: $\overline{\psi} = \pm\frac{1}{\sqrt{\varepsilon}}$; $\overline{y} = 0$

By substituting (2.26) and (2.27) in (2.21) and (2.22) we get

$$\frac{du_1}{d\tau} = \overline{y} + u_2 \tag{34}$$

$$\frac{du_2}{d\tau} = \varepsilon[(\overline{\psi})^3 + 3(\overline{\psi})^2 u_1 + 3(\overline{\psi})u_1^2 + u_1^3] - (\overline{\psi}) - u_1 \tag{35}$$

By neglecting products and second and higher powers of u_1 and u_2, the corresponding linearised perturbed equations are

$$\frac{du_1}{d\tau} = u_2 \tag{36}$$

$$\frac{du_2}{d\tau} = 2u_1 \tag{37}$$

Whose roots are $\pm\sqrt{2}$ both the roots are real and opposite. Hence the steady state is **unstable** .The solutions of equations (36) and (37) are given by

$$u_1(\tau) = \left(\frac{u_{10} + \frac{u_{20}}{\sqrt{2}}}{2}\right)e^{\sqrt{2}\tau} + \left(\frac{u_{10} - \frac{u_{20}}{\sqrt{2}}}{2}\right)e^{-\sqrt{2}\tau} \text{ and } u_2(\tau) = \left(\frac{u_{10} + \frac{u_{20}}{\sqrt{2}}}{\sqrt{2}}\right)e^{\sqrt{2}\tau} - \left(\frac{u_{10} - \frac{u_{20}}{\sqrt{2}}}{\sqrt{2}}\right)e^{-\sqrt{2}\tau} \tag{38}$$

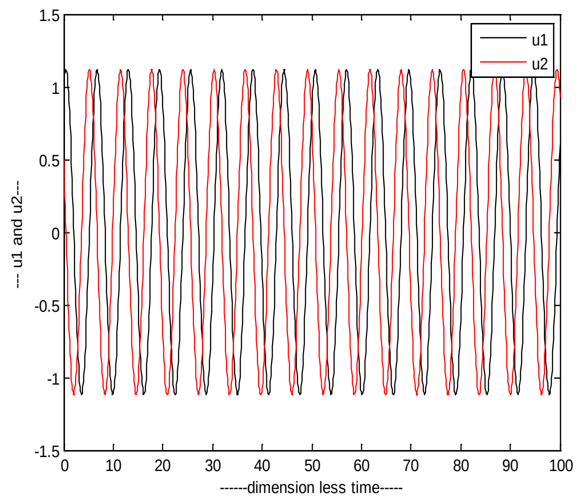

Fig. 28: variation of $u_1(\tau)$ and $u_2(\tau)$ versus dimensionless time at equilibrium E_1

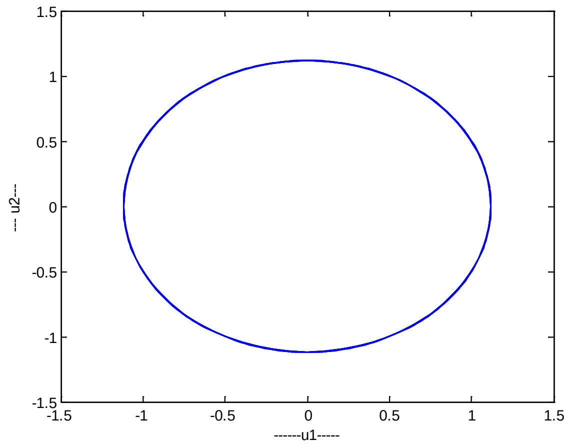

Fig. 29: Trajectories of perturbed angular displacement $u_1(\tau)$ and angular velocity $u_2(\tau)$ at E_1

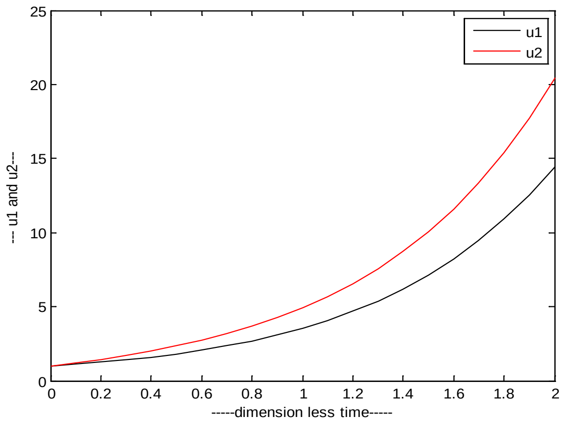

Fig. 30: variation of $u_1(\tau)$ and $u_2(\tau)$ versus dimensionless time at equilibrium $E_3$and E_4

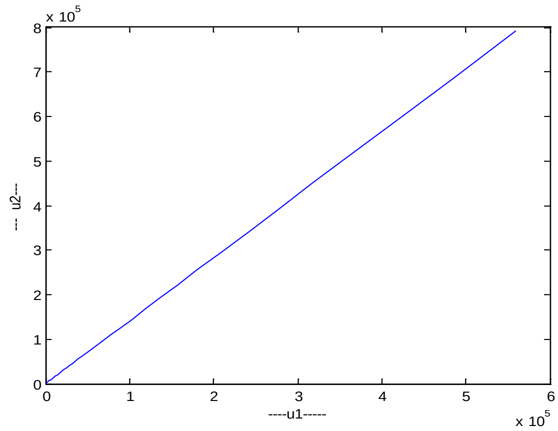

Fig. 31: Trajectories of perturbed angular displacement $u_1(\tau)$ and angular velocity $u_2(\tau)$ at E_2 and E_3

CONCLUSIONS

(1) The resonating character in both the dysfunction dynamic parameters (angular displacement, angular velocity) is increasing with increasing of initial data and the coefficient of nonlinear restoring force(ε).

(2) The Lagrangian interpolating patterns showing that the jump in peak to peak amplitude is Linear for lower values of initial data and lower values of non linear restoring coefficient, the nonlinearity appears in early stages with increase of parameters (a,b,ε).

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