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STRONG COMMUTATIVITY – PRESERVING DERIVATIONS ON NEAR-RINGS

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ABSTRACT

Bell and Mason [1] studies derivations in Near-rings and Near-fields. In this paper we present some properties of Strong Commutativity- Preserving derivations in left near-rings. We prove that if N is left near-ring with U as a right ideal which contains no zero devisors of N and d is a nonzero commuting scp – derivation on U, then N is commutative.

***Key words:** Prime near-ring, derivation, ideal, strong commutativity-preserving derivation.*

I. INTRODUCTION

Throughout this paper N denotes a zero- symmetric left near-ring with a derivation d satisfying d(xy) = xd(y) + d(x)y for all x, y in N and Z denotes the center of N. A left near-ring is a set N with two operations **+** and **.** such that (N,+) is a group and (N, .) is a semigroup satisfying the left distributive law x(y+z) = xy+yz for all x, y, z in N. According the Bell and Mason [2] a near-ring N is said to be prime if xNy = {0} for x, y in N implies x=0 or y=0. A derivation d is said to be strong commutativity-preserving derivation on N if [x, y] = [d(x), d(y)] for all x, y in N. An element c ϵ N for which d(c) = 0 is called a constant and a derivation d is said to be commuting on N if [x, d(x)=0 for all x ϵ N.

II. Main Theorems

To prove the main theorem we require the following Lemmas.

Lemma 3.2.1: If d is a scp-derivation on N, then constants are in Z. If N also has 1, then $(N, +)$ is abelian.

Proof: For c constant, we have $[c, y] = [d(c), d(y)] = [0, d(y)] =0$, for all $y \in N$.

In particular, if N has 1, then $1+1 \in Z$.

Hence $[1+1, x + y] = 0$, for all $x, y \in N$, from which we have that $(N, +)$ is abelian.

Lemma 3.2.2: Let d be be a derivation on N, and suppose $u \in N$ is not a left zero divisor. If $[u, d(u)]=0$, then $(x, u) = x+u-x-u$ is a constant for every $x \in N$.

Proof: From $u(u+x) = u^2+ux$, we obtain

$$ud(u+x)+d(u)(u+x) = ud(u) + d(u)u +ud(x)+d(u)x.$$

This implies that $ud(x)+d(u)u = d(u)u +ud(x)$.

Since $d(u)u = ud(u)$, we obtain $u(d(x)+d(u)-d(x)-d(u)) = 0 = ud((x, u))$.

Thus $d((x, u)) = 0$.

Theorem 3.2.1: If N has no zero divisions and admits a nonzero commuting scp-derivation, then N is a commutative ring with no idempotents except 0 or 1.

Proof: For all $x, y \in N$, we have $[x, y] = [d(x), d(y)]$.

We replace y by xy in the above equation. Then we get $[x, xy] = [d(x), d(xy)]$. This implies that $x[x, y] = [d(x), xd(y) + d(x)y]$ for all $x, y \in N$.

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So, $x[x, y] = d(x)xd(y) + d(x)^2y - d(x)yd(x) - xd(y)\, d(x)$.

Since d is commuting and hence (N, +) is abelian. Therefore we have

$$x[x, y] = x[d(x),d(y)] + d(x)\,[d(x),y] = x[x, y] + d(x)\,[d(x),y].$$

Hence $d(x)\,[d(x), y] = 0$, for all $x, y \in N$; and since N has no zero divisors, we get $[d(x), y] = 0$, for all $x, y \in N$.

In particular, $[d(x), d(y)] = 0$; and therefore $[x, y] = 0$, for all $x, y \in N$.

Thus N is a commutative ring.

We know that if N admits a commuting scp-derivation, then all idempotents e are central.

Therefore, if $e^2 = e \neq 0$, then e is central.

Since $e(ex - x) = 0$ for all $x \in N$, e is a left identity element.

Since e $\in Z$, it follows that $e = 1$.

Now we also have the following.

Corollary 3.2.1: *A* near-field with a scp-derivation is a field.

Corollary 3.2.2: *A* near-domain admitting a nonzero scp-derivation is a commutative ring (and hence an ordinary integral domain).

Corollary 3.2.3: If N has no nonzero nilpotent elements and admits a commuting scp-derivation, then N is a commutative ring.

Proof: By Lemma 4 of [11], there exists a family of completely prime ideals $\{P_\alpha / \alpha \in \Lambda\}$ such that N is a subdirect product of the near-rings N/ P_α, and such that for each $\alpha \in \Lambda$, the definition $\tilde{d}_\alpha\,(x+ P_\alpha) = d(x) + P_\alpha$ yields a derivation $\tilde{d}_\alpha$ on N/ P_α. Let $\tilde{N}$ denote a typical N/ P_α ; and $\tilde{N}$ has no zero divisors of zero.

If $\tilde{d}_\alpha$ is nonzero, then $\tilde{N}$ is a commutative ring by Theorem 3.2.1.

If $\tilde{d}_\alpha$ is trivial, then from the definition of scp-derivation we have that $\tilde{N}$ is commutative, hence distributive.

But then $(\widetilde{N^2}, +)$ is abelian, so that

$\widetilde{x^2} + \tilde{x}\tilde{y} - \widetilde{x^2} - \tilde{x}\tilde{y} = 0$, for all $\tilde{x}, \tilde{y} \in \tilde{N}$; and cancelling $\tilde{x}$ we obtain that $(\tilde{N},+)$ is abelian.

Theorem 3.2.2: Let U be a nonzero ideal of N which contains no zero divisors of N. If N admits a nonzero derivation d such that $[x, d(x)] = 0$, for all $x \in U$ and $[x, y] = [d(x), d(y)]$, for all $x, y \in U$, then N is a commutative ring.

Proof: By Lemma 3.2.2, we have the additive group commutator

$(x, a) = x + a - x - a$ is constant for all $a \in U$ and $x \in N$.

Since U is an ideal, we have $(x, a)y = (xy, ay)$ is also constant for arbitrary $y \in N$.

Hence $(x, a)\, d(N) = \{0\}$.

Since U has no zero divisors and $(x,a) \in U$, we obtain $(x, a) =0$; and therefore $(U,+)$ is abelian.

Now for any arbitrary $a \in U/ \{0\}$ and $x, y \in N$, we have $(ax, ay) = a(x, y) = 0$; and hence $(N, +)$ is abelian.

Now by the proof of theorem 3.2.1, we have

$$d(x)\,[d(x),y] = 0, \text{ for all } x, y \in U.$$

Since $[d(x), y] \in U$, we conclude that $[d(x), y] = 0$ or $d(x) = 0$.

Thus $[d(x), y] = 0$, for all $x, y \in U$.

In particular, for all $x, y \in U$ we have $[d(x), yd(y)] = 0 = y[d(x), d(y)]$.

Therefore, $0 = [d(x), d(y)] = [x, y]$, for all $x, y \in U$.

Using this, if $a \in A/\{0\}$ and $x, y \in N$, then we have

$axay - ayax = 0 = a^2(xy - yx) = a^2[x, y]$; so $[x, y] = 0$.

Therefore, N is a commutative ring.

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