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ON CERTAIN KINDS OF CHARACTERIZATIONS OF ALMOST PRIMARY IDEALS IN BOOLEAN LIKE SEMI RINGS

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ABSTRACT

In this paper we introduce the notion of weekly primary ideals and almost primary ideals in a Boolean like semi-ring and obtain various properties of these ideals. Some characterizations have been given for almost primary ideals in terms of semi prime ideals, 2-potent prime ideals and radical of ideals. Finally we established that an ideal I of a Boolean like semi ring R is almost primary if and only if I/I^2 is a weakly primary ideal of R/I.

Key words: *Prime ideal, weakly prime ideal, Almost Prime ideal and Almost Primary ideal, 2-potent Prime ideal, semi prime ideal*

Mathematics Subject Classification: *16Y30, 16Y60.*

INTRODUCTION

It is Foster A. L. [3] who has initiated the study of Boolean like rings. A Boolean like ring R is a commutative ring with unity in which $a + a = 0$ and $ab(1+a)(1+b) = 0$ for all a, b in R. It is clear that the class of Boolean like rings is a wider class than the class of Boolean rings. Venkateswarlu et al [9] has introduced the notion of Boolean like semi-rings by generalizing the concept of Boolean like rings of Foster. Thus the class of Boolean like semi-rings is a wider class of Boolean like rings. Venkateswarlu and Murthy have made extensive study on Boolean like semi-rings in [10, 11]. In fact Boolean like semi rings are special classes of left near rings.

Recently Adil Kadir Jabbar and Chwas Abas Ahmed [1] introduced a new class of ideals namely Almost Primary ideals in commutative rings. They obtained interesting properties and characterizations of almost primary ideals. The aim of this paper is to generalize the results on almost primary ideals of commutative rings to Boolean like semi-rings (which are special class of near rings). Even though the proofs seemingly to appear similar to the case of commutative rings with unity but it is not. It can easily be seen in the proofs of this paper.

This paper is divided into 3 sections. The first section is devoted to recollect certain definitions and results concerning Boolean like semi-rings. In section 2, we introduce the notion of Almost prime ideal and almost primary ideal in Boolean like semi-rings imitating as in the case of commutative rings of Adil Kadir Jabbar and Chwas Abas Ahmed in [1]. We prove that if I is an ideal of a Boolean like semi ring R and P is an almost primary ideal of R such that $I \subseteq P$, then P/I is an almost primary ideal of R/I(see theorem 2.7). In theorem 2.9, we establish that $b^n I \subseteq I^2$ for some positive integer n if I is an almost primary ideal of R and $b + I$ be a right zero divisor in R/I. Further we give a characterization for almost primary ideal in terms of ideal quotients and radical of an idea l(see theorem 2.10).

In section 3, we introduce the notion of 2-potent prime ideal in a Boolean like semi ring and establish various characterization of almost primary ideals in terms of 2-potent prime ideals, radical of an ideal. Finally we give a characterization theorem for almost prime ideal of Boolean like semi ring and weakly prime ideal of the quotient Boolean like semi ring (see theorem 3.16).

1. PRELIMINARIES

We collect certain definitions and results from [9, 10]

Definition 1.1. A non empty set R together with two binary operations + and satisfying the following conditions is called Boolean like semi-ring:

1. $(R, +)$ is an abelian group;
2. $(R, \cdot)$ is a semi group;

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3. $a \cdot (b + c) = a \cdot b + a \cdot c$;
4. $a + a = 0$ for all a in R;
5. $ab(a + b + ab) = ab$ for all $a, b, c \in R$.

Lemma 1.2. Let R be a Boolean like semi-ring. Then $a.0 = 0$ for all a in R.

Definition 1.3. A Boolean like semi-ring R is said to be weak commutative if $abc = acb$ for all a, b and $c \in R$.

Lemma 1.4. Let R be weak commutative Boolean like semi-ring. Then $0.a = 0$ for all $a \in R$.

Lemma 1.5. Let R be a Boolean like semi-ring and let m and n be integers. Then
i. $a^m a^n = a^{m+n}$
ii. $(a^m)^n = a^{mn}$
iii. $(ab)^n = a^n b^n$ if R is weak commutative

Definition 1.6. **[3]** A Boolean like ring R is a commutative ring with unity and is of characteristic 2 in which $ab(1 + a)(1 + b) = 0$ for all a, b in R.

Definition 1.7. A non empty subset I of a Boolean like semi ring R is called an ideal if
(a) I is a subgroup of R (b) ra $\in$ I for all r$\in$ R and a $\in$ I (c) For all r, s $\in$ R and for all a $\in$I , (r + a) s + rs $\in$ I.

Definition 1.8. Let I be an ideal of a Boolean like semi ring R then radical of I denoted by r(I) or rad (I) is defined as {x $\in$ R / x^n for some positive integer n}

Remark.1.9. In [9], it is observed that in any Boolean like semi-ring R, $a^n = a$ or a^2or a^3 for every positive integer n.

Definition.1.10. Let R be a Boolean like semi ring R and I, J be ideals of R. Then the quotient $(I: J)$ of I by J is defined as $(I: J) = \{x \in R : xJ \subseteq I\}$.

Remark 1.11. Let I and J be ideals of R, then the product of I and J, denoted by IJ, is defined as $\{\sum_{i=1}^{n} x_i\, y_i : x_i \in I, y_i \in J\}$. Further IJ is a left ideal of R (satisfying the conditions (a) and (b) of definition 1.7 only)

Definition 1.12. If P is a proper ideal of R, then P is called **prime** if for $a, b \in R$ such that $ab \in P$ then $a \in P$. or $b \in P$.

Definition 1.13. If P is a proper ideal of R, then it is called **primary** if for $a, b \in R$ such that $ab \in P$ then $a^n \in P$ or b $\in P$, for some positive integer n.

Theorem 1.14. A proper ideal I is primary $\Leftrightarrow R/I \neq \{0\}$ and every zero divisor (either left or right) in R/I is nilpotent.

Definition 1.15. A Proper ideal I of a Boolean like semi ring R is called **semi prime** if for $a \in R$, $a^2 \in I$ then $a \in I$

2. ALMOST PRIMARY IDEALS

Adopting the same as in the case of commutative rings, we introduce the notions of different classes of ideals in a Boolean like semi-ring R here under. Here after R stands for Boolean like semi ring unless otherwise stated.

We begin with the following

Definition 2.1. A proper ideal P of R is called **weakly prime** if for $a, b \in R$ such that $0 \neq ab \in P$, then $a \in P$ or $b \in P$.

Definition 2.2. If P is a proper ideal of R, then it is called **weakly primary** if for $a, b \in R$ such that $0 \neq ab \in P$, then $a^n \in$P or b $\in$ P, for some positive integer n.

Definition 2.3. A proper ideal P of R is called **almost prime** if for $a, b \in R$ such that $ab \in P - P^2$, then $a \in P$ or $b \in P$.

Definition 2.4. A proper ideal P of R is called **almost primary** if for $a, b \in R$ such that $ab \in P - P^2$, then $a^n \in P$ or $b \in P$, for some positive integer n.

Remark 2.5. It is clear that every prime ideal is primary and primary is almost Primary.

Example 2.6. Let R ={0,a,b,c,d,e,f,1 }. The binary operations + and. are defined as follows

+	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	0	c	b	e	d	1	f
b	b	c	0	a	f	1	d	e
c	c	b	a	0	1	f	e	d
d	d	e	f	1	0	a	b	c
e	e	d	1	f	a	0	c	b
f	f	1	d	e	b	c	0	a
1	1	f	e	d	c	b	a	0

.	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	a	a	a	a
b	0	0	b	b	0	0	b	b
c	0	0	b	b	a	a	c	c
d	0	0	0	0	d	d	d	d
e	0	a	0	a	d	e	d	e
f	0	0	b	b	d	d	f	f
1	0	a	b	c	d	e	f	1

Then R is a Boolean like semi ring. Now we have the following

Observations
1. R is not weak commutative since efc = 0 $\neq$ a = ecf.
2. {0} is almost primary but not primary since dc $\in$ {0} but neither $d^n \in \{0\}$ for all positive integer n nor c $\in$ {0}.
3. If I = {0, a, b, c} then I^2 = {0, b}. Now I – I^2 = {a, c}. Clearly I is almost primary and as well as semi prime.

Theorem 2.7. Let I be an ideal of R. If P is an almost primary ideal of R such that $I \subseteq P$, then P/I is an almost primary ideal of R/I.

Proof: Let $a+I, b+I \in R/I$ such that $(a+I)(b+I) \in P/I-(P/I)^2$. If $b+I \in P/I$, then we are done. Suppose $b+I \notin P/I$. Then, $b \notin P$. Since $(a+I)(b+I) \in P/I$, it is clear that $ab+I \in P/I \Rightarrow ab \in P$. If $ab \in P^2$, then $ab=\sum_{i-1}^{n} a_i b_{i,}\ a_i, b_i \in P$ for all $i=1,\cdots,n$. This implies $ab+I=\sum_{i=1}^{n}(a_i b_i+I)=\sum_{i=1}^{n}(a_i+I)(b_i+I) \in (P/I)^2$, which is a contradiction. Thus $ab \notin P^2$. Hence ab $\in P-\mathrm{P}^2$. Now $a^n \in P$ since $b \notin P$ .Hence $a^n+I \in P/I$ which is in turn $(\mathrm{a}+\mathrm{I})^{\mathrm{n}} \in P/I$ .Thus P/I is almost primary ideal.

Theorem 2.8. If I is an almost primary ideal of R and Q is a weakly primary ideal of R/I, then there exist an almost primary ideal P of R such that $I \subseteq P$ and $Q = P/I$.

Proof: Let I be an almost primary ideal of R and Q be a weakly primary ideal of R/I.

Then there exists an ideal P with $I \subseteq P$ such that $Q = P/I$ (see [4]). Now it is enough to show that P is almost primary. For that let $a, b \in R$ with $ab \in P-P^2$ and b $\notin$ P. We have the following

Case 1. If $ab \in I$, then $ab \in I-I^2$, since ab $\notin$ P^2 and hence ab $\notin$ I^2.
$\Rightarrow$ $a^n \in I$ or b $\in I$ since I is almost primary.
$\Rightarrow$ $a^n \in I$ or b $\in I$ in turn $a^n \in P$

Case 2. Suppose ab $\notin$ I then I $\neq$ (a+I)(b+I) $\in$P/I-{0} in R/I = Q – {0}
$\Rightarrow$ $a^n \in P$ or b $\in P$. In both the cases P is almost primary and hence the theorem

Theorem 2.9. Let I be an almost primary ideal of R and $b+I$ be a left zero divisor in R/I. Then, there exists a positive integer n such that $b^n I \subseteq I^2$.

Proof: Since $b+I$ is a zero-divisor in R/I , there exists $c \notin I$ such that $(b+I)\,(c+I)=I$.Thus $bc \in I$. If $b^n \in I$ for some positive integer n, then $b^n I \subseteq II=I^2$. If $b^n \notin I$ for all $n \in Z^+$, then we must have $bc \in I^2$. Otherwise, $bc \notin I^2$, then $bc \in I-I^2$ and hence either $b^n \in I$ or $c \in I$ which is a contradiction.

Now let y $\in bI$ then $y = bi$ for some $i \in I$ .Then $bc+bi=b(c+i) \in I$ .Further $i+c \notin I$ (Otherwise, $i+c=j \in I \Rightarrow c=j+i \in I, a\ contradiction$). If $b(i+c) \notin I^2$ then $bc \in I$ and hence $b(x+c) \in I$.

Now suppose $b(x+c) \notin I^2$. Then $b(x+c) \in I - I^2$.

Thus $b(i+c) \in I - I^2$ .Hence $b^n \in I$ or $i+c \in I$ which is a contradiction .Thus $b(i+c) \in I^2 \Rightarrow bi \in I^2$, since $bc \in I^2$ hence $bI \subseteq I^2$.

The following is a characterization of almost primary ideal in terms of ideal quotients and radicals.

Theorem 2.10. A proper ideal I of R is almost primary $\Leftrightarrow$ $(I{:}\,x) = (I^2 : x)$ or $(I : x) \subseteq r(I)$, for all $x \in R - I$.

Proof: Let I be an almost primary ideal of R and $x \notin I$. Since $I^2 \subseteq I$, it is clear that $(I^2 : x) \subseteq (I{:}\,x)$. Let $y \in (I : x)$ then $yx \in I$. If $yx \in I^2$, then $y \in (I^2 : x)$ and hence $(I{:}\,x) = (I^2 : x)$. Suppose $yx \notin I^2$ then $yx \in I - I^2$
$\Rightarrow$ $y^n \in I$ or $x \in I$ for some $n \in Z^+$
$\Rightarrow$ $y^n \in I \Rightarrow$ $y \in r(I)$

Conversely, let $xy \in I - I^2$ and $y \notin I$, then we have $x \in (I : y)$
Suppose $(I : y) = (I^2 : x)$ then $x \in (I^2 : x)$, a contradiction . Hence $x \in r(I)$ in turn $x^n \in I$, for some $n \in Z^+$. Thus I is almost primary.

3. SEMI PRIME IDEALS AND 2-POTENET PRIME IDEALS

We begin with the following

Definition 3.1. A Proper ideal I of a Boolean like semi-ring R is called **2-potent prime ideal** if, for $a, b \in R$, $ab \in I^2$ implies either $a \in I$ or $b \in I$.

Remark 3.2. It is clear that every prime ideal is 2-potent prime.

Now we give some characterization of almost prime ideals in terms of 2-potent Prime ideal, semi prime ideal and radical of ideal.

We have the following

Theorem 3.3. If I is a 2-potent prime ideal of R, then I is almost primary $\Leftrightarrow$ I is primary.

Proof: Let I be an almost primary ideal. Let $a, b \in R$ with $ab \in I$ but $b \notin I$. If $ab \in I^2$, since I is 2-potent we have that $a \in I$. Suppose $ab \notin I^2$ then $ab \in I - I^2$. Hence $a^n \in I$ for some $n \in Z^+$ and thus I is a primary ideal. The proof of the converse is obvious.

Theorem 3.4. If I is a 2-potent prime ideal of R, then I is almost primary $\Leftrightarrow$ I is almost prime ideal.

Proof: Let I be an almost primary ideal. Let $a, b \in R$ with $ab \in I - I^2$ but $b \notin I$. If $ab \in I^2$, since I is 2-potent we have that $a^n \in I$. Since $a^n = a \text{ or } a^2 \in I$. If $a \in I$ we are done. If $a^2 \in I$ then $a \text{ or } a^2 \in I^2$. Hence $a \in I$ since I is 2-potent prime. Converse is obvious.

Theorem 3.5. Let I be an ideal of R with $r(I) = I$. Then I is almost primary $\Leftrightarrow$ if I is almost prime.

Proof: Suppose I is almost primary. Let $a, b \in R$ with $ab \in I - I^2$. If $b \notin I$, then we have $a^n \in I$ for some $n \in Z^+$ and hence $a \in r(I) = I$. Hence I is almost prime. The proof of the converse is obvious.

Theorem 3.6. Let I is semi prime then I is almost primary $\Leftrightarrow$ I is almost prime .

Proof: Suppose I is almost primary. Let $a, b \in R$ with $ab \in I - I^2$. If $b \notin I$, then we have $a^n \in I$ for some $n \in Z^+$ and hence $a^n = a \text{ or } a^2 \in I$. If $a \in I$ we are done. If $a^2 \in I$ then $a \in I$ since I is semi prime. The proof of the converse is obvious.

Theorem 3.7. If I is a 2-potent prime ideal of R, then $r(I) = I$

Proof. is obvious

Example 3.8. Let I = {0,b} in the above example 2.6, it is clear that $r(I) = I$ but I is not 2-potent prime ideal.

Now we have the following theorems whose proofs are straightforward and hence we omit them.

Theorem 3.9. I is semi prime $\Leftrightarrow$ $r(I) = I$.

Theorem 3.10. If I is a 2-potent prime ideal of R, then I is semi prime.

Theorem 3.11. If I is prime ideal of R, then I is 2- potent prime ideal.

Theorem 3.12. If $I^2 = I$ then I is 2-potent prime ideal $\Leftrightarrow I$ prime.

Theorem 3.13. If I is almost primary and 2-potent prime ideal , J and K are ideals of R such that $JK \subseteq I\text{-}I^2$ then either $J \subseteq I$ or $K \subseteq r(I)$.

Proof. If $K \subseteq$ r(I) then we are done. Suppose not. Then there exists $y \in K$ but $y \notin r(I)$. Let $z \in$ J. Then $zy \in JK \subseteq I\text{-}I^2$.

Hence $z^n \in I$ or $y \in I$ since I is almost primary. If $z \in I$ we are done.

Now $z^2 = z^4 = z^2 z^2 \in II$. This gives that $z \in I$ since I is 2-potent Prime. Thus in any case $z \in I$.Hence $J \subseteq I$

Corollary 3.14. If I is 2-potent prime ideal and almost prime and J and K are ideals of R such that $JK \subseteq I\text{-}I^2$ then either $J \subseteq I$ or $K \subseteq r(I)$

Proof. follows from theorems 3.4 and theorem 3.13.

Also we have an easy consequence of theorem 3.13 in the following

Corollary 3.15. If I is almost prime, $I = r(I)$ and J , K are ideals of R such that $JK \subseteq I\text{-}I^2$ then either $J \subseteq I$ or $K \subseteq r(I)$.

Finally we conclude this in the following:

Theorem 3.16. A proper ideal I of R is almost primary $\Leftrightarrow$ if I/I^2 is a weakly primary ideal of R/I.

Proof. (⇒) Let I be almost primary ideal and $(a + I^2), (b + I^2) \in R$ such that $(a + I^2)(b + I^2) \in I/I^2 - \{0\}$. Then $ab \in I$ and $ab \notin I^2$, but since I is almost primary either $a^n \in I$ or $b \in I$ for some $n \in Z^+$. If $b \in I$ then $b + I^2 \in I/I^2$ and if $a^n \in I$ then $(a^n + I^2) = (a + I^2)^n \in I/I^2$.

(⇐) Let I/I^2 be a weakly primary ideal of R and $ab \in I - I^2$ where a, b $\in$R, then $ab + I^2 \in I/I^2$ and $ab + I^2 \neq I^2$ and hence $(a + I^2)(b + I^2) \in I/I^2 - \{0\}$. Thus, either $(a + I^2) \in I/I^2$ or $(b + I^2)^n \in I/I^2$ for some $n \in Z^+$ and hence either $a \in I$ or $b^n \in I$.

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