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ON THE OREDER AND TYPE OF ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

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ABSTRACT

In this paper we consider entire functions represented by Dirichlet series. We obtained some relationships involving orders and types of two or more entire Dirichlet series [2].

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1. INTRODUCTION:

Let ' f ' be an entire function represented by the Dirichlet series

$$f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \tag{1.1}$$

Where $a_n \in C$ and λ_n's satisfy the following conditions:

(i) $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ as $n \to \infty$.

(ii) $\limsup\limits_{n\to\infty} \dfrac{\log n}{\lambda_n} = 0.$

If ' ρ ' is the Ritt order [3] of ' f ', then

$$\rho = \limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} \qquad (0 \le \rho \le \infty). \tag{1.2}$$

Further λ_n satisfy the conditions:

(iii) $\lambda_{n+1} \sim \lambda_n$ as $n \to \infty$ and

(iv) $\left\{ \log \left| \dfrac{a_n}{a_{n+1}} \right| / \left(\lambda_{n+1} - \lambda_n \right) \right\}$, eventually, is non-decreasing sequence.

If ' λ ' is the lower order of ' f ', then

$$\lambda = \liminf_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} \qquad (0 \le \lambda \le \infty). \tag{1.3}$$

Let ' f ' be an entire function of finite, positive order ' ρ '.

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If $'T'$ is the Ritt type of $'f'$ then

$$T = \limsup_{n\to\infty} \frac{\lambda_n}{\rho e} |a_n|^{\rho/\lambda_n} \tag{1.4}$$

If $'t'$ is lower type of $'f'$, then P. K. Kamthan[1] showed that

$$t = \liminf_{n\to\infty} \frac{\lambda_n}{\rho e} |a_n|^{\rho/\lambda_n} \tag{1.5}$$

In this paper we investigate certain relationships between two or more entire functions.

2. MAIN RESULTS

Theorem 1: Let $\{\mu_n\}$ and $\{\nu_n\}$ be real sequences such that conditions (i) and (ii) are satisfied with $'\mu_n'$ in the place of $'\lambda_n'$ or $'\nu_n'$ in the place of $'\lambda_n'$, for every n. f_1 and f_2 are entire functions represented by the Dirichlet series

$$f_1(z) = \sum_{n=1}^{\infty} b_n e^{\mu_n z}, \quad f_2(z) = \sum_{n=1}^{\infty} c_n e^{\nu_n z} \ \ \forall z \in C,$$

of orders ρ_1, ρ_2; lower orders λ_1, λ_2, types T_1, T_2 and lower types t_1, t_2 respectively, each being positive and finite.

Further condition (iv) is satisfied with $'b_n'$ and $'c_n'$ in the place of $'a_n'$. Then the function $'f'$ defined by

$$f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \quad \forall z \in C,$$

where $\{\lambda_n\}$ satisfies (i) and (ii),

(v) $\mu_n \sim \lambda_n \sim \nu_n$ as $n \to \infty$ and

(vi) For some positive reals α_1, α_2 with $\alpha_1 + \alpha_2 = 1,$

$$\log|a_n|^{-1} \sim \left(\log|b_n|^{-1}\right)^{\alpha_1} \sim \left(\log|c_n|^{-1}\right)^{\alpha_2} \text{ as } n \to \infty,$$

is an entire function. Further if $'\rho', '\lambda'$ are the order and lower order of f and $'T', 't'$ are the type and lower type of $'f'$ respectively, then

(a) $\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \le \lambda \le \rho \le \rho_1^{\alpha_1} \rho_2^{\alpha_2}$,

(b) $t_1^{\alpha_1} t_2^{\alpha_2} \le t$ and

(c) $T \le T_1^{\alpha_1} T_2^{\alpha_2}$, provided $\rho = \rho_1^{\alpha_1} \rho_2^{\alpha_2}$.

Proof: In view of the hypothesis it follows that

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = \limsup_{n\to\infty} \frac{\log n}{\mu_n} = \limsup_{n\to\infty} \frac{\log n}{\nu_n} = 0.$$

Let $k > 0$. Since f_1 and f_2 are entire functions it follows that

$$\frac{\log|b_n|^{-1}}{\mu_n} > k \text{ and } \frac{\log|c_n|^{-1}}{\nu_n} > k \quad \text{eventually.}$$

Hence, for sufficiently large 'n',

$$\left(\frac{\log|b_n|^{-1}}{\mu_n}\right)^{\alpha_1} \left(\frac{\log|c_n|^{-1}}{\nu_n}\right)^{\alpha_2} > k^{\alpha_1+\alpha_2} = k.$$

By (v) and (vi) we get that

$\dfrac{\log|a_n|^{-1}}{\lambda_n} > k$ eventually. Thus 'f' is an entire function.

Let $0 < \in < \min\left\{\dfrac{1}{\rho_1}, \dfrac{1}{\rho_2}\right\}$;

By the definition of $\rho_j\,(j = 1,2)$, we have

$$\left(\log|b_n|^{-1}\right)^{\alpha_1} > (\mu_n \log \mu_n)^{\alpha_1}\left(\rho_1^{-1} - \in\right)^{\alpha_1}$$

and

$$\left(\log|c_n|^{-1}\right)^{\alpha_2} > (\nu_n \log \nu_n)^{\alpha_2}\left(\rho_2^{-1} - \in\right)^{\alpha_2} \text{ eventually.}$$

Hence $\left(\log|b_n|^{-1}\right)^{\alpha_1}\left(\log|c_n|^{-1}\right)^{\alpha_2} > (\mu_n \log \mu_n)^{\alpha_1}(\nu_n \log \nu_n)^{\alpha_2}\left(\rho_1^{-1} - \in\right)^{\alpha_1}\left(\rho_2^{-1} - \in\right)^{\alpha_2}$

$$\Rightarrow \log|a_n|^{-1} > (\lambda_n \log \lambda_n)\left(\rho_1^{-1} - \in\right)^{\alpha_1}\left(\rho_2^{-1} - \in\right)^{\alpha_2}$$

$$\Rightarrow \limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log|a_n|^{-1}} = \rho \le \frac{1}{\left(\rho_1^{-1} - \in\right)^{\alpha_1}\left(\rho_2^{-1} - \in\right)^{\alpha_2}};$$

Since $\in > 0$ is arbitrary, follows that

$$\rho \le \frac{1}{\left(\rho_1^{-1}\right)^{\alpha_1}\left(\rho_2^{-2}\right)^{\alpha_2}} = \rho_1^{\alpha_1}\rho_2^{\alpha_2}.$$

It is trivial that $\lambda \le \rho$.

Let $\in > 0$; by the definition of $\lambda_j\,(j = 1,2)$, we have

$$\left(\log|b_n|^{-1}\right)^{\alpha_1} > (\mu_n \log \mu_n)^{\alpha_1}\left(\lambda_1^{-1} + \in\right)^{\alpha_1}$$

and

$$\left(\log|c_n|^{-1}\right)^{\alpha_2} > (\nu_n \log \nu_n)^{\alpha_2}\left(\lambda_2^{-1} + \in\right)^{\alpha_2} \text{ eventually.}$$

$$\Rightarrow \log|a_n|^{-1} > (\lambda_n \log \lambda_n)\left(\lambda_1^{-1} + \in\right)^{\alpha_1}\left(\lambda_2^{-1} + \in\right)^{\alpha_2} \text{ eventually.}$$

$$\Rightarrow \lambda_1^{\alpha_1}\lambda_2^{\alpha_2} = \frac{1}{\left(\lambda_1^{-1}\right)^{\alpha_1}\left(\lambda_2^{-2}\right)^{\alpha_2}} \le \liminf_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log|a_n|^{-1}} = \lambda.$$

Thus $\lambda_1^{\alpha_1}\lambda_2^{\alpha_2} \le \lambda \le \rho \le \rho_1^{\alpha_1}\rho_2^{\alpha_2}$. This proves (a).

Let $0 < \in < \min\{t_1, t_2\}$.

By the definition of $t_j\,(j = 1,2)$, we have

$\dfrac{\mu_n}{e\rho_1}|b_n|^{\rho_1/\mu_n} > t_1 - \in$ and $\dfrac{\nu_n}{e\rho_2}|c_n|^{\rho_2/\nu_n} > t_2 - \in$ eventually; these imply

$$\left(\log|b_n|^{-1}\right)^{\alpha_1} < \left[\frac{\mu_n}{\rho_1}\log\left\{\frac{\mu_n}{e\rho_1(t_1 - \in)}\right\}\right]^{\alpha_1}$$

and

$$\left(\log|c_n|^{-1}\right)^{\alpha_2} < \left[\frac{\nu_n}{\rho_2}\log\left\{\frac{\nu_n}{e\rho_2(t_2 - \in)}\right\}\right]^{\alpha_2} \text{ eventually;}$$

$$\Rightarrow \log|a_n|^{-1} < \frac{\lambda_n}{\rho}\left[\log\left\{\frac{\lambda_n}{A}\right\}^{\alpha_1} \log\left\{\frac{\lambda_n}{B}\right\}^{\alpha_2}\right]$$

$$= \frac{\lambda_n}{\rho}\left[\left(1-\frac{\log A}{\log\lambda_n}\right)^{\alpha_1}\left(1-\frac{\log B}{\log\lambda_n}\right)^{\alpha_2}\right]\log\lambda_n$$

$$< \frac{\lambda_n}{\rho}\left[1-\frac{\log(A^{\alpha_1}B^{\alpha_2})}{\log\lambda_n}+O((\log\lambda_n)^{-2})\right]\log\lambda_n,$$

where $A = e\rho_1(t_1-\in),\ B = e\rho_2(t_2-\in)$.

$$\Rightarrow |a_n|^{\rho/\lambda_n} > \lambda_n^{-\left[1-\frac{\log(A^{\alpha_1}B^{\alpha_2})}{\log\lambda_n}+O((\log\lambda_n)^{-2})\right]}.$$

$\Rightarrow \dfrac{\lambda_n}{\rho e}|a_n|^{\rho/\lambda_n} \geq \dfrac{1}{\rho e}A^{\alpha_1}B^{\alpha_2}\left(e^{M}\right)^{\frac{1}{\log\lambda_n}}$, where M is a constant.

$$\Rightarrow t = \liminf_{n\to\infty}\frac{\lambda_n}{\rho e}|a_n|^{\rho/\lambda_n} \geq \frac{1}{\rho e}\left[\left(e\rho_1(t_1-\in)\right)^{\alpha_1}\left(e\rho_2(t_2-\in)\right)^{\alpha_2}\right](1)$$

$$= \frac{\rho_1^{\alpha_1}\rho_2^{\alpha_2}}{\rho}\left[(t_1-\in)^{\alpha_1}(t_2-\in)^{\alpha_2}\right]$$

$$\geq (t_1-\in)^{\alpha_1}(t_2-\in)^{\alpha_2}$$

$\Rightarrow t \geq t_1^{\alpha_1}t_2^{\alpha_2}$ or $t \leq t_1^{\alpha_1}t_2^{\alpha_2}$. This proves (b).

Let $\in > 0$ and $A_1 = e\rho_1(T_1+\in),\ B_1 = e\rho_2(T_2+\in);$ as in (b) we get that

$\dfrac{\lambda_n}{\rho e}|a_n|^{\rho/\lambda_n} < \dfrac{1}{\rho e}A_1^{\alpha_1}B_1^{\alpha_2}\left[\left(e^{M_1}\right)^{\frac{1}{\log\lambda_n}}\right]$ eventually, where M_1 is a constant;

$$\Rightarrow T = \limsup_{n\to\infty}\frac{\lambda_n}{\rho e}|a_n|^{\rho/\lambda_n} \leq \frac{1}{\rho e}A_1^{\alpha_1}B_1^{\alpha_2}(1)$$

$$= \frac{\rho_1^{\alpha_1}\rho_2^{\alpha_2}}{\rho}(T_1+\in)^{\alpha_1}(T_2+\in)^{\alpha_2}$$

$= (T_1+\in)^{\alpha_1}(T_2+\in)^{\alpha_2}$, provided $\rho = \rho_1^{\alpha_1}\rho_2^{\alpha_2}$

$\Rightarrow T \leq T_1^{\alpha_1}T_2^{\alpha_2}$. This proves (c).

Thus if $\rho = \rho_1^{\alpha_1}\rho_2^{\alpha_2}$, we have $t_1^{\alpha_1}t_2^{\alpha_2} \leq t \leq T \leq T_1^{\alpha_1}T_2^{\alpha_2}$.

Remark: Theorems (1), (2) and (3) of P. K. Kamthan [1] can be deduced from our theorem by taking $\alpha_1 = \alpha_2 = \dfrac{1}{2}$.

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