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SOME FIXED POINT THEOREMS IN FUZZY METRIC SPACE

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ABSTRACT

The aim of the present paper is to extend the study of non compatible maps in fuzzy metric space by using the notion of R-weak commutativity of type (Ag) in fuzzy metric space. Simultaneously, we provide an answer in fuzzy metric space, perhaps maiden, to the problem of Rhoades ([18]). We also constructed few results on recently introduced concept of Property (E.A) by Aamri and Moutawakil [1].

Keywords: *R-weakly commuting of type (Ag), fuzzy metric space.*

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1. INTRODUCTION:

The concept of fuzzy sets was coined by Zadeh [22] in his seminal paper in 1965. There after the concept of a fuzzy metric space has been introduced and generalized in different ways by Deng [3], Erceg [5], Kaleva and Seikkala [9], Kramosil and Michalek [7], George and Veeramani [6] etc. It has also been shown that every metric induces a fuzzy metric (see Example1 below). In 1988, Grabiec [7] extended the fixed point theorems of Banach [2] and Edelstein [4] to fuzzy metric spaces. Following Grabiec [7] and Kramosil and Michalek [8], Mishra et al. [10] obtained common fixed point theorems for compatible maps and asymptotically commuting maps on fuzzy metric space. The utility of study of non compatible maps can be understood from the fact that while studying the common fixed point theorems of compatible maps we often require assumptions on the completeness of the space or the continuity of the mappings involved besides some contractive condition but the study of non compatible maps can be extended to the class of non expansive or Lipschitz type mapping pairs even without assuming continuity of the mappings involved or completeness of the space. Recently Aamri and Moutawakil [1] introduced the property (E.A) and thus generalized the concept of non compatible maps. In the present paper we prove common fixed point theorems for R-weakly commuting maps of type (Ag) in fuzzy metric space by using the concept of non compatibility or the property (E.A), however, without assuming either the completeness of the space or the continuity of the mappings involved. Simultaneously, we also find an answer in fuzzy metric space to the problem of Rhoades [18].In 1994 Pant [13] introduced the concept of R-weakly commuting maps in metric spaces. Later Pathak et al. [17] generalized this concept and gave the concept of R-weakly commuting maps of type (Ag). Vasuki [20] proved some common fixed point theorems for R-weakly commuting maps in the fuzzy metric space. Here we define the concept of R-weakly commuting maps of type (Ag) and the property (E.A) in the fuzzy metric space and then prove common fixed point theorems for a pair of selfmaps. Before we start we give some preliminaries.

Definition 1.1 ([22]) Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 1.2 ([20]) A binary operation $*: [0,1]\times[0,1] \to [0,1]$ is called a *continuous t –norm* if ([0,1], $*$) is an Abelian topological monoid with the unit 1 such that $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Definition 1.3 ([6]) The 3-tuple $(X,M,*)$ is called a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t -norm and M is a fuzzy set in $X^2 \times[0,\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$.

(i) $M(x, y,0) > 0$

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$

(iii) $M(x, y, t) = M(y,x, t)$

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(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$

(v) $M(x, y, \cdot) : [0,1) \to [0,1]$ is continuous.

Definition 1.4 ([10]) Let A and B maps from a fuzzy metric space $(X,M, *)$ into itself. The maps A and B are said to be *compatible* (or *asymptotically commuting*) if, for all $t > 0$, $\lim n\, M(ABx_n, BAx_n, t) = 1$.Whenever $\{x_n\}$ is a sequence in X such that $\lim n\to\infty\, A\, x_n = \lim n\to\infty B\, x_n = z$ for some $z \in X$.From the above definition it is inferred that A and B are non compatible maps from a fuzzy metric space $(X,M, *)$ into itself if $\lim n\to\infty\, A\, x_n = \lim n\to\infty B\, x_n = z$ for some $z \in X$, but either $\lim nM(AB\, x_n, B\, A\, x_n, t) = 1$ or the limit does not exists.

Definition 1.5 ([13]) Two maps A and S are called *R-weakly commuting at a point x* if $d(ASx,SAx) \leq Rd(Ax,Sx)$ for some $R > 0$. A and S are called *pointwise R-weakly commuting on X* if given x in X, there exists $R > 0$ such that

$$d(ASx,SAx) \leq Rd(Ax,Sx).$$

Definition: 1.6 ([20]) Two mappings A and S of a fuzzy metric space $(X,M, *)$ into itself are *R-weakly commuting* provided there exists some real number R such that

$$M(ASx,SAx,t) \geq M(Ax,Sx, t/R)$$

for each $x \in X$ and $t > 0$.

Definition: 1.7 ([17]) Two selfmappings A and S of a metric space (X,d) are called *R-weakly commuting of type* (Ag) if there exists some positive real number R such that

$$d(AAx,SAx) \leq Rd(Ax,Sx) \text{ for all } x \text{ in } X.$$

Definition: 1.8 ([1]) Let f and g be two selfmappings of a metric space (X,d). We say that f and g satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such thatlim $n\to\infty\, f\, x_n = \lim\, n\to\infty\, g\, x_n = t$ for some $t \in X$.

Definition: 1.9 Two mappings A and S of a fuzzy metric space $(X,M, *)$ into itself are *R-weakly commuting of type* (Ag) provided there exists some real number R such that $M(AAx,SAx, t) \geq M(Ax,Sx, t/R)$ for each $x\in X$ and $t > 0$.

Definition: 1.10 Let f and g be two self mappings of a fuzzy metric space $(X,M, *)$. We say that A and S satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that $\lim n\to\infty\, f\, x_n = \lim n\to\infty g\, x_n = t$ for some $t \in X$.

MAIN RESULTS:

Theorem: 1 *Let f and g be pointwise R-weakly commuting selfmaps of type* (Ag) *of a fuzzy metric space* $(X,M, *)$ *such that*

(i) $f\, X \subset g\, X$,

(ii) $M(f\, x, f\, y, t) > \min\{M(g\, x, g\, y, th),], M(f\, x, g\, x, th), [M(f\, y, g\, y, th)+ M(f\, y, g\, x, th)]/2, M(f\, x, g\, y, th)\}$, (1)

$0 \leq h < 1$, $t > 0$.

If f and g satisfy the property (*E.A*) *and the range of either of f or g is a complete subspace of X, then f and g have a unique common fixed point.*

Proof: Since f and g are satisfy the property (E.A), there exists a sequence $\{x_n\}$ in X such that

(i) $\lim_{n\to\infty} f\, x_n = \lim_{n\to\infty} g\, x_n = p$ for some p in X. Since $p\in f\, X$ and $f\, X \subset g\, X$, there exists some point u in X such that $p = gu$ where $p = \lim_{n\to\infty} g\, x_n$ If $f\, u \neq gu$, the inequality

$M(f\, x_n, f\, u, t) \geq \min\{M(g\, x_n, gu, th), M(fx_n, g\, x_n, th), [M(fy,gu,th)+M(fu,gx_n,th)]/2, M(fx_n, gu, th)\}$on letting $n\to\infty$ yields

$$M(gu, f\, u, t) \geq \min\{M(gu, gu, th), M(gu, gu, th), [M(fu,gu,th)+M(fu, gu, th)]/2, M(gu, gu, th)\}$$

$$= M(gu, f\, u, th)$$

Hence $fu=gu$. Since f and g are R-weak commutating of type (Ag), there exists $R > 0$ such that

$M(ffu, gfu, t) \geq M(fu, gu, t/R) = 1$, that is, $ffu = gfu$ and $ffu = fgu = gfu = ggu$. If $fu \neq ffu$, using (ii), we get

$M(fu, ffu, t) > \min\{M(gu, gfu, th), M(fu, gu, th), [M(ffu, gfu, th)+M(ffu, gfu, th)]/2, M(gu, ffu, th)\}$

$= M(fu, ffu, th)$,

a contradiction. Hence, $fu = ffu$ and $fu = ffu = fgu = gfu = ggu$. Hence fu is a common fixed point of f and g . The case when fX is a complete subspace of X is similar to the above case since $fX \subset gX$. Hence we have the theorem.

Theorem: 2 *Let f and g be non compatible pointwise R-weakly commuting selfmaps of type* (Ag) *of a fuzzy metric space* $(X,M,*)$ *such that*

(i) $fX \subset gX$,

(ii) $M(fx, fy, t) > \min\{M(gx, gy, th), M(fx, gx, th), [M(fy, gy, th) + M(fy, gx, th)]/2, M(fx, gy, th)\}$ *(2)*

$0 \leq h < 1$, $t > 0$.

If the range of f or g is a complete subspace of X, then f and g have a unique common fixed point and the fixed point is the point of discontinuity.

Proof: Since f and g are non compatible maps, there exists a sequence $\{x_n\}$ in X such that (2)

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some p in X, but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exist. Since $p \in fX$ and $fX \subset gX$, there exists some point u in X such that $p = gu$ where $p = \lim_{n\to\infty} gx_n$. If $fu \neq gu$,

the inequality

$M(fx_n, fu, t) > \min\{M(gx_n, gu, th), M(fx_n, gx_n, th), [M(fu, gu, th)+ M(fu, gx_n, th)]/2, M(fx_n, gu, th)\}$ on letting $n\to\infty$ yields

$M(gu, fu, t) \geq M(gu, gu, th)$.

Hence $fu = gu$. Since f and g are R-weak commutating of type (Ag), there exists $R > 0$ such that

$M(ffu, gfu, t) \geq M(fu, gu, t/R) = 1$,

that is, $fu = gfu$ and $ffu = fgu = gfu = ggu$. If $fu \neq ffu$, using (iii), we get

$M(fu, ffu, t) > \min\{M(gu, gfu, th), M(fu, gu, th), [M(ffu,gfu,th)+M(ffu, gfu, th)]/2, M(gu, ffu, th)\}$

$= M(fu, ffu, th)$,

a contradiction. Hence $fu = ffu$ and $fu = ffu = fgu = gfu = ggu$. Hence fu is a common fixed point of f and g . The case when fX is a complete subspace of X is similar to the above case since $fX \subset gX$. We now show that f and g are discontinuous at the common fixed point $p = fu = gu$. If possible, suppose f is continuous. Then considering the sequence $\{x_n\}$ of (2) we get $\lim n\, ffx_n = fp = p$. R-weak commutativity of type (Ag) implies that

$M(ffx_n, gfxn, t) \geq M(fx_n, gxn, t/R) = 1$ which on letting $n \to\infty$this yields $\lim_{n\to\infty} gfx_n = fp = p$. This, in turn, yields

$\lim n \to\infty M(fgx_n, gfx_n, t) = 1$.

This contradicts the fact that

$\lim_{n\to\infty} M(fgx_n, gfx_n, t)$ is either nonzero or nonexistent for the sequence $\{x_n\}$ of (1). Hence f is discontinuous at the fixed point. Next, suppose that g is continuous.

Then for the sequence $\{x_n\}$ of (1), we get $\lim n\, gfx_n = gp = p$ and $\lim n\, ggx_n = gp = p$. In view of these limits, the inequality

$$M(f x_n, f g x_n, t) \geq \min\{M(g x_n, g g x_n, th), M(f x_n, g g x_n, th), M(f g x_n, g g x_n, th)+M(f g x_n, g x_n, th)]/2, M(f x_n, g g x_n, th)\}$$

$$= M(f x_n, f g x_n, th)$$

yields a contradiction unless $\lim_{n\to\infty} f g x_n = f p = g p$. But $\lim_{n\to\infty} f g x_n = g p$ and $\lim_{n\to\infty} g f x_n = g p$ contradicts the fact that $\lim_{n\to\infty} M(f g x_n, g f x_n)$ is either nonzero or nonexistent. Thus both f and g are discontinuous at their common fixed point. Hence we have the theorem.

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