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COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACE FOR SINGLE AND SET-VALUED M-MAPS

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ABSTRACT

In this paper, we introduce the notion of M-maps w.r.to a single map and a pair of maps in fuzzy metric spaces and obtain common fixed point theorems for two pairs of sub compatible maps satisfying implicit relations. Our results generalize and extend several comparable results in existing literature like Abbas.et.al [2], Kumar et.al. [9] to the setting of single and set-valued maps and also by relaxing some more conditions.

Key words: *Fuzzy Metric space , set-vlaued M-maps , sub compatible mappings.*

AMS Mathematics Subject Classification: *47H10, 54H25.*

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1. INTRODUCTION AND PRELIMINARIES:

Zadeh [17] introduced the concept of fuzzy sets in 1965 and Kramosil and Michalek [8] introduced the concept of fuzzy metric space in 1975. George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [8]. Later several authors Grabiec[6], Mishra, Sharma and Singh[10], Singh and Jain[15], Subrahmanyam[16], Altun and Turkoglu[3], Sedghi, K.P.R.Rao and shobe[14], Y.J.Cho, Sedghi and Shobe[4], Pant[11], Abbas, Altun and Gopal[2], Kumar and Fisher[9] etc. obtained fixed and common fixed point theorems satisfying various contractive conditions in fuzzy metric spaces. Aamri and Moutawakil [1] introduced the concept of property (E.A) for a pair of maps and studied fixed and comon fixed point theorems.

In this paper we obtain two common fixed point theorems for two pairs of single and multi valued mappings by using property (E.A) type definition.

First, we give some known preliminaries.

Definition 1.1 ([17]): Let X be any set. A fuzzy set A in X is a function with domain X and values in [0,1].

Definition 1.2 ([13]): A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm if ([0,1], $*$) is an abelian topological monoid with the unit 1 such that $a*b\le c*d$, whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Note that among a number of possible choices for $*$, '$*=\min$' is the strongest possible universal t-norm (see [13]).

Definition 1.3 ([8]) : The triplet $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $t, s, > 0$,

(i) $M(x,y,t)>0$, $M(x,y,0)=0$;

(ii) $M(x,y,t)=1$ forall $t>0$ if and only if $x=y$;

(iii) $M(x,y,t)=M(y,x,t)$;

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(iv) $M(x,y,t) \;*\; M(y,z,s) \le M(x,z,t+s)$;

(v) $M(x,y,.) : [0,\infty) \to [0,1]$ is a continuous function .

Lemma 1.4 ([6]): Let $(X,M,*)$ be a fuzzy metric space. Then $M(x,y,t)$ is non-decreasing with respect to t for all $x, y \in X$.

Lemma 1.5 ([12]): Let $(X,M,*)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0,\infty)$.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x,r,t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by $B(x,r,t) = \{ y \in X / M(x,y,t) > 1\text{-}r \}$.

The collection of $\{B(x,r,t) : x \in X, \; 0 < r < 1, \; t > 0\}$ is a neighborhood system for a topology τ on X induced by the fuzzy metric. This topology is Hausdorff.

A sequence $\{x_n\}$ in X converges to x if and only if for each $\varepsilon > 0$ and each $t > 0$ there exists $n_0 \in N$ such that $M(x_n,y,t) > 1\text{-}\varepsilon$ for all $n \ge n_0$.

Lemma 1.6 ([10]) : Let $(X,M,*)$ be a fuzzy metric space and $M(x,y,t) \to 1$ as $t \to \infty$ for all $x,y \in X$. If $M(x,y,kt) \ge M(x,y,t)$ for all $x,y \in X, \; t > 0$ and for a number $k \in (0,1)$ then $x = y$.

Here afterwards, we assume that $B(X)$ is the set of nonempty bounded subsets of fuzzy metric space $(X,M,*)$.

For $A,B \in B(X)$ and for every $t > 0$, define

$$M(A,B,t) = \inf\{M(a,b,t) : a \in A, \; b \in B\}.$$

If $A = \{a\}, \; \delta_M(A,B,t) = M(a,B,t)$.

If $A = \{a\}$, $B = \{b\}$ then $\delta_M(A,B,t) = \delta_M(a,b,t)$.

From the definition it follows that

$$\delta_M(A,B,t) = \delta_M(B,A,t) \ge 0,$$

$$\delta_M(A,B,t) = 1 \Leftrightarrow A = B = \{ \sin gleton \},$$

$$\delta_M(A,B,t+s) \ge \delta_M(A,C,t) * \delta_M(C,B,s) \text{ for all } A,B,C \in B(X) \text{ and for all } s, t > 0 .$$

Definition 1.7: A sequence $\{A_n\}$ in $B(X)$ is said to be convergent to a set $A \in B(X)$ if $\delta_M(A_n, A, t) \to 1$ as $n \to \infty$ for all $t > 0$.

Immediately one can prove the following

Lemma 1.8: Let $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively. Then $\delta_M(A_n, B_n,t) \to \delta_M(A,B,t)$ as $n \to \infty$ for all $t > 0$.

Lemma 1.9: If $\delta_M(A,B,kt) \ge \delta_M(A,B,t)$ for all $A, B \in B(X)$ and for all $t > 0$, $0 < k < 1$ then $A = B = \{ \sin gleton \}$ provided $M(x, y, t) \to 1$ as $t \to \infty \;\; \forall \; x, y \in X$.

Definition 1.10 ([7]): The maps $F : X \to B(X)$ and $f : X \to X$ are said to be weakly compatible or sub compatible if they commute at coincidence points, i.e., for each point $u \in X$ such that $Fu = \{fu\}$, we have

$$Ffu = fFu .$$

Generally to prove common fixed point theorems for two pairs of maps or Gungck type maps using property (E.A), introduced in [1] one can tempt to assume that the range set of one mapping is closed or one of the mappings is surjective. In this paper, we relax some conditions by introducing the following two definitions.

Definition 1.11: Let $(X, M, *)$ be a fuzzy metric space and $f : X \to X$ and $F : X \to B(X)$. Then (f, F) is said to be a pair of M-maps with respect to f if there exists a sequence $\{x_n\}$ in X such that for every $t > 0$, $M(fx_n, z, t) \to 1$ and $\delta_M(Fx_n, \{z\}, t) \to 1$ as $n \to \infty$ for some $z \in f(X)$.

Definition 1.12: Let $(X, M, *)$ be a fuzzy metric space and $f, g : X \to X$ and $F : X \to B(X)$. Then (f, F) is said to be a pair of M-maps with respect to (f, g) if there exists a sequence $\{x_n\}$ in X such that for every $t > 0$, $M(fx_n, z, t) \to 1$ and $\delta_M(Fx_n, \{z\}, t) \to 1$ as $n \to \infty$ for some $z \in f(X) \cap g(X)$.

2. IMPLICIT RELATION:

Let Φ denote the class of all continuous functions $\varphi : [0,1]^6 \to R_+$ satisfying

$\varphi(u, 1, 1, v, v, 1) \geq 0$ or $\varphi(u, 1, v, 1, 1, v) \geq 0$ or $\varphi(u, v, 1, 1, v, v) \geq 0$ Implies $u \geq v$.

These are the some examples of implicit relations.

Example 2.1: $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$.

Example 2.2: $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_3 t_5, t_4 t_6\}$.

Example 2.3: $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \dfrac{t_3 t_4 + t_5 t_6}{1 + t_2}$.

Example 2.4: $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi(t_2, t_3, t_4, t_5, t_6)$ where ψ is increasing in each coordinate and $\psi(t,t,t,t,t) > t \quad \forall \quad t \in [0,1)$.

3. MAIN RESULTS:

Theorem 3.1: Let $(X, M, *)$ be a fuzzy metric space and $f, g : X \to X$ and $F, G : X \to B(X)$ be maps satisfying

$$\varphi\begin{pmatrix} \delta_M(Fx, Gy, kt), & M(fx, gy, t), & \delta_M(fx, Fx, t), \\ \delta_M(gy, Gy, t), & \delta_M(fx, Gy, t), & \delta_M(gy, Fx, t) \end{pmatrix} \geq 0 \tag{3.1.1}$$

for all $x, y \in X$, $t > 0$ and $k \in (0,1)$ where $\varphi \in \Phi$,

the pairs (f, F) and (g, G) are sub compatible , (3.1.2)

(a) (f, F) is a pair of M-maps with respect to f and $Fx \subseteq g(X)$ forall $x \in X$. (3.1.3)

or

(b) (g, G) is a pair of M-maps with respect to g and $Gx \subseteq f(X)$ forall $x \in X$. (3.1.3)

Then f, g, F and G have a unique common fixed point $z \in X$ such that

$$Fz = Gz = \{z\} = \{fz\} = \{gz\}.$$

Proof: Suppose (3.1.3)(a) is holds.

Since (f, F) is a pair of M-maps with respect to f there exists a sequence $\{x_n\}$ in X such that for every $t > 0$,

$$\lim_{n\to\infty} M(fx_n, z, t) = 1 \text{ and } \lim_{n\to\infty} \delta_M(Fx_n, \{z\}, t) = 1 \text{ for some } z \in f(X).$$

Hence there exists $u \in X$ such that $z = fu$.

Since $Fx \subseteq g(X)$ for all $x \in X$ there exist $\alpha_n \in Fx_n$ and $y_n \in X$ such that $\alpha_n = gy_n$ for all n.

Also $M(gy_n, z, t) = M(\alpha_n, z, t) \geq \delta_M(Fx_n, \{z\}, t) \to 1$ as $n \to \infty$.

Hence $\lim_{n\to\infty} M(\mathrm{gy_n}, \mathrm{z}, \mathrm{t}) = 1$.

Now,

$$\varphi\begin{pmatrix} \delta_M(Fx_n, Gy_n, kt), & M(fx_n, gy_n, t), & \delta_M(fx_n, Fx_n, t), \\ \delta_M(gy_n, Gy_n, t), & \delta_M(fx_n, Gy_n, t), & \delta_M(gy_n, Fx_n, t) \end{pmatrix} \geq 0.$$

Letting $n \to \infty$ we have,

$$\varphi\left(\delta_M(\{z\}, \lim_{n\to\infty} Gy_n, kt), 1, 1, \delta_M(\{z\}, \lim_{n\to\infty} Gy_n, t), \ \delta_M(\{z\}, \lim_{n\to\infty} Gy_n, t), \ 1\right) \geq 0.$$

Hence

$$\delta_M(\{z\}, \lim_{n\to\infty} Gy_n, kt) \geq \delta_M(\{z\}, \lim_{n\to\infty} Gy_n, t) \quad \text{from property of } \varphi.$$

From Lemma 1.9, we have $\lim_{n\to\infty} Gy_n = \{z\}$.

Thus $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gy_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \{z\} = \{fu\}$.

Now,

$$\varphi\begin{pmatrix} \delta_M(Fu, Gy_n, kt), M(fu, gy_n, t), \delta_M(fu, Fu, t), \\ \delta_M(gy_n, Gy_n, t), \delta_M(fu, Gy_n, t), \delta_M(gy_n, Fu, t) \end{pmatrix} \geq 0.$$

Letting $n \to \infty$ we have,

$$\varphi\left(\delta_M(Fu, \{z\}, kt), 1, \delta_M(\{z\}, Fu, t), 1, 1, \delta_M(\{z\}, Fu, t)\right) \geq 0.$$

$$\delta_M(\mathrm{Fu}, \{\mathrm{z}\}, \mathrm{kt}) \geq \delta_M(\{\mathrm{z}\}, \mathrm{Fu}, \mathrm{t}).$$

Hence $Fu = \{z\}$. Thus $Fu = \{z\} = \{fu\}$.

Since $\{z\} = Fu \subseteq g(X)$, there exists $v \in X$ such that $z = gv$.

Now,

$$\varphi\begin{pmatrix} \delta_M(Fx_n, Gv, kt), & M(fx_n, gv, t), & \delta_M(fx_n, Fx_n, t), \\ \delta_M(gv, Gv, t), & \delta_M(fx_n, Gv, t), & \delta_M(gv, Fx_n, t) \end{pmatrix} \geq 0.$$

Letting $n \to \infty$ we have,

$$\varphi\left(\delta_M(\{z\}, Gv, kt), 1, 1, \delta_M(\{z\}, \mathrm{Gv}, \mathrm{t}), \ \delta_M(\{z\}, Gv, t), 1\right) \geq 0.$$

$$\delta_M(\{z\}, Gv, kt) \geq \delta_M(\{z\}, Gv, t).$$

Hence $Gv = \{z\}$. Thus $Gv = \{z\} = \{gv\}$.

Since the pairs (f, F) and (g, G) are sub compatible, we have

$Fz = \{fz\}$ and $Gz = \{gz\}$.

$$\varphi\begin{pmatrix}\delta_M(Fz, Gv, kt), & M(fz, gv, t), & \delta_M(fz, Fz, t), \\ \delta_M(gv, Gv, t), & \delta_M(fz, Gv, t), & \delta_M(gv, Fz, t)\end{pmatrix} \geq 0.$$

$$\varphi\left(M(fz, z, kt),\ M(fz, z, t),\ 1,\ 1,\ M(fz, z, t),\ M(z, fz, t)\right) \geq 0.$$

$$M(fz, z, kt) \geq M(fz, z, t).$$

Hence $fz = z$. Thus $Fz = \{fz\} = \{z\}$.

Also,

$$\varphi\begin{pmatrix}\delta_M(Fu, Gz, kt), & M(fu, gz, t), & \delta_M(fu, Fu, t), \\ \delta_M(gz, Gz, t), & \delta_M(fu, Gz, t), & \delta_M(gz, Fu, t)\end{pmatrix} \geq 0.$$

$$\varphi\left(M(z, gz, kt),\ M(z, gz, t),\ 1,\ 1, M(z, gz, t), M(gz, z, t)\right) \geq 0.$$

$$M(z, gz, kt) \geq M(z, gz, t).$$

Hence $gz = z$. Thus $Gz = \{gz\} = \{z\}$.

Thus z is a common fixed point of f, g, F and G such that $Fz = Gz = \{z\} = \{fz\} = \{gz\}$. Uniqueness of common fixed point follows easily from (3.1.1). The following example illustrates Theorem 3.1.

Example 3.2: Let $X = [0,1]$ and $d(x, y) = |x - y|$ and $a * b = \min\{a, b\}$ $\forall\ a,b \in [0,1]$. Define $M(x,y,t) = \dfrac{t}{t + d(x,y)}$ $\forall\ t > 0$ and $\forall\ x, y \in X$.

Then $(X,M,*)$ is a fuzzy metric space.

Define $F,G: X \to B(X)$ and $f,g : X \to X$ as follows,

$$fx = \begin{cases} \frac{1}{2} & \text{if } x \in [0,\frac{1}{2}], \\ \frac{x+1}{4} & \text{if } x \in (\frac{1}{2},1] \end{cases}, \qquad gx = \begin{cases} 1 - x & \text{if } x \in (0,\frac{1}{2}], \\ 0 & \text{if } x \in (\frac{1}{2},1] \cup \{0\} \end{cases}$$

$$Fx = \left\{\frac{1}{2}\right\}, \quad Gx = \begin{cases} \{\frac{1}{2}\} & \text{if } x \in [0,\frac{1}{2}], \\ [\frac{3}{8}, \frac{1}{2}] & \text{if } x \in (\frac{1}{2},1] \end{cases}$$

Then $Fx = \left\{\frac{1}{2}\right\} \subseteq g(X) = [\frac{1}{2},1) \cup \{0\}$ $\forall\ x \in X$, $Gx = [\frac{3}{8}, \frac{1}{2}] \not\subset f(X) = (\frac{3}{8}, \frac{1}{2}]$ for all $x \in (\frac{1}{2},1]$.

Case (i): if $x \in X$, $y \in [0, \frac{1}{2}]$ then, $\delta_M(Fx,Gy, kt) = 1$

Case (ii): if $x \in X$, $y \in (\frac{1}{2},1]$ then,

$$\delta_M(Fx,Gy, kt) = \inf\{M(a, b, kt) : a \in Fx,\ b \in Gy\}$$

$$= \inf\left\{\frac{kt}{kt+d(a,b)} : a\in\left\{\tfrac{1}{2}\right\},\ b\in\left[\tfrac{3}{8},\tfrac{1}{2}\right]\right\}.$$

$$= \frac{kt}{kt+\frac{1}{8}}$$

$$M(fx,\ gy,\ t) = \frac{t}{t+d(fx,gy)} = \frac{t}{t+d(fx,0)} > \frac{t}{t+\frac{1}{2}}.$$

Thus $\delta_M(Fx,Gy,\ kt) \geq M(fx,gy,\ t)\ \forall\ \ x,y \in X,\ \forall\ \ t>0$ where $k = \frac{1}{2}$.

(3.1.1) is satisfied with $k = \frac{1}{2}$ and $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$.

Clearly (f, F) and (g, G) are sub compatible.

For $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{2n}\right\}$ we have $\{fx_n\} \to \frac{1}{2}$, $Fx_n = \frac{1}{2}$, $\frac{1}{2} \in f(X)$, $Fx \subseteq g(X)\ \forall\ x \in X$ So (3.1.1)(a) is satisfied. But (3.1.1) (b) is not satisfied since $Gx \not\subseteq f(X)\ \ \forall\ \ \ x\in (\frac{1}{2}, 1]$.

Thus all the conditions of Theorem 3.1 are satisfied and $\frac{1}{2}$ is the unique common fixed point of f, g, F and G.

In this example note that the maps f and g are not surjective and the sets $f(X)$ and $g(X)$ are not closed.

Theorem 3.3: Let $(X,M,*)$ be a fuzzy metric space and $f, g : X \to X$ and $F, G : X \to B(X)$ be maps satisfying (3.1.1), (3.1.2) and (3.3.1) (f, F) or (g, G) is a pair of M-maps with respect to (f, g).
Then f, g, F and G have a unique common fixed point $z \in X$ such that $Fz = Gz = \{z\} = \{fz\} = \{gz\}$.

Proof: Suppose (f, F) is a pair of M-maps with respect to (f, g). Then there exists a sequence $\{x_n\}$ in X such that for every $t > 0$,

$$M(fx_n, z, t) \to 1 \text{ and } \delta_M(Fx_n, \{z\}, t) \to 1 \text{ for some } z \in f(X) \cap g(X).$$

Hence there exist $u, v \in X$ such that $z = fu = gv$.

Now

$$\varphi\begin{pmatrix} \delta_M(Fx_n,Gv,kt), M(fx_n,gv,t), \delta_M(fx_n,Fx_n,t), \\ \delta_M(gv,Gv,t),\ \delta_M(fx_n,Gv,t), \delta_M(gv,Fx_n,t) \end{pmatrix} \geq 0 .$$

Letting $n \to \infty$, we have

$$\varphi\left(\delta_M(\{z\},Gv,kt),\ 1,\ 1, \delta_M(\{z\},Gv,t), \delta_M(\{z\},Gv,t),\ 1\right) \geq 0 .$$

$$\delta_M(\{z\},Gv,kt) \geq \delta_M(\{z\},Gv,t).$$

Hence $Gv = \{z\}$. Thus $Gv = \{z\} = \{gv\}$.

$$\varphi\begin{pmatrix} \delta_M(Fu,Gv,kt),\ M(fu,gv,t),\ \delta_M(fu,Fu,t), \\ \delta_M(gv,Gv,t),\ \delta_M(fu,Gv,t), \delta_M(gv,Fu,t)\}. \end{pmatrix} \geq 0 .$$

$$\varphi\left(\delta_M(Fu,\{z\},kt),\ 1,\ \delta_M(\{z\},Fu,t),\ 1,\ 1,\ \delta_M(\{z\},Fu,t)\right) \geq 0 .$$

$$\delta_M(Fu,\{z\},kt) \geq \delta_M(\{z\},Fu,t).$$

Hence $Fu = \{z\}$. Thus $Fu = \{z\} = \{fu\}$.

From (3.1.2), we have $Fz = \{fz\}$ and $Gz = \{gz\}$.

The rest of the proof follows as in Theorem 3.1. The following example illustrates Theorem 3.3.

Example 3.4: Let $X = [0,1]$ and $d(x, y) = |x - y|$ and $a * b = \min\{a, b\}$ $\forall$ $a,b \in [0,1]$ Define

$$M(x,y,t) = \frac{t}{t + d(x,y)} \;\forall\; t > 0 \text{ and } \forall\; x, y \in X$$

Then $(X,M,*)$ is a fuzzy metric space.

Define $F,G: X \to B(X)$ and $f, g : X \to X$ as follows

$$fx = \begin{cases} \frac{1}{2} & \text{if } x \in [0,\frac{1}{2}], \\ \frac{x+1}{4} & \text{if } x \in (\frac{1}{2},1] \end{cases}, \qquad gx = \begin{cases} 1 - x & \text{if } x \in (0,\frac{1}{2}], \\ 0 & \text{if } x \in (\frac{1}{2},1] \cup \{0\} \end{cases}$$

$$Fx = \begin{cases} \{\frac{1}{2}\} & \text{if } x \in [0,\frac{1}{2}], \\ [\frac{7}{16}, \frac{1}{2}] & \text{if } x \in (\frac{1}{2},1] \end{cases}, \quad Gx = \begin{cases} \{\frac{1}{2}\} & \text{if } x \in [0,\frac{1}{2}], \\ [\frac{3}{8}, \frac{1}{2}] & \text{if } x \in (\frac{1}{2},1] \end{cases}$$

$$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$$

Case (i): if $x, y \in [0,\frac{1}{2}]$ then, $\delta_M(Fx,Gy, kt) = 1$

Case (ii): if $x \in [0,\frac{1}{2}]$, $y \in (\frac{1}{2},1]$ then,

$$\begin{aligned} \delta_M(Fx,Gy, kt) &= \inf\{M(a, b, kt) : a \in Fx, b \in Gy\} \\ &= \inf\{\tfrac{kt}{kt+d(a,b)} : a \in \{\tfrac{1}{2}\}, b \in [\tfrac{3}{8},\tfrac{1}{2}]\}. \\ &= \frac{kt}{kt + \frac{1}{8}} \end{aligned}$$

$$M(fx, gy, t) = \frac{t}{t + d(fx,gy)} = \frac{t}{t + \frac{1}{2}}.$$

Case (iii): if $x \in (\frac{1}{2},1]$, $y \in [0,\frac{1}{2}]$ then,

$$\delta_M(Fx,Gy, kt) = \inf\left\{\frac{kt}{kt + d(a,b)} : a \in [\tfrac{7}{16},\tfrac{1}{2}],\; b = \tfrac{1}{2}\right\} = \frac{kt}{kt + \frac{1}{16}}$$

$$\begin{aligned} \delta_M(fx,Fx, t) &= \inf\left\{\frac{kt}{kt + d(a,b)} : a \in \tfrac{x+1}{4},\; b \in [\tfrac{7}{16},\tfrac{1}{2}]\right\} \\ &= \frac{t}{t + \frac{1}{8}} \end{aligned}$$

Case (iv): if $x, y \in (\frac{1}{2},1]$ then,

$$\delta_M(Fx,Gy, kt) = \inf\left\{\frac{kt}{kt + d(a,b)} : a \in [\tfrac{7}{16},\tfrac{1}{2}],],\; b \in [\tfrac{3}{8},\tfrac{1}{2}]\right\}$$

$$= \frac{kt}{kt + \frac{1}{8}}$$

$$M(fx, gy, t) = \frac{t}{t + d(fx, gy)} = \frac{t}{t + d(\frac{x+1}{4}, 0)} = \frac{t}{t + \frac{x+1}{4}} .$$

In all cases

$$\delta_M(Fx, Gy, \tfrac{1}{2}t) \geq \min\left\{\begin{array}{l} M(fx,gy,t), \delta_M(fx,Fx,t), \delta_M(gy,Gy,t), \\ \delta_M(fx,Gy,t),\ \delta_M(gy,Fx,t) \end{array}\right\} \ \forall\ x, y \in X,\ \forall\ t > 0 .$$

Clearly (f, F) and (g, G) are sub compatible.

For $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{2n}\right\}$ we have $\{gx_n\} \to \frac{1}{2}$, $Gx_n = \left\{\frac{1}{2}\right\}$, $\left\{\frac{1}{2}\right\} \in f(X) \cap g(X)$

Hence (g, G) is a pair of M-maps with respect to (f, g).

Thus all the conditions of Theorem 3.3 are satisfied and $\frac{1}{2}$ is the unique common fixed point of f, g, F and G.
In this example note that

(i) the maps f and g are not surjective.

(ii) $f(X)$ and $g(X)$ are not closed.

(iii) $Fx \not\subseteq g(X)\ \ \forall\ x \in (\frac{1}{2}, 1]$ and $Gx \not\subseteq f(X)\ \ \forall\ x \in (\frac{1}{2}, 1]$.

Recently Abbas et.al. [2] used the following implicit relation. Let Ψ be a class of all continuous functions $\varphi : [0,1]^5 \to [0,1]$ which are increasing in each coordinate and $\varphi(t, t, t, t, t) > t$ for all $t \in [0,1)$. Using this implicit relation Abbas et.al. [2] proved the following:

Theorem: 3.5 (Theorem 2.1, [2])**:** Let $(X, M, *)$ be a fuzzy metric space. Let f, g, F and G be self maps on X satisfying

(i) $F(X) \subseteq g(X)$ and $G(X) \subseteq f(X)$ and there exists a constant $k \in (0, \frac{1}{2})$ such that

(ii) $M(Fx, Gy, kt) \geq \varphi\left\{\begin{array}{l} M(fx,gy,t),\ M(fx,Fx,t),\ M(gy,Gy,t), \\ M(Fx,gy,\alpha t),\ M(Gy, fx, (2-\alpha)t). \end{array}\right\}$

for all $x, y \in X$, $t > 0$ and $\alpha \in (0,2)$ where $\varphi \in \Psi$

Then F, G, f and g have a unique common fixed point in X provided

(iii) the pair (F, f) or (G, g) satisfies (E.A) property,

(iv) one of $F(X)$, $G(X)$, $f(X)$, $g(X)$ is a closed subset of X and

(v) the pairs (F, f) and (G, g) are weakly compatible.

Theorem: 3.6 (Theorem 2.3, [2])**:** Let $(X, M, *)$ be a fuzzy metric space. Let f, g , F and G be self maps on X satisfying (ii), (v) ,

(vi) The pairs (F, f) and (G, g) satisfy common (E.A) property and,

(vii) $f(X)$ and $g(X)$ are closed subsets of X.

Then F, G, f and g have a unique common fixed point in X.

These two theorems can be extended to two pairs of multi valued and single valued maps and improved respectively as follows.

Theorem: 3.7 Let $(X,M,*)$ be a fuzzy metric space and $f, g : X \to X$ and $F, G : X \to B(X)$ be maps satisfying

(3.7.1) $$\delta_M(Fx,Gy,t) \geq \varphi\left\{\begin{array}{l} M(fx,gy,t), \delta_M(fx,Fx,t), \delta_M(gy,Gy,t), \\ \delta_M(fx,Gy,t), \delta_M(gy,Fx,t)\}. \end{array}\right\}$$

for all $x, y \in X$, $t > 0$ where $\varphi \in \Psi$,

(3.7.2) the pairs (f, F) and (g, G) are sub compatible (3.7.3)(a) (f, F) is a pair of M-maps with respect to f and $Fx \subseteq g(X)\ \forall\ x \in X$.
or
(3.7.3)(b) (g, G) is a pair of M-maps with respect to g and $Gx \subseteq f(X)\ \forall\ x$ \in X.

Then F, G, f and g have a unique common fixed point $z \in X$ such that $Fz = Gz = \{z\} = \{fz\} = \{gz\}$.

Theorem: 3.8 Let $(X,M,*)$ be a fuzzy metric space and $f, g : X \to X$ and $F, G : X \to B(X)$ be maps satisfying (3.7.1), (3.7.2) and (3.8.1) (f, F) or (g, G) is a pair of M-maps with respect to (f, g).
Then F, G, f and g have a unique common fixed point $z \in X$ such that $Fz = Gz = \{z\} = \{fz\} = \{gz\}$.

REFERENCES:

[1] Aamri.M and El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J.math. Anal.Appl.,270(2002),181-188.
[2] Abbas Mujahid,Altun Ishak and Gopal Dhananjay, Common fixed point theorems for non compatible mappings in fuzzy metric spaces, Bulletin of Mathematical Analysis and Applications, vol.1,issue 2,(2009), 47-56.
[3] Altun.I and Turkoglu.D, Some fixed point theorems on fuzzy metric spaces with implicit relations, Commun. Korean.Math.Soc.,23(1)(208), 111-124.
[4] Cho.Y.j, Sedghi.S and Shobe.N, Generalized fixed point theorems for compatible mappings with types in fuzzy metric spaces, Chaos, Solitons and fractals, 39(2009),2233-2244.
[5] George.A and veeramani.P, On some results in fuzzy metric spaces, Fuzzy sets and systems, 64(1994) 395-399.
[6] Grabiec.M, Fixed points in fuzzy metric space, Fuzzy sets and systems, 27(1988),385-389.
[7] Jungck.G and Rhoades B.E, Fixed points for set-valued mappings with out continuity, Indian J. Pure Appl. Math., 29(3),1998,227-238.
[8] Kramosil.I and Michalek.J , Fuzzy metric and statistical metric spaces, Kybernetika, 11(1975),326-334 .
[9] Kumar.S and Fisher.B, A common fixed point theorem in fuzzy metric space using property(E.A) and implicit relation, Thai Journal of Mathematics, vol.8(2010),No.3, 439-446.
[10] Mishara.S.N, Shrma.N and Singh.S.L, Common fixed points of maps in fuzzy metric spaces, Internat. J. Math. Sci., 29(3),(1998),227-238.
[11] Pant.V, Contractive conditions and common fixed points in fuzzy metric spaces, J. fuzzy. Math., 14(2)(2006),267-272.
[12] Rodriguez Lopez.J and Ramaguera.S, The Hausdorff fuzzy metric on compact sets, Fuzzy sets and systems, 147 (2004), 273-283.
[13] Schweizer.B and Sklar.A, Probabilistic metric spaces, North Holland, Amsterdam,1983.
[14] Sedghi.S, Rao.K.P.R and Shobe.N, A general common fixed point theorems for multi-maps satisfying an implicit relation in fuzzy metric spaces, Filomat, 22(1)(2008),1-10.
[15] Singh.B and Jain.S, Weak compatibility and fixed point theorems in fuzzy metric space, Ganita, 56(2) (2005), 167-176.
[16] Subrahmanyam, P.V, A common fixed point theorem in fuzzy metric spaces, Information Sciences, 83(1995), 109-112.
[17] Zadeh. L.A, Fuzzy sets, Inform and Control, 8(1965), 338-353.

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