A NOTE ON NEAR-RING GROUP OF QUOTIENTS

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ABSTRACT

In this paper we present near-ring group structure of quotients of a near-ring group and is seen that near-ring of quotients may appear as a particular case in some cases. Bringing the notion of fractions of a near-ring group into our picture, we introduce the notion to meet our purpose in a broad aspect so that near-ring of quotients may appear as a particular case of what is available already.

Keywords: Near-ring; Near-ring group; Near-ring group of quotients.

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1. INTRODUCTION

In case of a right near-ring $N$ usually an $N$-group $E$ is an algebraic structure such that $(E,+)$ is a group (not necessarily abelian) together with an external composition $N \times E \rightarrow E$ operating $E$ where $N$ operates on $E$ from left. In contrast what has been stated above we begin with the formal definition of an unusual module (N-group) structure because of the fact that to get the structure of right near-ring group of quotients or a near-ring group of quotients are more well behaved. Suppose $(E, +)$ is a group and $(N, +, \cdot)$ is a right near-ring. A complementary representation of $N$ on $E$ is a semigroup homomorphism $\theta : (N,\cdot) \rightarrow (\text{End} (E), o)$. Suppose that $(N, +, \cdot)$ is a right near-ring. Then an unusual near-ring $N$ module is a pair $( (E,+), *)$ where $(E, +)$ is a group and $\cdot : E \times N \rightarrow E$ is a function which satisfies (i) $x*(a\cdot b) = (x*a)*b$ for all $x \in E$ and $a, b \in N$, and (ii) $(x+y)*a = x*a + y*a$ for all $x, y \in E$ and $a \in N$.

Suppose $( (E,+), *)$ is an unusual near-ring $N$-module. Then a function $\theta : N \rightarrow \text{End}(E)$
\[ a \rightarrow a\theta \]
given by $x \rightarrow x(a\theta) = x*a$ is a semigroup homomorphism.

Thus ‘*’ induces the complimentary representation.

Conversely if $\theta : N \rightarrow \text{End}(E)$ is a complementary representation then define a right scalar multiplication $* : E \times N \rightarrow E$ by $x*a = x(a\theta)$ for all $a \in N$ and $x \in E$. Thus $\theta$ induces a right scalar multiplication that makes $E$ an unusual near-ring $N$-module.

Here in this paper, our motivation to explore some unusual algebraic structure (other than usual ones) may lead us to some normally expected more well be have structures that appear as a very natural one in some specific extended generalize structures to give a better explanation of what is stated above may be read as follows.

In case of a ring we have the notion of ring of right quotients as well as that of a left quotients. Similarly we have the structures of module of quotients where that is a right module or a left module. In case of near-ring we have the structure of the right near-ring of right quotients and the right near-ring of left quotients. It is interesting to note that in case of right near-ring of right quotients incidentally some unusual module theoretic or near-ring group theoretic structures come up to the front line. It is very interesting to note that with a matching to Barua’s [[1]] work we find Grainger[[5]] has presented in his Doctoral thesis specifically in his discussion on unusual near-ring module structures.

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In contrast what is available in ring theory it is here observed that to have the structure of near-ring group of quotients it is difficult to give up the notion of what Grainger [5] has termed as unusual near-ring module structures. One may hope for very interesting structure theory in such type of near-ring group of quotients.

The definition of unusual near-ring group structure of an additive group (not necessarily Abelian) over a right near-ring with identity 1 is as follows.

Let \((E, +)\) be a group and \(N\) be near-ring with a map \(\mu : E \times N \to E, (x, q) \to xq\)

such that for all \(x, y \in E\) and \(q, r \in N\), we have

\[
(x + y)q = xq + yq \\
x(qr) = (xq)r \\
x0 = 0
\]

and \(x1 = x\), where zero in the left is the zero of \(N\) and the zero in the right is the zero in \(E\).

Then \(E_N = (E, +, \mu )\) is called the near-ring group.

As for example we consider the near-ring \(N (=\mathbb{Z}_8)\) without unity w. r. t. addition modulo 8 and multiplication defined by the following table.

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Here \(N\) is a near-ring group over itself.

A subset \(A\) of a near-ring group \(E_N\) is a sub-near-ring group of \(E_N\), if \(x - y, xn \in A\) for all \(x, y \in A, n \in N\).

The notion immediately leads us to the following,

If \(E\) and \(F\) are near-rings groups, then, a mapping \(f : E \to F\) is an \(N\)-homomorphism if

i) \(f\) is a group homomorphism

ii) \(f(en) = f(e)n\), for \(e \in E\) and \(n \in N\)

in a usual way the notion of kernel of \(f\) follows.

The notion of an \(N\)-map follows when the condition (i) is absent.

In what follows it contains the notion of essential as well as rational extensions together with some relevant results.

An \(N\)-subgroup \(A(\neq 0)\) of \(E\) (i.e. \((A, +)\) is a subgroup of \((E, +)\), with \(AN \subseteq A\)) is an essential \(N\)-subgroup of \(E\), or \(E\) is an essential extension of \(A\), if for every \(N\)-subgroup \(X \neq 0\) of \(E\), we have \(A \cap X \neq 0\) and is denoted by \(A \subseteq_e E\).

An \(N\)-subgroup \(D\) of \(E\) is a dense (or rational) \(N\)-subgroup of \(E\) or \(E\) is a rational extension of \(D\), written \(D \subseteq_r E\) if given \(u, v \in E\) with \(u \neq 0\) there exists \(t \in N\) such that \(vt \in D\) and \(ut \neq 0\).

An \(N\)-subset \(D\) of \(L\) where \(N\) is a sub-near-ring of the near-ring \(L\) is \(D \subseteq_L L\) if and only if given \(k, l \in L\) with \(l \neq 0\) there exists \(x \in N\) such that \(kx \in D\) and \(lx \neq 0\).

An \(N\)-subgroup \(D\) of \(F\) where \(F\) is a near-ring group over \(L\) (where \(N\) is a sub-near-ring of near-ring \(L\)) is \(D \subseteq_F F\) if and only if given \(p, q \in L\) with \(q \neq 0\) there exists \(x \in D\) such that \(px \in D\) and \(qx \neq 0\).
If \( D \subseteq F \subseteq E \), where \( E \) is a near-ring group, \( F \) is a sub-near-ring group and \( D \) is a normal sub-near-ring group of \( F \) such that \( D \subseteq E \), then zero homomorphism is the only homomorphism from \( F/D \) to \( E \) \( \text{[i.e., } \text{Hom} ( F/D , E ) = (0) \text{]} \).

Clearly, a rational extension is an essential extension. Moreover if \( D \subseteq G \subseteq E \), where \( E \) is a near-ring group, \( G \), an \( N \)-subgroup of \( E \) and \( D \) an \( N \)-subset of \( G \) then, \( D \subseteq E \) implies \( D \subseteq G \subseteq E \).

In this paper we mainly present the formal structure of near-ring group of quotients of a near-ring group \( E_N \) as mentioned above.

**A fraction of** \( E_N \) **is a** \( N \)-**map** \( f : A_N \rightarrow E_N \) **where** \( A_N \) **is a dense** \( N \)-**subgroup of** \( E_N \).**

Given two fractions \( f \) and \( g \) of \( E_N \), \( f \) and \( g \) are ‘\(~\)’ related (denoted ‘\( f \sim g \)’) if and only if they agree on the common part of their domains.

It is to be noted that if \( f \) and \( g \) are fractions of \( E_N \), then \( f \sim g \) if and only if there exists a dense \( N \)-sub-near-ring group \( D \) of \( E_N \) such that \( f(x) = g(x) \) for all \( x \in D \) and the relation ‘\(~\)’ is an equivalence relation on the set of all fractions of \( E_N \).

Let \( Q(E) \) be the set of all equivalence classes of \( f, g , \ldots \) in to which the fractions \( f, g , \ldots \) of \( E_N \) are partitioned by the relation \( \sim \).

Defined in \( Q(E) \) by the rule \( f + g = f + g \).

We see that ‘\( + \)’ in \( Q(E) \) is justified. For, \( f \sim h \) and \( g \sim l \) let \( x \in \text{Dom } f \cap \text{Dom } g \cap \text{Dom } h \cap \text{Dom } l \).

Now \( (f + g) (x) = f(x) + g(x) = h(x) + l(x) = (h + l)(x) \).

Hence \( f + g \sim h + l \). Thus if \( f = h \) and \( g = l \), we have \( f + g = h + l \).

Again, let \( Q(N) \) be the set of all equivalence classes \( \frac{n}{1} , \frac{\alpha}{1} , \ldots \) into which the fractions \( \frac{n}{1} , \frac{\alpha}{1} , \ldots \) of \( N \) are partitioned by the relation \( \sim \). Then define in \( Q(E) \) the rule as follows \( \frac{f}{n} \frac{n}{1} = \frac{f}{1} \).

We see that it is well-defined in \( Q(E) \). Suppose \( f \sim g \) and \( \frac{n}{1} \sim \frac{\alpha}{1} \).

Let \( x \in \text{Dom } \frac{n}{1} \cap \text{Dom } \frac{\alpha}{1} \cap \left( \frac{n}{1} \right)^{-1} \left( \text{Dom } f \right) \cap \left( \frac{\alpha}{1} \right)^{-1} \left( \text{Dom } g \right) \).

Then \( \left( f \frac{n}{1} \right)(x) = f \left( \frac{n}{1}(x) \right) = f \left( \frac{\alpha}{1}(x) \right) = g \left( \frac{\alpha}{1}(x) \right) = g \left( \frac{n}{1}(x) \right) \).
Hence $f \frac{n}{1} - g \frac{\alpha}{1}$, thus if $\frac{f}{1} = \frac{g}{1}$ and $\frac{n}{1} = \frac{\alpha}{1}$ we have $\frac{f}{1} = \frac{g}{1}$.

A near-ring group $F_L$ with $E_N$ as its sub-near-ring group is a near-ring group of quotients of $E_N$ if $E_N \subseteq F_L$, where $N$ is a sub-near-ring of near-ring $L$. In a domain $R$, let $S = R - \{0\}$. Then $S$ is a multiplicatively closed set. Then we can obtain the quotient ring $S^{-1} = Q(R)$ which is a rational extension of $R$.

The near-ring group $E_N$ satisfies the N-Ore condition w.r.t a subset $S$ of $N$ if given $\left( x, r \right) \in E \times S$, there exists a common right multiple $x r' = r x'$ such that $(x', r') \in E \times S$.

An ordered family $\{A_1, A_2, \ldots, A_n\}$ of sub-near-ring groups of $E_N$ is called an independent family if $(A_1 + \ldots + A_t + \ldots + A_n) \cap A_t = \emptyset$, for some $1 \leq t \leq n$. The symbol $^\wedge$ denotes omission of $A_t$.

2. PRELIMINARIES

Lemma 2.1. Given two fractions $f$ and $g$ of $E_N$ with domains $\text{Dom } f$ and $\text{Dom } g$ respectively, then the map $f + g$: $\text{Dom } f \cap \text{Dom } g \rightarrow E_N$ $x \rightarrow f(x) + g(x)$ is also a fraction of $E_N$.

Proof: Since $\text{Dom } f$ and $\text{Dom } g$ are dense $N$-sub-near-ring group of $E_N$, $\text{Dom } f \cap \text{Dom } g$ is also dense $N$-sub-near-ring group of $E_N$. Now for any $x \in E_N$, $n \in N$ we have,

$$(f + g)(x n) = f(x n) + g(x n) = f(x)n + g(x)n = (f(x) + g(x))n = (f + g)(x)n.$$ 

Thus $f + g$ is a $N$-map. Hence $f + g$ is a fraction of $E_N$.

Lemma 2.2. Given two fractions $f$ and $\frac{g}{n}$ with domains $\text{Dom } f$ and $\text{Dom } \frac{g}{n}$, and also $\frac{g}{n}: \text{Dom } \frac{g}{n} \rightarrow E$, $x \rightarrow nx$. Then the map $f \frac{n}{n}: (\frac{n}{n})^{-1}(\text{Dom } f) \rightarrow E_N$ $x \rightarrow f(\frac{n}{n}(x))$ is also a fraction of $E_N$.

Proof: Let $u, v \in E_N$, $v \neq 0$. Since $\text{Dom } \frac{n}{n}$ is dense $N$-sub-near-ring group of $E_N$, there exists $t \in N$ such that $ut \in \text{Dom } \frac{n}{n}$ and $vt \neq 0$. We note that $\frac{n}{n}(ut), vt \in E_N$.

Since $\text{Dom } f$ is dense in $E_N$, it follows that $\exists p \in N$ such that $\frac{n}{n}(ut)p \in \text{Dom } f$, $(vt)p \neq 0$.

Again since $\frac{n}{n}$ is $N$-map, $\frac{n}{n}(ut)p = \frac{n}{n}(utp)$ thus given any $u, v \in E_N$, $v \neq 0$, $\exists p \in N$ such that $(ut)p \in (\frac{n}{n})^{-1}(\text{Dom } f)$ and $(vt)p \neq 0$.

Thus, $(\frac{n}{n})^{-1}(\text{Dom } f)$ is dense $N$-sub-near-ring group of $E_N$. Further, $x \in (\frac{n}{n})^{-1}(\text{Dom } f)$, $m \in N$

$$\left( f \frac{n}{n} \right)(xm) = f\left( \frac{n}{n}(x)m \right) = (f \frac{n}{n}(x))m = (f \frac{n}{n})(xm).$$

Thus $f \frac{n}{n}$ is a fraction of $E_N$.
Lemma 2.3: If f and g are fractions of $E_N$, then $f \sim g$ if and only if there exists a dense N-sub-near-ring group D of $E_N$ such that $f(x) = g(x)$ for all $x \in D$.

**Proof:** If $D = \text{Dom } f \cap \text{Dom } g$, then ‘only if’ follows.

To see ‘if’ part, let $x \in \text{Dom } f \cap \text{Dom } g$. Suppose $f(x) \neq g(x)$. Then we have $x, f(x) - g(x) \in E$ and $f(x) - g(x) \neq 0$.

Hence, there exists $y \in N$ such that $xy \in D$ and $(f(x) - g(x))y \neq 0$

But we have, $f(x)y = f(xy) = g(xy)$, since $xy \in D$

= $g(x)y$,

or $(f(x) - g(x))y = 0$. Which is a contradiction. From this contradiction it follows that $f(x) = g(x)$ for all $x \in \text{Dom } f \cap \text{Dom } g$.

Lemma 2.4: The relation ‘~’ is an equivalence relation on the set of all fractions of $E_N$.

**Proof:** The relation ~ is reflexive and symmetric trivially. To see that it is transitive, let $f, g, h$ be fractions of $E_N$ such that $f \sim g$ and $g \sim h$. Then $f(x) = g(x)$ for all $x \in \text{Dom } f \cap \text{Dom } g$ and $g(x) = h(x)$ for all $x \in \text{Dom } g \cap \text{Dom } h$.

Consequently we get $f(x) = h(x)$ for all $x \in \text{Dom } f \cap \text{Dom } g \cap \text{Dom } h$.

Where Dom $f \cap \text{Dom } g \cap \text{Dom } h$ is a dense N –sub-near-ring group of $E_N$. Thus $f \sim h$.

Lemma 2.5: Let $F_L$ be a near-ring group with $E_N$ as a sub-near-ring group (where N is a sub-near-ring of near-ring L). Then $F_L$ is a near-ring group of quotients of $E_N$ if and only if for every $0 \neq q \in L$ we have $q^{-1}E_N \subseteq E_N, q(q^{-1}E_N) \supseteq (0)$ where $q^{-1}E_N = \{x \in E_N | qx \in E_N\}$.

**Proof:** Suppose $E_N \subseteq r F_L$. Given $z \in q^{-1}E_N$ and $n \in N$. Then we get,

$q(zn) \in E_N$

$\Rightarrow zn \in q^{-1}E_N$

Hence $q^{-1}E_N$ is subset of $F_L$.

$u, v \in F_L, v \neq 0$. Then $qu \in F_L$. Since $E_N \subseteq r F_L$, there exists $t \in N$ such that $(qt)t \in E_N$ and $vt \neq 0$

The first condition implies that $ut \in q^{-1}E_N$. Thus, given $u, v \in F_L, v \neq 0$, there exists $t \in N$ such that $ut \in q^{-1}E_N$ and $vt \neq 0$ leading there by $q^{-1}E_N \subseteq r F_L$.

Since we have $q^{-1}E_N \subseteq E_N \subseteq F_L$ and $q^{-1}E_N \subseteq r F_L$, it follows that $q^{-1}E_N \subseteq r E_N$.

Now $a \in E_N \cap qE_N$ gives $a \in E_N, qE_N$ and hence there exists $b \in E_N$ such that $a = qb$. Since $qb = a \in E_N$,

we have $b = q^{-1}E_N$. Hence, $a(\neq qb) \in q(q^{-1}E_N)$. Thus $q(q^{-1}E_N) \supseteq E_N \cap qE_N$.

Recalling $E_N \subseteq r F_L$ and $E_N \subseteq r F_L$, we see that $E_N \cap qE_N \supseteq (0)$, which gives $q(q^{-1}E_N) \supseteq (0)$.

Suppose $q^{-1}E_N \subseteq r E_N$ and $q(q^{-1}E_N) \supseteq (0)$ hold and $p, q \in L$ with $q \neq 0$.

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Now we show that there exists \( x \in E_N \) for which \( px \in E_N \) and \( qx \neq 0 \). Keeping in note \( q(q^{-1}E_N) \subseteq qE_N \) and \( q(q^{-1}E_N) = \{qx \mid x \in q^{-1}E_N\} \)
\[
= \{qx \mid qx \in E_N\} \subseteq E_N.
\]

We get \( q(q^{-1}E_N) \subseteq E_N \cap qE_N \) which in turn gives \( E_N \cap qE_N \supseteq (0) \).

Thus, there exists \( b \in E_N \) such that \( a = qb \). We note that \( qb \neq 0 \).

1. Suppose \( p = 0 \) and \( x = b \). Then we get \( px (= 0b = 0) \in E_N \) and \( qx (= qb) \neq 0 \).

2. Suppose \( p \neq 0 \). Then \( q^{-1}E_N \subseteq E_N \) gives \( p^{-1}E_N \subseteq E_N \).

And we have \( b, qb \in E_N \) with \( qb \neq 0 \). Hence there exists \( y \in N \) such that \( by \in p^{-1}E_N \) and \( (qb)y \neq 0 \).

Again \( x = y \) gives \( x \in E_N \) such that \( px \in E_N \).

Thus in both the cases \( p = 0 \) and \( p \neq 0 \), there exists \( x \in E_N \) with \( px \in E_N \) and \( qx \neq 0 \). Hence \( E_N \subseteq F_L \).

As in Lemma 2.1.1 \[2\] we have

**Lemma 2.6.** Let \( C(Q(N)) \) be the complete near-ring of quotients of \( N \) and \( C(Q(N)) \) exists. If \( s_1, s_2, \ldots, s_n \in S \), then there exists \( x_1, x_2, \ldots, x_n \in N \) and \( s \in S \) such that \( s_i^{-1} = x_i s^{-1}, i = 1, 2, \ldots, n \)

### 3. MAIN RESULTS

Now we present the important notion of what we are intending.

We see in this section that the fractions of \( E_N \) yield a near-ring group of quotients of \( E_N \).

**Theorem 3.1.** The set \( Q(E) \) of all equivalence classes of \( \overline{f}, \overline{g}, \ldots \) in to which the fractions \( f, g, \ldots \) of \( E_N \) are partitioned by the relation \( \sim \) defined by the rule \( \overline{f} + \overline{g} = \overline{f + g} \) and \( \overline{f} \alpha = \overline{f \alpha} \) is a near-ring group. Where \( \overline{\alpha} \in Q(N) \), the near-ring of right quotients.

**Proof:** Let us define \( \mu: Q(E) \times Q(N) \rightarrow Q(E), (\overline{f}, \overline{\alpha}) = \overline{f \alpha} = \overline{f \alpha} \), for all \( \overline{\alpha} \in Q(N), \overline{f} \in Q(E) \).

Now for all \( \overline{\alpha}_1, \overline{\alpha}_2 \in N, \overline{f}, \overline{g} \in Q(E) \) we have

\[
(\overline{f} + \overline{g})(\overline{\alpha}_1) = \overline{(f + g)\alpha_1} = \overline{f\alpha_1 + g\alpha_1} = \overline{f\alpha_1} + \overline{g\alpha_1} = \overline{f\alpha_1} + \overline{g\alpha_1},
\]

\[
\overline{f}(\overline{\alpha}_1 \overline{\alpha}_2) = \overline{f(\alpha_1 \alpha_2)} = \overline{(f\alpha_1)\alpha_2} = \overline{f\alpha_1} \overline{\alpha}_2.
\]

\( Q(E) \) has an additive identity \( \overline{0} \) given by the fraction \( 0: E_N \rightarrow E_N, x \rightarrow 0 \).
For every $f \in Q(E)$ we get a fraction
\[- f : \text{Dom} f \to E_N, \ x \to - f(x)\]

Given $x \in \text{Dom} f$, we have
\[(f + (- f))(x) = f(x) - f(x) = 0 \quad 0(x) = 0,
\[
(( - f) + (f))(x) = - f(x) + f(x) = 0
\]
Hence, $f + (- f) = 0 = ( - f) + f$ or, $f + (- f) = 0 = ( - f) + f$
Thus, every $f \in Q(E)$ has an inverse $( - f) \in Q(E)$.

And so we have $0(0) = 0$.
Hence $Q(E) = (Q(E), +, \mu)$ is a near-ring group over $Q(N)$.

In particular $Q(N) = (Q(N), +, \cdot)$ is a near ring and $Q(E)$ is a near-ring $Q(N)$ group.

**Theorem 3.2.** $E_N$ is embedded in the near-ring group $Q(E)_{Q(N)}$ (and so $N$ is embedded in the near-ring $Q(N)$).

**Proof:** For every $n \in N$, we get a left multiplication in the near-ring group of transformations of $E_N$,\[
\frac{n}{1} : E_N \to E_N, \ x \to nx .
\]

Given $x \in E_N, \ p \in N$ we see that
\[
\left(\frac{n}{1}\right)(xp) = n(xp)
\]
\[
= (nx)p \\
= \left(\frac{n}{1}\right)(xp)
\]
Thus the left multiplication $\left(\frac{n}{1}\right)$ is a N-map and hence a fraction of $E_N$.

Consider the map $\alpha : E_N \to Q(E)_{Q(N)}, \ x \to \frac{x}{1}$, for $x \in E_N$.

Then for $x, y \in E_N$ we have,
\[
\alpha(x + y) = \frac{x + y}{1} = \frac{x}{1} + \frac{y}{1} = \alpha(x) + \alpha(y)
\]

Again, for $n \in N, x \in E_N$ we have,
\[\alpha (xn) = \frac{\frac{x}{n}}{1} = \frac{x}{n} \]

Thus \(\alpha\) is a \(N\)-homomorphism.

Now, Kernel \(\alpha = \{ x \in E_N \mid \frac{x}{1} = 0 \}\)

\[= \{ x \in E_N \mid x = 0 \}\]

\[= (0)\]

Hence \(\alpha\) is a \(N\)-monomorphism.

**Note 3.3.** Since the map \(\alpha: E_N \to Q(E)_{Q(N)}\) is a monomorphism, we shall identify \(\alpha(E_N)\) with \(E_N\) and \(\frac{n}{1}\) with \(n\), for simplicity of notation.

**Theorem 3.4.** If \(D \subset E_N\) and \(q \in Q(N), q \neq 0\), then \(qD = (0)\) implies \(q = 0\).

**Proof:** Let \(q = \frac{t}{r}\), where \(t\) is a fraction of \(N\), and \(r \in N\). Then \(qD = (0)\) implies that

\[\frac{t}{r(n/1)} = (0)\]

Or, \(t((n/1)(x)) = 0\) for every \(x \in (n/1)^{-1}(\text{Dom}t)\)

\[\Rightarrow t(\text{Dom}t) = 0\]

Thus \(q = \frac{t}{r} = 0\).

**Theorem 3.5.** If \(q \in Q(N)\), \(t\) is a fraction of \(N\), and \(f\) is a fraction of \(E_N\) such that \(q = \frac{t}{r}\), then \(\text{Dom}f \subseteq q^{-1}E_N\).

**Proof:** Let \(r \in \text{Dom}f\), then

\[qr = \frac{t}{r} = \frac{t(r/1)}{1} \in E_N\]

\[\Rightarrow r \in q^{-1}E_N\]

Hence \(\text{Dom}f \subseteq q^{-1}E_N\).

As a corollary we get

**Corollary 3.6.** If \(q \in Q(N)\), then \(q^{-1}E_N \subseteq E_N\), where \(q^{-1}E_N = \{ x \in E_N \mid qx \in E_N \}\). Also

**Theorem 3.7.** If \(q \in Q(N), q \neq 0\), then

\[q(q^{-1}E_N) \supset (0)\]

The proof is immediately follows from Theorem 3.4.
Using the last two results and the Lemma 2.5., we have the

**Theorem 3.8.** \( Q(E)_{Q(N)} \) is a near-ring group of right quotients of \( E_N \).

**Theorem 3.9.** Let \( E_N \) satisfy the N-Ore condition with respect to a multiplicatively closed subset \( S \) of \( N \) and have \( N \)-homomorphisms

\[
\alpha : N \rightarrow Q(N) \\
\beta : E_N \rightarrow Q(E)_{Q(N)}
\]

such that \( r \in S \) implies \( \alpha (r)^{-1} \in Q(N) \). Then the subset

\[
ES^{-1} = \{\beta(a)\alpha (r)^{-1} | (a,r) \in E \times S\}
\]

is a sub-near-ring group of \( Q(E)_{Q(N)} \).

**Proof:** Let \( \beta(a)\alpha (r)^{-1}, \beta(b)\alpha (s)^{-1} \in ES^{-1} \). Then we have \( (a,r),(b,s) \in E \times S \). Since \( E_N \) satisfies the N-Ore condition w.r.t. \( S \) and \( (a,r),(b,s) \in E \times S \), we get that there exists \( (r',s'), (b',r'') \in E \times S \) such that \( rs' = sr' \) and \( br'' = rb'' \) and hence (i) \( \beta(rs') = \beta(sr') \) and (ii) \( \beta(br'') = \beta(rb'') \)

Again, as \( N \) satisfies the Ore condition w.r.t. \( S \) and \( (r',s'), (p',r'') \in N \times S \), we therefore get that there exists \( (r'',s'') \in N \times S \) such that \( rs'' = sr'' \) and \( pr'' = rp'' \)

And hence (iii) \( \alpha (rs') = \alpha (sr') \) and (iv) \( \alpha (pr'') = \alpha (rp'') \)

Now \( \beta(a)\alpha (r)^{-1} - \beta(b)\alpha (s)^{-1} = \beta(a)(\alpha (s')\alpha (sr')^{-1} - \beta(b)(\alpha (r')\alpha (rs')^{-1} \) [using (iii)]

\[
= \beta(a)\alpha (s')\alpha (rs')^{-1} - \beta(b)\alpha (r')\alpha (rs')^{-1}
\]

\[
= (\beta(a)\alpha (s') - \beta(b)\alpha (r'))\alpha (rs')^{-1} \in ES^{-1}
\]

Again, let \( \alpha (n)\alpha (r)^{-1} \in NS^{-1} \).

Then,

\[
\beta(b)\alpha (s)^{-1} \alpha (n)\alpha (r)^{-1} = \beta(b)\alpha (rs')\alpha (r')^{-1} \alpha (n)\alpha (r)^{-1} [\text{using (iii)}]
\]

\[
= \beta(b)\alpha (rs')\alpha (n')\alpha (r')^{-1}, \text{ where } \alpha (r')^{-1} \alpha (n) = \alpha (n')\alpha (r')^{-1}
\]

\[
= \beta(b)\alpha (rs'n')\alpha (rr')^{-1} \in ES^{-1}
\]

Hence, \( ES^{-1} \) is a sub-near-ring group of \( Q(E)_{Q(N)} \).

**Remark 3.10.** We note that if in Theorem 3.9., \( 1 \in S \), then the near-ring group \( ES^{-1} \) has \( \beta (1) \) as its identity.

Thus we get

**Theorem 3.11.** Let \( E_N \) be near-ring group and \( S \) be a multiplicatively closed subset of \( N \) containing 1, \( \alpha : N \rightarrow Q(N) \), \( \beta : E_N \rightarrow Q(E)_{Q(N)} \) are homomorphisms satisfying the condition \( \alpha (r)^{-1} \in Q(N) \), for \( r \in S \) and the condition \( \beta (a) = 0 \), for \( a \in E_N \) implies that there exists \( t \in S \) with \( at = 0 \).

Also if the subset \( ES^{-1} \) is a sub-near-ring group of \( Q(E)_{Q(N)} \), then \( E_N \) satisfies the \( N \)-Ore condition with respect to \( S \).
Proof: Let \((a, r) \in E \times S\). Then \(a \in E^{-1}_N\) and \(r \in S\). Since \(r \in S\) implies \(\alpha(r)^{-1} \in Q(N)\), \(\alpha(r)^{-1}\) and \(\alpha(1)^{-1}\) exist and as \(ES^{-1}\) is a sub-near-ring group, we get,

\[
\beta(1)\alpha(r)^{-1} \beta(a)\alpha(1)^{-1} \in ES^{-1} \Rightarrow \alpha(r)^{-1} \beta(a) \in ES^{-1}
\]

It follows from the definition of \(ES^{-1}\) there exists \((b, s) \in E \times S\) such that

\[
\alpha(r)^{-1} \beta(a) = \beta(b)\alpha(s)^{-1} \Rightarrow \beta(a\alpha(s)) = \beta(\alpha(r)b) \Rightarrow \beta(a\alpha(s)) = \alpha(r)\beta(b)
\]

\[
\Rightarrow \beta((a\alpha(s)) - \alpha(r)b) = 0
\]

Thus there exists \(t \in S\) such that \((a\alpha(s)) - \alpha(r)b)t = 0 \Rightarrow a\alpha(st) - \alpha(r)bt = 0 \Rightarrow \alpha(t) = \alpha(s)\).

Putting \(bt = a'\) and \(st = r\), we see that given \((a, r) \in E \times S\), there exists \((a', r') \in E \times S\) such that \(\alpha(r)a' = \alpha(r)^{-1}\), i.e., \(E_N\) satisfies N-Ore condition with respect to \(S\).

Theorem 3.12. Let \(S\) be the set of non-zero divisors of \(N\). If \(E_N\) satisfies the N-Ore condition with respect to \(S\) and \(s \in S\), then \(sE_N \subseteq E_N\).

Proof: Clearly \(sE_N\) is an \(N\)-subgroup of \(E_N\). Let \(a, b \in E_N\), \(b \neq 0\). Then \((a, s) \in E \times S\) and hence there exists a common right multiple \(as' = sa'\) such that \((a', r') \in E \times S\). Since \(sa' \in sE_N\) and since \(s'\) is a non-zero divisor and \(b \neq 0\), we have \(bs' \neq 0\). Thus \(sE_N \subseteq E_N\).

Because of the note 3.3., we regard \(E_N\) as a sub near-ring group of \(Q(E)_{Q(N)}\).

Following result gives how the N-Ore condition is connected with classical near-ring group of quotients

Theorem 3.13. If \(E_N\) satisfies the N-Ore condition w.r.t. \(S\), then the subset \(C(Q(E)) = \{xr^{-1} \in Q(E) \mid (x, r) \in E \times S\}\) is a sub-near-ring group of \(Q(E)_{Q(N)}\). \(C(Q(E))\) is the classical near-ring group of quotients of \(E_N\).

Proof: Let \(\alpha\) and \(\beta\) be the embedding \(\alpha : N \rightarrow Q(N)\) and \(\beta : E_N \rightarrow Q(E)_{Q(N)}\)

\[
r \rightarrow r, \ x \rightarrow \bar{x}
\]

Also for any \(r \in S, \alpha(r)^{-1} \in Q(N)\)

Thus \(C(Q(E)) = \{xr^{-1} \in Q(E) \mid (x, r) \in E \times S\}\)

\[
= \{ \beta(x)\alpha(r)^{-1} \in Q(E) \mid (x, r) \in E \times S\}
\]

\[
= ES^{-1}
\]

As \(ES^{-1}\) sub near-ring group of \(Q(E)_{Q(N)}\), \(C(Q(E))_{C(Q(N))}\) is a sub-near-ring group of \(Q(E)_{Q(N)}\) (where \(C(Q(N))\) is the classical near-ring of quotients of near-ring \(N\)).

Theorem 3.14. If \(J\) be a right \(N\)-sub-near-ring group of \(E_N\), then the subset \(JS^{-1} = \{xs^{-1} \in Q(E) \mid (x, s) \in J \times S\}\) is a right \(C(Q(N))\) sub-near-ring group of \(C(Q(E))_{C(Q(N))}\)

Proof: Let \(p \in C(Q(E))_{C(Q(N))}, q \in C(Q(N))\).
Then \( p = ax^{-1}, q = xt^{-1} \) where \( a \in J, x \in N, s, t \in S \)
And \( pq = (ax^{-1})(xt^{-1}) = a(s^{-1}xt^{-1}) \)

Since \( xt^{-1} \in C(Q(N)), s^{-1} = 1s^{-1} \in NS^{-1} = C(Q(N)) \), we get
\( s^{-1}(xt^{-1}) \in C(Q(N)) \)

Let \( s^{-1}(xt^{-1}) = bu^{-1}, b \in N, u \in S \). Then, \( ab \in J \) gives \( pq \in JS^{-1} \).

Hence the result.

**Theorem 3.15.** If \( \{ J_1, J_2, \ldots, J_t \} \) be an independent family of right N-sub-near-ring groups of \( E_N \), then \( \{ J_1S^{-1}, J_2S^{-1}, \ldots, J_tS^{-1} \} \) is an independent family of right \( C(Q(N)) \)–sub-near-ring group of \( C(Q(E)) \)

**Proof:** If possible, let \( \{ J_1S^{-1}, J_2S^{-1}, \ldots, J_tS^{-1} \} \) be not an independent family. Then there is an \( m, 1 \leq m \leq t \) such that, \( J_mS^{-1} \cap \sum_{n \neq m} J_nS^{-1} \neq 0 \).

Then there is a non-zero element,
\( j_m^s = j_1s_1^{-1} + \ldots + j_ms_m^{-1} + \ldots + j_ts_t^{-1} \) (\(^\wedge\)stands for deletion of the term underneath) in the intersection.

By Lemma 2.6., for \( s_1, \ldots, s_t \in S \), we get \( x_1, \ldots, x_t \in N \) and \( s \in S \) such that
\( s_i^{-1} = x_is^{-1}, 1 \leq i \leq t \)

Now, \( j_mx^s = j_1x_1s^{-1} + \ldots + j_ms_m^{-1} + \ldots + j_tx_t^{-1} \)
\( = (j_1x_1 + \ldots + j_ms_m + \ldots + j_tx_t)s^{-1} \)
And this gives,
\( j_mx = j_1x_1 + \ldots + j_ms_m + \ldots + j_tx_t \neq 0 \).

So, \( J_m \cap (\sum_{n \neq m} J_n) \neq (0) \) and is a contradiction, for \( \{ J_1, \ldots, J_t \} \) is an independent family of N-sub-near-ring group of \( E_N \).

Therefore \( \{ J_1S^{-1} \ldots J_tS^{-1} \} \) is an independent family of right \( C(Q(N)) \)–sub-near-ring group of \( C(Q(E)) \).

**REFERENCES**

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