



X-DOMATIC PARTITION OF GRAPH $VV^+(G)$ AND CHROMATIC NUMBER OF G

*Y. B. Venkatakrishnan¹ and ² V. Swaminathan

¹ Department of Mathematics, Sastra University, Tanjore 613 402. India

² Coordinator, Ramanujan Research Centre, S. N. College, Madurai 625 002. India

E-mail: ¹ sulanesri@yahoo.com, ² Venkatakrish2@maths.sastra.edu

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ABSTRACT

Let $G^1 = (X, Y, E)$ be a bipartite graph. A X -domatic partition of G^1 is a partition of X , all of whose classes are X -dominating set in G^1 . The X -domatic number of G^1 is the maximum number of classes of X -domatic partition of G^1 . The X -domatic number of G^1 is denoted by $d_X(G^1)$.

Let G be a simple undirected graph. We obtain a sharp upper bound for the sum of chromatic number of G and X -domatic number of the bipartite graph VV^+ obtained from G and characterize the corresponding extremal graphs.

Keywords: Bipartite graphs, Chromatic number, X -domatic number.

MSC 2000: 05C69.

1. INTRODUCTION:

Let $G = (V, E)$ be a simple undirected graph. The chromatic number $\chi(G)$ is defined to be the minimum number of colors required to color all the vertices such that adjacent vertices do not receive the same color. A domatic partition of G is a partition of $V(G)$, all of whose classes are dominating set in G . The domatic number of G is the maximum number of classes of a domatic partition of G . The domatic number is denoted by $d(G)$.

Given a graph $G = (V, E)$, the bipartite graph $VV^+(G) = (V, V^1, E^1)$ is defined by the edges $E^1 = \{(u, v^1) / (u, v) \in E\}$ together with $\{(u, u^1) / u \in V\}$. Let $G^1 = (X, Y, E)$ be a bipartite graph. A subset $D \subseteq X$ is an X -dominating set if for every $x \in X - D$, there exists atleast one vertex $u \in D$ such that x and u are adjacent to a common vertex $y \in Y$. The minimum cardinality taken over all the minimal X -dominating set is called X -domination number and is denoted by $\gamma_X(G^1)$. A X -domatic partition of G^1 is a partition of X , all of whose classes are X -dominating set in G^1 . The X -domatic number of G^1 is the maximum number of classes of a X -domatic partition of G^1 . The X -domatic number of G^1 is denoted by $d_X(G^1)$.

Bipartite theory of graphs was proposed in [2] and [3]. Given any problem say A, on an arbitrary graph G , there is a corresponding problem B on a bipartite graph G^1 , such that solution for B provides a solution for A.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph

*Corresponding author: *Y.B. Venkatakrishnan¹, E-mail: Venkatakrish2@maths.sastra.edu

theoretic parameter and characterized the corresponding extremal graphs. In [5] Paulraj Joseph J and Arumugam proved $\gamma_c + \chi \leq p + 1$. They also characterized the class of graphs for which the upper bound is attained. In [4] Mahadevan et al proved $\gamma_{dd}(G) + \chi(G) \leq 2n$, and characterized the corresponding extremal graphs. In this paper, we obtain sharp upper bound for the sum of the chromatic number of a graph G and the X-domatic number of bipartite graph $VV^+(G)$ constructed from G and characterize the corresponding extremal graphs. By the theory of bipartite graph, the above gives an upper bound for the sum of the chromatic number of a graph G and the domatic number of G^2 . [2] defines square of a graph $G^2 = (V, E^2)$ as $(u, v) \in E^2$ if and only if distance $d(u, v) \leq 2$ in G .

Proposition 1.1: $d_X(VV^+(K_n)) = n$.

Proof: K_n is a complete graph on n vertices. Every vertex is adjacent to other vertices in G . In VV^+ , every vertex is X-adjacent to other vertices. Every vertex is a X-dominating set. Therefore, $d_X(VV^+(K_n)) = n$.

Theorem: 1.2 In a bipartite graph $G = (X, Y, E)$, $|X| = n$ then $d_X(G) \leq n$.

Theorem: 1.3 [1] For any connected graph G , $\chi(G) \leq \Delta + 1$.

2. MAIN RESULT:

Theorem: 2.1 For any connected graph G , $d_X(VV^+(G)) + \chi(G) \leq 2n$ and equality holds if and only if $G \cong K_n$.

Proof: $d_X(VV^+(G)) + \chi(G) \leq n + \Delta + 1 \leq n + n - 1 + 1 = 2n$. If $d_X(VV^+(G)) + \chi(G) = 2n$, then the possible case is $d_X(VV^+) = n$ and $\chi(G) = n$. Since, $\chi(G) = n$, $G \cong K_n$ and $d_X(VV^+) = n$. Hence, $G \cong K_n$. The converse of the above is obvious.

G_1 is the family of graphs on n vertices which contains a clique $K = K_{n-1}$ and the other vertex is adjacent to α vertices of K , where $1 \leq \alpha \leq n - 2$.

Theorem: 2.2 For any connected graph G , $d_X(VV^+(G)) + \chi(G) = 2n - 1$ if and only if $G \in G_1$.

Proof: Assume $d_X(VV^+(G)) + \chi(G) = 2n - 1$. This is possible only if $d_X(VV^+(G)) = n$ and $\chi(G) = n - 1$ or $d_X(VV^+(G)) = n - 1$ and $\chi(G) = n$.

Case: (a) $d_X(VV^+(G)) = n$ and $\chi(G) = n - 1$.

Since, $\chi(G) = n - 1$, G contains a clique $K = K_{n-1}$. Let x be any vertex of G other than the vertices of K . Since G is connected x is to a vertex of K . Let us assume x is adjacent to α vertices of K , $1 \leq \alpha \leq n - 2$. The vertex in $N(x)$ of the graph G is of degree n in $VV^+(G)$. Hence, $G \in G_1$

Case : (b) $d_X(VV^+(G)) = n - 1$ and $\chi(G) = n$.

Since, $\chi(G) = n$; $G \cong K_n$ and $d_X(VV^+(G)) = n \neq n - 1$, a contradiction. So no graph exists.

If $G \in G_1$, then $d_X(VV^+(G)) + \chi(G) = 2n - 1$.

G_2 is a family of graphs on n vertices which contains a clique $K = K_{n-2}$, and the other vertices are adjacent to same vertex of K and degree of two vertices are α, β where $1 \leq \alpha, \beta \leq n-3$.

G_3 is a family of graph on n vertices which contains a clique $K = K_{n-2}$ and the other vertices of G is K_2 and the ends of K_2 are adjacent to same vertex of K , the degree of two vertices are α, β where $1 \leq \alpha, \beta \leq n-2$.

Theorem: 2.3 For a connected graph G , $d_X(VV^+) + \chi(G) = 2n-2$ if and only if G belongs to G_2 or G_3 .

Proof: If G belongs to G_2 or G_3 , then $d_X(VV^+) + \chi(G) = 2n-2$ is obvious. Conversely, assume that $d_X(VV^+) + \chi(G) = 2n-2$. This is possible only if

(a) $d_X(VV^+) = n; \chi(G) = n-2$ or

(b) $d_X(VV^+) = n-1; \chi(G) = n-1$ or

(c) $d_X(VV^+) = n-2; \chi(G) = n$.

Case: (a) $d_X(VV^+) = n; \chi(G) = n-2$

Since $\chi(G) = n-2$, G contains a clique $K = K_{n-2}$. Let $S = \{x, y\}$ be other vertices of G . $\langle S \rangle = K_2$ or $\overline{K_2}$.

Sub case: (i) $\langle S \rangle = \overline{K_2}$.

Since G is connected. Vertices of $\langle S \rangle$ are adjacent to vertices of K . If x and y are adjacent to the same vertex of K , we get $d_X(VV^+) = n$. We also get $d_X(VV^+) = n$, if x and y are adjacent to α and β vertices of K where $1 \leq \alpha, \beta \leq n-3$. $G \in G_2$.

Sub case: (ii) $\langle S \rangle = K_2$

G is connected, one vertex of K_2 is adjacent to a vertex of K , $d_X(VV^+) < n$ a contradiction.

If both ends of K_2 are adjacent to same vertex of K , we get $d_X(VV^+) = n$. The vertices x and y can be of adjacent to α and β $1 \leq \alpha, \beta \leq n-2$. Which is graph G_3 .

Case: (b) $d_X(VV^+) = n-1; \chi(G) = n-1$

Since $\chi(G) = n-1$, G contains a clique $K = K_{n-1}$. Let x be a vertex of G other than the vertices of K . Since G is connected, x should be adjacent to atleast one vertex of K . Hence, $d_X(VV^+(G)) = n \neq n-1$. A contradiction. Therefore, no such graph exists.

Case: (c) $d_X(VV^+) = n-2; \chi(G) = n$.

Since $\chi(G) = n, G \cong K_n$. But $d_X(VV^+(K_n)) = n \neq n-2$. A contradiction.

We define certain family of graphs

G_4 is the family of graphs on n vertices, which contain a clique $K = K_{n-3}$ and other vertices of G are adjacent to a vertex of K and the degree of the vertices are α, β, γ where $1 \leq \alpha, \beta, \gamma \leq n-4$.

G_5 is the family of graphs on n vertices, which contain a clique $K = K_{n-3}$ and other vertices of G forms a P_3 and all the vertices of P_3 are adjacent to same vertex of K . The degree of the vertices of P_3 are α, β, γ where $1 \leq \alpha, \gamma \leq n-3; 1 \leq \beta \leq n-2$.

G_6 is the family of graphs on n vertices, which contain a clique $K = K_{n-3}$ and other vertices of G form $K_1 \cup K_2$. All the vertices in $K_1 \cup K_2$ are adjacent to a vertex of K . The degree of these vertices are α, β, γ where $1 \leq \alpha \leq n-4; 1 \leq \beta, \gamma \leq n-3$.

G_7 is the family of graphs on n vertices, which contain a clique $K = K_{n-3}$ and other vertices of G form a K_3 and the vertices of K_3 are adjacent to different vertices of K .

G_8 is the family of graphs on n vertices, which contain a clique $K = K_{n-2}$ and the other two vertices are adjacent to different vertices of K .

G_9 is the family of graphs on n vertices, which contain a clique $K = K_{n-2}$ and the other vertices forms a K_2 in which the ends are adjacent to different vertices of K .

Theorem: 2.4 For a connected graph G , $d_X(VV^+) + \chi(G) = 2n - 3$ if and only if $G \in G_i$ $4 \leq i \leq 9$.

Proof: If $G \in G_i$ $4 \leq i \leq 9$, then clearly $d_X(VV^+) + \chi(G) = 2n - 3$. Conversely, assume that $d_X(VV^+) + \chi(G) = 2n - 3$. This is possible only if

- (a) $d_X(VV^+) = n - 3; \chi(G) = n$ or
- (b) $d_X(VV^+) = n - 2; \chi(G) = n - 1$ or
- (c) $d_X(VV^+) = n - 1; \chi(G) = n - 2$ or
- (d) $d_X(VV^+) = n; \chi(G) = n - 3$.

Case (a): $d_X(VV^+) = n - 3; \chi(G) = n$

Since, $\chi = n$ $G \cong K_n$. But $d_X(VV^+(K_n)) = n \neq n - 3$ a contradiction. Hence, no such graph exists.

Case: (b) $d_X(VV^+) = n - 2; \chi(G) = n - 1$

Since $\chi(G) = n - 1$, G contains a clique $K = K_{n-1}$. Let x be a vertex of G other than the vertices of K . Since G is connected, x is adjacent to atleast one vertex of K . $d_X(VV^+) = n \neq n - 2$, a contradiction. Hence, no such graphs exist.

Case: (c) $d_X(VV^+) = n - 1; \chi(G) = n - 2$

Since $\chi(G) = n - 2$, G contains a clique $K = K_{n-2}$. Let $S = \{x, y\}$ be vertices of G other than the vertices of K . Then $\langle S \rangle = K_2 \text{ or } \overline{K_2}$.

Sub case: (i) If $\langle S \rangle = \overline{K_2}$.

If vertices in $\langle S \rangle$ are adjacent to same vertex of K , we get a contradiction to $d_X(VV^+) = n - 1$. Hence no such graphs exists. If the vertices of $\langle S \rangle$ are adjacent to different vertices of K , we get $G \in G_8$.

Sub case: (ii) If $\langle S \rangle = K_2$.

If the end of K_2 are adjacent to a vertex of K , we get a contradiction to $d_X(VV^+) = n - 1$. Hence no such graph exists. If the vertices of $\langle S \rangle$ are adjacent to different vertices of K , we get $G \in G_9$.

Case: (d) $d_X(VV^+) = n$; $\chi(G) = n - 3$.

Since $\chi(G) = n - 3$, G contains a clique $K = K_{n-3}$. Let $S = \{x, y, z\}$ be vertices of G other than the vertices of K . Then $\langle S \rangle = K_3; \overline{K_3}; P_3; K_2 \cup K_1$.

Sub case: (i) $\langle S \rangle = K_3$

Since G is connected, a vertex of K_3 is adjacent to a vertex of K . If the vertices of $\langle S \rangle$ are adjacent to different vertices of K , we get $d_X(VV^+) = n$ and therefore, $G \in G_7$.

Sub case: (ii) $\langle S \rangle = \overline{K_3}$.

Since G is connected, all the vertices of $\overline{K_3}$ are adjacent to a vertex of K . If the vertices x, y, z are of degree α, β, γ where $1 \leq \alpha, \beta, \gamma \leq n - 4$, then $d_X(VV^+) = n$. Therefore, $G \in G_4$.

Sub case: (iii) $\langle S \rangle = P_3$

Since G is connected, a vertex of P_3 is adjacent to a vertex of K , a contradiction to $d_X(VV^+) = n$. If all the vertices of P_3 are adjacent to a vertex of K . If the degree of the vertices are α, β, γ where $1 \leq \alpha, \gamma \leq n - 3; 1 \leq \beta \leq n - 2$. Then, $d_X(VV^+) = n$. Hence, $G \in G_5$.

Sub case : (iv) $\langle S \rangle = K_2 \cup K_1$

If a vertex of K_2 and K_1 are adjacent to same vertex of K . If the degree of these vertices are α, β, γ where $1 \leq \alpha \leq n - 4; 1 \leq \beta, \gamma \leq n - 3$. Then, $d_X(VV^+) = n$. Hence $G \in G_6$.

Theorem: 2.5 For a connected graph G , $d(G^2) = d_X(VV^+(G))$.

Proof: Let V_1, V_2, \dots, V_n be domatic partition of G .² Each V_i is a dominating set in G^2 . Therefore, V_i is a X-dominating set in the bipartite graph $VV^+(G)$. Hence, V_1, V_2, \dots, V_n is a X-domatic partition of $VV^+(G)$. Hence, $d_X(VV^+(G)) \geq d(G^2)$.

Conversely, V_1, V_2, \dots, V_n be X-domatic partition of $VV^+(G)$. Then, V_1, V_2, \dots, V_n is a domatic partition in G^2 . Therefore, $d(G^2) \geq d_X(VV^+)$. Hence, $d(G^2) = d_X(VV^+(G))$.

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