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# A NOTE ON NEAR-RING GROUP OF QUOTIENTS

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*ABSTRACT*

*In this paper we present near-ring group structure of quotients of a near-ring group and is seen that near-ring of quotients may appear as a particular case in some cases. Bringing the notion of fractions of a near-ring group into our picture, we introduce the notion to meet our purpose in a broad aspect so that near-ring of quotients may appear as a particular case of what is available already.*

***Keywords**: Near-ring; Near-ring group; Near-ring group of quotients.*

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## 1. INTRODUCTION

In case of a right near-ring N usually an N-group E is an algebraic structure such that ( E,+ ) is a group ( not necessarily abelian ) together with an external composition $N \times E \to E$ operating E where N operates on E from left. In contrast what has been stated above we begin with the formal definition of an unusual module ( N-group ) structure because of the fact that to get the structure of right near-ring group of quotients or a near-ring group of quotients are more well behaved. Suppose (E, +) is a group and (N, +, .) is a right near-ring. A *complementary representation* of N on E is a semigroup homomorphism $\theta$: (N,.) → (End (E), o). Suppose that (N, +, .) is a right near-ring. Then an *unusual near-ring N module* is a pair ( ( E,+ ) , *) where ( E,+ ) is a group and $*: E \times N \to E$ is a function which satisfies (i) $x*(a.b) = (x*a)*b$ for all $x \in E$ and $a, b \in N$ and (ii) $(x+y)*a = x*a + y*a$ for all $x, y \in E$ and $a \in N$.

Suppose $((E,+),*)$ is an unusual near-ring N-module. Then a function

$$\theta : N \to End(E)$$
$$a \to a\theta$$

given by $\begin{array}{c} a\theta : E \to E \\ x \to x(a\theta) = x*a \end{array}$ is a semigroup homomorphism.

Thus ‘ $*$ ’ induces the complimentary representation.

Conversely if $\theta : N \to End(E)$ is a complementary representation then define a right scalar multiplication $*: E \times N \to E$ by $x*a = x(a\theta)$ for all $a \in N$ and $x \in E$. Thus $\theta$ induces a right scalar multiplication that makes E an unusual near-ring N-module.

Here in this paper, our motivation to explore some unusual algebraic structure ( other than usual ones) may lead us to some normally expected more well be have structures that appear as a very natural one in some specific extended generalize structures to give a better explanation of what is stated above may be read as follows .

In case of a ring we have the notion of ring of right quotients as well as that of a left quotients. Similarly we have the structures of module of quotients where that is a right module or a left module. In case of near-ring we have the structure of the right near-ring of right quotients and the right near-ring of left quotients. It is interesting to note that in case of right near-ring of right quotients incidentally some unusual module theoretic or near-ring group theoretic structures come up to the front line . It is very interesting to note that with a matching to Barua’s [[1]] work we find Grainger[[5]] has presented in his Doctoral thesis specifically in his discussion on unusual near-ring module structures.

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Incontrast what is available in ring theory it is here observed that to have the structure of near-ring group of quotients it is difficult to give up the notion of what Grainger [[5]] has termed as unusual near-ring module structures. One may hope fore very interesting structure theory in such type of near-ring group of quotients.

The definition of unusual near-ring group structure of an additive group (not necessarily Abelian) over a right **near-ring with identity 1 is** as follows.

Let (E, +) be a group and N be near- ring with a map

$$\mu : E \times N \rightarrow E, \ (x,q) \rightarrow xq$$

such that for all x , y $\in$ E and q, r $\in$ N , we have

$$(x + y)q = xq + yq$$
$$x(qr) = (xq)r$$
$$x0 = 0$$

and x1 = x , where zero in the left is the zero of N and the zero in the right is the zero in E.

Then $E_N$ = (E , + , $\mu$ ) is called the *near-ring group*.

As for example we consider the near-ring $N$ (=$Z_8$) without unity w. r. t. addition modulo 8 and multiplication defined by the following table.

| **•** | ***0*** | ***1*** | ***2*** | ***3*** | ***4*** | ***5*** | ***6*** | ***7*** |
|---|---|---|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| **2** | 0 | 2 | 0 | 2 | 2 | 2 | 2 | 2 |
| **3** | 0 | 3 | 0 | 3 | 3 | 3 | 3 | 3 |
| **4** | 0 | 4 | 0 | 4 | 4 | 4 | 4 | 4 |
| **5** | 0 | 5 | 0 | 5 | 5 | 5 | 5 | 5 |
| **6** | 0 | 6 | 0 | 6 | 6 | 6 | 6 | 6 |
| **7** | 0 | 7 | 0 | 7 | 7 | 7 | 7 | 7 |

Here N is a near-ring group over itself.

A subset *A* of a near-ring group $E_N$ is a sub-near-ring group of $E_N$ , if $x - y, xn \in A$ for all $x, y \in A, n \in N$ .

The notion immediately leads us to the following,

If E and F are near-rings groups, then, a mapping $f : E \rightarrow F$ is *an N-homomorphism* if

i) $f$ is a group homomorphism

ii) $f(en) = f(e)n$ , for $e \in E$ and $n \in N$

in a usual way the notion of kernel of $f$ follows.

The notion of *an N-map* follows when the condition (i) is absent.

In what follows it contains the notion of essential as well as rational extensions together with some relevant results.

An N-subgroup A($\neq$0) of E (i.e.(A,+) is a subgroup of (E,+), with AN$\subseteq$A) is an *essential N-subgroup* of E or E is an essential extension of A , if for every N-subgroup X ( $\neq$ 0) of E , we have A $\cap$ X $\neq$0 and is denoted by A $\subseteq_e$ E.

An N-subgroup D of E is a dense ( or rational ) N-subgroup of E or E is a *rational extension* of D , written D $\subseteq_r$ E if given u , v $\in$ E with u $\neq$ 0 there exists t $\in$ N such that vt $\in$ D and ut $\neq$ 0.

An N-subset D of L where N is a sub-near-ring of the near-ring L is $D \subseteq_r L$ if and only if given $k, l \in L$ with $l \neq 0$ there exists $x \in N$ such that $kx \in D$ and $lx \neq 0$ .

An N-subgroup D of F where F is a near-ring group over L (where N is a sub-near-ring of near-ring L) is $D \subseteq_r F$ if and only if given $p, q \in L$ with $q \neq 0$ there exists $x \in D$ such that $px \in D$ and $qx \neq 0$ .

 

If $D \subseteq F \subseteq E$, where E is a near-ring group, F is a sub-near-ring group and D is a normal sub-near-ring group of F such that $D \subseteq_r E$ , then zero homomorphism is the only homomorphism from F/D to E [ i.e., **Hom ( F/D , E ) = (0) ].**

Clearly, a rational extension is an essential extension. Moreover if $D \subseteq G \subseteq E$ , where E is a near-ring group, G, an N-subgroup of E and D an N-subset of G then, $D \subseteq_r E$ implies $D \subseteq_r G \subseteq_r E$ .

In this paper we mainly present the formal structure of near-ring group of quotients of a near-ring group $E_N$ as mentioned above.

A *fraction of* $E_N$ is a N- map f: $A_N \to E_N$ where $A_N$ is a dense N-subgroup of $E_N$ .

Given two fractions f and g of $E_N$ , f and g are '~' related (denoted 'f ~ g') if and only if they agree on the common part of their domains.

It is to be noted that if f and g are fractions of $E_N$ , then f ~ g if and only if there exists a dense N-sub-near-ring group D of $E_N$ such that f(x) = g(x) for all $x \in D$ and the relation '~' is an equivalence relation on the set of all fractions of $E_N$ .

Let $Q(E)$ be the set of all equivalence classes of $\overline{f}$ , $\overline{g}$ , . . . . in to which the fractions
*f*, g, . . . . of $E_N$ are partitioned by the relation ~ .
Defined in $Q(E)$ by the rule $\overline{f} + \overline{g} = \overline{f+g}$ .
We see that '+' in $Q(E)$ is justified. For, $f \sim h$ and $g \sim l$ let $x \in \text{Dom}\, f \cap \text{Dom}\, g \cap \text{Dom}\, h \cap \text{Dom}\, l$ .

$$\begin{aligned} \text{Now } (f+g)(x) &= f(x) + g(x) \\ &= h(x) + l(x) \\ &= (h+l)(x) \end{aligned}$$

Hence $f+g \sim h+l$ . Thus if $\overline{f} = \overline{h}$ and $\overline{g} = \bar{l}$ , we have $\overline{f} + \overline{g} = \overline{h} + \bar{l}$ .

Again, let $Q(N)$ be the set of all equivalence classes $\overline{\frac{n}{1}}, \overline{\frac{\alpha}{1}}$, . . . . into which the fractions $\frac{n}{1}, \frac{\alpha}{1}$,. . . . of N are partitioned by the relation ~ . Then define in $Q(E)$ the rule as follows

$$\overline{f}\,\overline{\frac{n}{1}} = \overline{f\frac{n}{1}}.$$

We see that it is well-defined in $Q(E)$. Suppose $f \sim g$ and $\frac{n}{1} \sim \frac{\alpha}{1}$

Let $x \in \text{Dom}\frac{n}{1} \cap \text{Dom}\frac{\alpha}{1} \cap \left(\frac{n}{1}\right)^{-1}(\text{Dom}\ f) \cap \left(\frac{\alpha}{1}\right)^{-1}(\text{Dom}\ g)$

$$\begin{aligned} \text{Then } (f\tfrac{n}{1})(x) &= f\left(\frac{n}{1}(x)\right) \\ &= f\left(\frac{\alpha}{1}(x)\right) \\ &= g\left(\frac{\alpha}{1}(x)\right) = (g\frac{\alpha}{1})(x) \end{aligned}$$

 

Hence $f\frac{n}{1} \sim g\frac{\alpha}{1}$. Thus if $\overline{f}=\overline{g}$ and $\overline{\frac{n}{1}}=\overline{\frac{\alpha}{1}}$ we have $\overline{f}\overline{\frac{n}{1}}=\overline{g}\overline{\frac{\alpha}{1}}$.

A near-ring group $F_L$ with $E_N$ as its sub-near –ring group is a *near-ring group of quotients of* $E_N$ if $E_N \subseteq_r F_L$, where N is a sub-near-ring of near-ring L. In a domain R, let S = R-{0}. Then S is a multiplicatively closed set. Then we can obtain the quotient ring $RS^{-1}$ = Q(R) which is a rational extension of R.

The near-ring group $E_N$ satisfies the *N-Ore condition w.r.t a subset S of N*, if given $(x,r)\in E\times S$, there exists a common right multiple

$$xr' = rx'$$

such that $(x', r') \in E \times S$.

An ordered family $\{A_1, A_2, ...., A_n\}$ of sub-near-ring groups of $E_N$ is called an independent family if $(A_1 + ....... + \hat{A_t} + ....... + A_n) \cap A_t = 0$, for some $1 \le t \le n$. The symbol ^ denotes omission of $A_t$.

## 2. PRELIMINARIES

**Lemma 2.1.** Given two fractions f and g of $E_N$ with domains Dom f and Dom g respectively, then the map

$$f + g: \text{Dom } f \cap \text{Dom } g \to E_N$$
$$x \to f(x) + g(x)$$

is also a fraction of $E_N$.

**Proof:** Since Dom f and Dom g are dense N-sub-near-ring group of $E_N$, Dom f $\cap$ Dom g is also dense N-sub-near-ring group of $E_N$. Now for any x $\in E_N$, n $\in$ N we have,

$$(f + g)(x\,n) = f(x\,n) + g(x\,n) = f(x)n + g(x)n = (f(x) + g(x))n = (f + g)(x)n.$$

Thus f + g is a N-map. Hence f + g is a fraction of $E_N$.

**Lemma 2.2.** Given two fractions f and $\frac{n}{1}$ with domains Dom f and Dom $\frac{n}{1}$, and also $\frac{n}{1}$: Dom $\frac{n}{1} \to E$, $x \to nx$. Then the map

$$f\tfrac{n}{1} : (\tfrac{n}{1})^{-1}(\text{Dom } f) \to E_N$$
$$x \to f(\tfrac{n}{1}(x))$$

is also a fraction of $E_N$.

**Proof:** Let u, v $\in E_N$, v $\neq 0$. Since Dom $\frac{n}{1}$ is dense N-sub-near-ring group of $E_N$, there exists t $\in$ N such that ut $\in$ Dom $\frac{n}{1}$ and vt $\neq$ 0. We note that

$$\tfrac{n}{1}(ut),\ vt \in E_N.$$

Since Dom f is dense in $E_N$, it follows that $\exists$ p $\in$ N such that

$$\tfrac{n}{1}(ut)p \in \text{Dom } f,\ (vt)p \neq 0.$$

Again since $\frac{n}{1}$ is N-map, $\frac{n}{1}(ut)p = \frac{n}{1}(utp)$. Thus given any u ,v $\in E_N$, v $\neq 0$, $\exists$ p $\in$ N such that (ut)p $\in (\frac{n}{1})^{-1}$ (Dom f) and (vt)p $\neq$ 0.

Thus, $(\frac{n}{1})^{-1}$(Dom f ) is dense N-sub-near-ring group of $E_N$. Further, x $\in (\frac{n}{1})^{-1}$( Dom f ), m $\in$ N

$$(f\tfrac{n}{1})(xm) = f(\tfrac{n}{1}(x)m) = f(\tfrac{n}{1}(x))m = (f\tfrac{n}{1})(x)m.$$

Thus $f\frac{n}{1}$ is a fraction of $E_N$.

 

**Lemma 2.3**.If f and g are fractions of $E_N$ , then f ~ g if and only if there exists a dense N-sub-near-ring group D of $E_N$ such that f(x) = g(x) for all x $\in$ D.

**Proof**: If D = Dom f $\cap$ Dom g , then 'only if ' follows.

To see 'if' part, let x $\in$ Dom f $\cap$ Dom g. Suppose f(x) $\neq$ g(x) . Then we have x, f(x) – g(x) $\in$ E and f(x) – g(x) $\neq$ 0.

Hence, there exists y $\in$ N such that xy $\in$ D and (f(x)-g(x)) y $\neq$ 0

But we have, f(x)y = f(xy) = g(xy), since xy $\in$ D
= g(x) y,

or (f(x) – g(x))y = 0. Which is a contradiction. From this contradiction it follows that f(x) = g(x) for all x $\in$ Dom f $\cap$ Dom g .

**Lemma 2.4.**The relation '~' is an equivalence relation on the set of all fractions of $E_N$ .

**Proof**: The relation ~ is reflexive and symmetric trivially. To see that it is transitive, let f, g, h be fractions of $E_N$ such that f ~ g and g ~ h. Then
f(x) = g(x) for all x $\in$ Dom f $\cap$ Dom g and g(x) = h(x) for all x $\in$ Dom g $\cap$ Dom h.

Consequently we get f(x) = h(x) for all x $\in$ Dom f $\cap$ Dom g $\cap$ Dom h.

Where Dom f $\cap$ Dom g $\cap$ Dom h is a dense N –sub-near-ring group of $E_N$ . Thus f ~ h.

**Lemma 2.5.** Let $F_L$ be a near-ring group with $E_N$ as a sub-near-ring group (where N is a sub-near-ring of near-ring L). Then $F_L$ is a near-ring group of quotients of $E_N$ if and only if for every $q \in L, q \neq 0$ we have $q^{-1}E_N \subseteq_r E_N, q(q^{-1}E_N) \supset (0)$ where $q^{-1}E_N = \{x \in E_N \mid qx \in E_N\}$.

**Proof:** Suppose $E_N \subseteq_r F_L$ . Given $z \in q^{-1}E_N$ and $n \in N$ . Then we get,
$q(zn) \in E_N$
$\Rightarrow zn \in q^{-1}E_N$

Hence $q^{-1}E_N$ is subset of $F_L$ .
$u, v \in F_L, v \neq 0$ . Then $qu \in F_L$ . Since $E_N \subseteq_r F_L$ , there exists t $\in$ N such that
$(qu)t \in E_N$ and vt $\neq$ 0

The first condition implies that $ut \in q^{-1}E_N$ . Thus, given $u, v \in F_L, v \neq 0$ , there exists $t \in N$ such that $ut \in q^{-1}E_N$ and $vt \neq 0$ leading there by $q^{-1}E_N \subseteq_r F_L$ .

Since we have $q^{-1}E_N \subseteq E_N \subseteq F_L$ and $q^{-1}E_N \subseteq_r F_L$ , it follows that $q^{-1}E_N \subseteq_r E_N$.

Now $a \in E_N \cap qE_N$ gives $a \in E_N, qE_N$ and hence there exists $b \in E_N$ such that *a=qb*. Since $qb = a \in E_N$ ,

we have $b = q^{-1}E_N$ . Hence, $a(= qb) \in q(q^{-1}E_N)$ . Thus $q(q^{-1}E_N) \supseteq E_N \cap qE_N$ .

Recalling $E_N \subseteq_r F_L$ , and $E_N \subseteq_e F_L$ , we see that $E_N \cap qE_N \supset (0)$, which gives $q(q^{-1}E_N) \supset (0)$ .

Suppose $q^{-1}E_N \subseteq_r E_N$ and $q(q^{-1}E_N) \supset (0)$ hold and $p, q \in L$ with $q \neq 0$ .

 

Now we show that there exists $x \in E_N$ for which $px \in E_N$ and $qx \neq 0$. Keeping in note $q(q^{-1}E_N) \subseteq qE_N$ and

$$
\begin{aligned}
q(q^{-1}E_N) &= \{qx \mid x \in q^{-1}E_N\} \\
&= \{qx \mid qx \in E_N\} \\
&\subseteq E_N
\end{aligned}
$$

We get $q(q^{-1}E_N) \subseteq E_N \cap qE_N$ which in turn gives $E_N \cap qE_N \supset (0)$.

Thus, there exists $b \in E_N$ such that $a = qb$. We note that $qb \neq 0$.

(1) Suppose p = 0 and x = b. Then we get px (= 0b = 0) $\in E_N$ and $qx(= qb) \neq 0$

(2) Suppose $p \neq 0$. Then $q^{-1}E_N \subseteq_r E_N$ gives $p^{-1} E_N \subseteq_r E_N$

And we have b, qb $\in E_N$ with $qb \neq 0$. Hence there exists y $\in$ N such that $by \in p^{-1}E_N$ and $(qb)y \neq 0.$

Again x = by gives $x \in E_N$ such that $px \in E_N$.

Thus in both the cases p = 0 and $p \neq 0$, there exists x $\in E_N$ with $px \in E_N$ and $qx \neq 0$. Hence $E_N \subseteq_r F_L$.

As in Lemma 2.1.1 [2] we have

**Lemma 2.6.** Let $C(Q(N))$ be the complete near-ring of quotients of N and $C(Q(N))$ exists.If $s_1, s_2, ... s_n \in S$, then there exists $x_1, x_2, ..., x_n \in N$ and $s \in S$ such that $s_i^{-1} = x_i s^{-1}, i = 1,2,...,n$

**3. MAIN RESULTS**

Now we present the important notion of what we are intending.

We see in this section that the fractions of $E_N$ yield a near-ring group of quotients of $E_N$.

**Theorem 3.1.** The set $Q(E)$ of all equivalence classes of $\overline{f}, \overline{g}$, . . . . in to which the fractions *f*, g, . . . . of $E_N$ are partitioned by the relation ~ defined by the rule $\overline{f} + \overline{g} = \overline{f+g}$ and $\overline{f}\ \overline{\alpha} = \overline{f\alpha}$ is a near-ring group. Where $\overline{\alpha} \in Q(N)$, the near-ring of right quotients.

**Proof**: Let us define μ: $Q(E) \times Q(N) \rightarrow Q(E)$, $(\overline{f}, \overline{\alpha}) = \overline{f}\ \overline{\alpha} = \overline{f\alpha}$, for all $\overline{\alpha} \in Q(N)$, $\overline{f} \in Q(E)$.

Now for all $\overline{\alpha_1}, \overline{\alpha_2} \in$ N, $\overline{f}, \overline{g} \in Q(E)$ we have

$$
\begin{aligned}
(\overline{f} + \overline{g})(\overline{\alpha_1}) &= \overline{(f+g)\alpha_1} \\
&= \overline{(f\alpha_1 + g\alpha_1)} \\
&= \overline{f\alpha_1} + \overline{g\alpha_1} \\
&= \overline{f}\ \overline{\alpha_1} + \overline{g}\ \overline{\alpha_1}
\end{aligned}
$$

$$
\begin{aligned}
\overline{f}(\overline{\alpha_1}\,\overline{\alpha_2}) &= \overline{f}\,\overline{(\alpha_1\alpha_2)} \\
&= \overline{f(\alpha_1\alpha_2)} \\
&= (\overline{(f\alpha_1)})\overline{\alpha_2} \\
&= (\overline{f}\,\overline{\alpha_1})\overline{\alpha_2}
\end{aligned}
$$

$Q(E)$ has an additive identity $\overline{0}$ given by the fraction 0: $E_N \rightarrow E_N$, x→ 0.

For every $\overline{f} \in Q(E)$ we get a fraction

$$- f : \text{Dom } f \to E_N \text{ , x} \to - f \text{ (x)}$$

Given x $\in$ Dom $f$, we have

$(f + (-f))(x) = f(x) - f(x) = 0$

$0(x) = 0,$

$((-f) + (f))(x) = -f(x) + f(x) = 0$

Hence, $\overline{f + (-f)} = \overline{0} = \overline{(-f) + f}$ Or, $\overline{f} + \overline{(-f)} = \overline{0} = \overline{(-f)} + \overline{f}$

Thus, every $\overline{f} \in Q(E)$ has an inverse $\overline{(-f)} \in Q(E)$.

And so we have

$$\overline{f}\, 0 = 0.$$

Hence $Q(E)$= ( $Q(E)$, +, μ) is a near-ring group over $Q(N)$ .

In particular $Q(N)$ =( $Q(N)$,+,.) is a near ring and $Q(E)$ is a near-ring $Q(N)$ group.

**Theorem 3.2.** $E_N$ is embedded in the near-ring group $Q(E)_{Q(N)}$.(and so N is embedded in the near-ring $Q(N)$

**Proof:** For every n $\in$ N, we get a left multiplication in the near-ring group of transformations of $E_N$ ,

$$\frac{n}{1} : E_N \to E_N \text{ , x} \to \text{n x .}$$

Given x $\in E_N$ , p $\in$ N we see that

$$\left(\frac{n}{1}\right)(xp) = n\,(xp)$$
$$= (nx)\, p$$
$$= \left(\frac{n}{1}\right)(x)p$$

Thus the left multiplication $\left(\frac{n}{1}\right)$ is a N-map and hence a fraction of $E_N$ .

Consider the map α: $E_N \to Q(E)_{Q(N)}$, x→ $\frac{\overline{x}}{1}$, for x $\in E_N$.

Then for x, y $\in E_N$ we have,

$$\alpha(x + y) = \overline{\frac{x+y}{1}}$$
$$= \frac{\overline{x}}{1} + \frac{\overline{y}}{1}$$
$$= \alpha(x) + \alpha(y)$$

Again, for n $\in$ N, x $\in E_N$ we have,

 

$$\alpha(xn) = \overline{\frac{xn}{1}} = \overline{\frac{x}{1}\frac{n}{1}}$$
$$= \overline{\frac{x}{1}n}$$
$$= \overline{\frac{x}{1}}\, n$$
$$= \alpha(x)\, n$$

Thus $\alpha$ is a N-homomorphism.

Now, Kernel $\alpha$ = { x $\in E_N$ | $\overline{\frac{x}{1}} = 0$ }

= {x $\in E_N$ | x = 0 }

= (0)

Hence $\alpha$ is a N-monomorphism.

**Note 3.3.** Since the map α: $E_N \rightarrow Q(E)_{Q(N)}$ is a monomorphism , we shall identify $\alpha(E_N)$ with $E_N$ and $\overline{\frac{n}{1}}$ with n, for simplicity of notation.

**Theorem 3.4.** If $D \subseteq_r E_N$ and $q \in Q(N),\ q \neq 0$, then $qD = (0)$ implies $q = 0$.

**Proof:** Let $q = \bar{t}$, where t is a fraction of N, and $n \in N$. Then $qD = (0)$ implies that

$\overline{t(n/1)}$ = (0)

Or, $t((n/1)(x)) = 0$ for every $x \in (n/1)^{-1}$(Domt)

$\Rightarrow t$ (Domt) = 0

Thus $q = \bar{t} = 0$.

**Theorem 3.5.** If $q \in Q(N)$, t is a fraction of N, and $f$ is a fraction of $E_N$ such that $q = \bar{t}$, then $Domf \subseteq q^{-1}E_N$.

**Proof**: Let $r \in Domf$ , then

$$qr = \bar{t}\overline{\frac{r}{1}} = \overline{t(r/1)} = \overline{t(r)/1} \in E_N$$
$$\Rightarrow r \in q^{-1}E_N$$

Hence $Domf \subseteq q^{-1}E_N$.

As a corollary we get

**Corollary 3.6.** If $q \in Q(N)$, then $q^{-1}E_N \subseteq_r E_N$ ,where $q^{-1}E_N = \{ x \in E_N \mid qx \in E_N \}$. Also

**Theorem 3.7.** If $q \in Q(N),\ q \neq 0$, then
$$q(q^{-1}E_N) \supset (0).$$

The proof is immediately follows from Theorem 3.4.

 

Using the last two results and the Lemma 2.5. , we have the

**Theorem3.8.** $Q(E)_{Q(N)}$ is a near-ring group of right quotients of $E_N$ .

**Theorem 3.9.** Let $E_N$ satisfy the N-Ore condition with respect to a multiplicatively closed subset S of N and have N-homomorphisms

$$\alpha : N \to Q(N)$$
$$\beta : E_N \to Q(E)_{Q(N)}$$

such that r∈S implies $\alpha(r)^{-1} \in Q(N)$ . Then the subset
$ES^{-1} = \{\beta(a)\alpha(r)^{-1} \mid (a,r) \in E \times S\}$ is a sub-near-ring group of $Q(E)_{Q(N)}$.

**Proof**: Let $\beta(a)\alpha(r)^{-1}, \beta(b)\alpha(s)^{-1} \in ES^{-1}$. Then we have $(a,r),(b,s) \in E \times S$ . Since $E_N$ satisfies the N-Ore condition w.r.t. S and $(a,r),(b,s) \in E \times S$ , we get that there exists $(r',s'),(b',r'') \in E \times S$ such that $rs' = sr'$ and $br'' = rb'$ and hence (i) $\beta(rs') = \beta(sr')$ and (ii) $\beta(br'') = \beta(rb')$

Again, as N satisfies the Ore condition w.r.t S and $(r,s),(p,r) \in N \times S$ , we therefore get that there exists $(r',s'),(p',r') \in N \times S$ such that $rs' = sr'$ and $pr'' = rp'$
And hence (iii) $\alpha(rs') = \alpha(sr')$ and (iv) $\alpha(pr'') = \alpha(rp')$

Now $\beta(a)\alpha(r)^{-1} - \beta(b)\alpha(s)^{-1} = \beta(a)(\alpha(s')\alpha(sr')^{-1} - \beta(b)(\alpha(r')\alpha(rs')^{-1}))$ [ using (iii)]
$= \beta(a)\alpha(s')\alpha(rs')^{-1} - \beta(b)\alpha(r')\alpha(rs')^{-1}$
$= (\beta(a)\alpha(s') - \beta(b)\alpha(r'))\alpha(rs')^{-1} \in ES^{-1}$

Again, let $\alpha(n)\alpha(r)^{-1} \in NS^{-1}$.

Then,
$\beta(b)\alpha(s)^{-1}\alpha(n)\alpha(r)^{-1} = \beta(b)\alpha(rs')\alpha(r')^{-1}\alpha(n)\alpha(r)^{-1}$ [using (iii)]
$= \beta(b)\alpha(rs')\alpha(n')\alpha\left(r_1'\right)^{-1}$, where $\alpha(r')^{-1}\alpha(n) = \alpha(n')\alpha\left(r_1'\right)^{-1}$
$= \beta(b)\alpha(rs'n')\alpha(rr_1')^{-1}$
$\in ES^{-1}$

Hence, $ES^{-1}$ is a sub-near ring group of $Q(E)_{Q(N)}$.

**Remark 3.10.** We note that if in Theorem 3.9., 1∈S , then the near-ring group $ES^{-1}$ has $\beta(1)$ as its identity.

Thus we get

**Theorem 3.11.** Let $E_N$ be near-ring group and S be a multiplicatively closed subset of N containing 1, $\alpha : N \to Q(N), \beta : E_N \to Q(E)_{Q(N)}$ are homomorphisms satisfying the condition $\alpha(r)^{-1} \in Q(N)$, for r ∈S and the condition $\beta(a) = 0$, for $a \in E_N$ implies that there exists t ∈S with $at = 0$.

Also if the subset $ES^{-1}$ is a sub-near-ring group of $Q(E)_{Q(N)}$, then $E_N$ satisfies the *N-Ore* condition with respect to S.

**Proof**: Let $(a,r)\in E\times S$. Then $a\in E_N$ and r $\in$S . Since r $\in$S implies $\alpha(r)^{-1}\in Q(N)$ , $\alpha(r)^{-1}$ and $\alpha(1)^{-1}$ exist and as $ES^{-1}$ is a sub-near-ring group, we get,

$\beta(1)\alpha(r)^{-1}\beta(a)\alpha(1)^{-1}\in ES^{-1} \Rightarrow \alpha(r)^{-1}\beta(a)\in ES^{-1}$

It follows from the definition of $ES^{-1}$ there exists $(b,s)\in E\times S$ such that

$$\alpha(r)^{-1}\beta(a)=\beta(b)\alpha(s)^{-1} \Rightarrow \beta(a\alpha(s))=\beta(\alpha(r)b) \Rightarrow \beta(a)\alpha(s)=\alpha(r)\beta(b)$$
$$\Rightarrow \beta((a\alpha(s))-\alpha(r)b)=0$$

Thus there exists t $\in$S such that $(a(\alpha(s))-\alpha(r)b)t=0$

$$\Rightarrow a\alpha(st)-\alpha(r)bt = 0$$

Putting bt = $a'$ and st = $r'$, we see that given $(a,r)\in E\times S$, there exists $(a',r')\in E\times S$ such that $\alpha(r)a'=a\alpha(r')$ , i.e., $E_N$ satisfies N-Ore condition with respect to S.

**Theorem 3.12.** Let S be the set of non-zero divisors of N. If $E_N$ satisfies the N-Ore condition with respect to S and s $\in$S, then $sE_N\subseteq_r E_N$ .

**Proof**: Clearly s $E_N$ is an N-subgroup of $E_N$ . Let $a,b\in E_N, b\neq 0$. Then $(a,s)\in E\times S$ and hence there exists a common right multiple $as'=sa'$ such that $(a',s')\in E\times S$ . Since $sa'\in sE_N$ and since $s'$ is a non-zero divisor and $b\neq 0$, we have $bs'\neq 0$. Thus $sE_N\subseteq_r E_N$ .

Because of the note 3.3., we regard $E_N$ as a sub near-ring group of $Q(E)_{Q(N)}$

Following result gives how the N-ore condition is connected with classical near-ring group of quotients

**Theorem 3.13.**If $E_N$ satisfies the N-Ore condition w.r.t. S, then the subset $C(Q(E))=\{xr^{-1}\in Q(E)\,|\,(x,r)\in E\times S\}$ is a sub- near-ring group of $Q(E)_{Q(N)}$. $C(Q(E))$ is the *classical near-ring group of quotients of* $E_N$ .

**Proof:** Let α and β be the embedding

$\alpha: N\rightarrow Q(N)$ and $\beta: E_N\rightarrow Q(E)_{Q(N)}$

r → r, $x\rightarrow \frac{\bar{x}}{1}$

Also for any $r\in S, \alpha(r)^{-1}\in Q(N)$

Thus $C(Q(E))=\{xr^{-1}\in Q(E)\,|\,(x,r)\in E\times S\}$

$= \{\beta(x)\alpha(r)^{-1}\in Q(E)\,|\,(x,r)\in E\times S\}$

$=ES^{-1}$

As $ES^{-1}$ sub near-ring group of $Q(E)_{Q(N)}$ , $C(Q(E))_{c(Q(N))}$ is a sub-near-ring group of $Q(E)_{Q(N)}$ (where $C(Q(N))$ is the classical near-ring of quotients of near-ring N ).

**Theorem 3.14.** If J be a right N-sub-near-ring group of $E_N$, then the subset $JS^{-1}=\{xs^{-1}\in Q(E)\,|\,(x,s)\in J\times S\}$ is a right $C(Q(N))$ sub-near-ring group of $C(Q(E))_{C(Q(N))}$

**Proof:** Let p $\in C(Q(E))_{C(Q(N))}$ , q $\in C(Q(N))$.

 

Then p = $as^{-1}$, q = $xt^{-1}$ where $a \in J$ , $x \in N$ , $s, t \in S$

And $pq = \left(as^{-1}\right)\left(xt^{-1}\right)$

$= a\left(s^{-1}xt^{-1}\right)$

Since $xt^{-1} \in C(Q(N))$ , $s^{-1} = 1\, s^{-1} \in NS^{-1} = C(Q(N))$, we get

$s^{-1}\left(xt^{-1}\right) \in C(Q(N))$

Let $s^{-1}\left(xt^{-1}\right) = bu^{-1}$, $b \in N$ , $u \in S$ . Then, $ab \in J$ gives $pq \in JS^{-1}$ .

Hence the result.

**Theorem 3.15.** If { $J_1, J_2, \ldots, J_t$ } be an independent family of right N-sub-near-ring groups of $E_N$ , then { $J_1S^{-1}, J_2S^{-1}, \ldots, J_tS^{-1}$ } is an independent family of right $C(Q(N))$ –sub-near-ring group of $C(Q(E))$

**Proof:** If possible, let {$J_1S^{-1}, J_2S^{-1}, \ldots, J_tS^{-1}$ } be not an independent family. Then there is an m , $1 \le m \le t$ such that , $J_mS^{-1} \cap \sum_{n \neq m} J_nS^{-1} \neq 0$.

Then there is an non-zero element,

$j_ms_m^{-1} = j_1s_1^{-1} + \ldots + \widehat{j_m s_m} + \ldots + j_ts_t^{-1}$ (^stands for deletion of the term underneath) in the intersection.

By Lemma 2.6., for $s_1, \ldots, s_t \in S$ , we get $x_1, \ldots, x_t \in N$ and $s \in S$ such that

$$s_i^{-1} = x_is^{-1}, \quad 1 \le i \le t$$

Now, $j_mx_ms^{-1} = j_1x_1s^{-1} + \ldots + \widehat{j_m x_m s^{-1}} + \ldots + j_tx_ts^{-1}$

$= (j_1x_1 + \ldots + \widehat{j_m x_m} + \ldots + j_tx_t)s^{-1}$

And this gives,

$j_mx_m = j_1x_1 + \ldots + \widehat{j_m x_m} + \ldots + j_tx_t$

$\neq 0.$

So, $J_m \cap (\sum_{n \neq m} J_n) \neq (0)$ and is a contradiction, for { $J_1, \ldots, J_t$ } is an independent family of N-sub-near-ring group of $E_N$ .

Therefore {$J_1S^{-1} \ldots J_tS^{-1}$} is an independent family of right $C(Q(N))$–sub-near-ring group of $C(Q(E))$.

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