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ON A MAP OF XIA ET.AL AND SOME FIXED POINT THEOREMS FOR A CLASS OF CONTRACTIVE MAPPINGS IN G-DISLOCATED METRIC SPACES

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ABSTRACT

In Xia et.al [Common fixed points for two self-mappings on symmetric sets and complete metric spaces, Advances in Mathematics, vol. 36, no. 4, pp. 415–420, 2007.], the following map is introduced: Let $R_+ = [0,\infty)$, *let* $T_1 : R_+^2 \to R_+$, *and* $T_2 : R_+^3 \to R_+$, *satisfy the following: (i) if* $w \le T_1(u,v)$, *then there exist* $c \in (0,1)$, *such that* $w \le c\max\{u,v\}$; *(ii) if* $w \le T_2(u,v,r)$, *then there exists* $c \in (0,1)$, *such that* $w \le c\max\{u,v,r\}$. *The authors use these maps to prove the following fixed point result: Let* (X,d) *be a complete metric space, and let* $f,g : X \to X$ *be continuous mappings, and for all* $x,y \in X$ *such that*

$$d(f(x),g(y)) \le T_1\big(d(x,f(x)),d(y,g(y))\big), \text{ or } d(f(x),g(y)) \le T_2\big(d(x,y),d(x,f(x)),d(y,g(y))\big),$$

then, f,g *have a unique common fixed point. In the present paper we define an analogous map in the setting of G-dislocated metric spaces and use it to obtain fixed point theorems.*

AMS Subject Classification: *47H10, 54H25.*

I. INTRODUCTION

Fixed point theory, a pivotal branch of analysis, has several applications. One of the most celebrated fixed point theorems is due to Banach [1], and several generalizations of it have appeared in the literature, see [2-9] for examples. In this paper we extend the map of Xia *et. al* [Common fixed points for two self-mappings on symmetric sets and complete metric spaces, Advances in Mathematics, vol. 36, no. 4, pp. 415–420, 2007], and use it to obtain fixed point theorems in the setting of G-dislocated metric spaces.

II. BASIC NOTIONS AND NOTATIONS

In analogy to Zeyada *et.al* [A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces, The Arabian Journal for Science and Engineering Section A, vol. 31, no. 1, pp. 111–114, 2006], we introduce the following .

Definition 1: Let X be a non-empty set. We will say $G : X \times X \times X \to R_+$ is a distance function if for all $x,y,z,w \in X$

(a) $G(x,y,z) \ge 0$
(b) $G(x,y,z) = 0 \Leftrightarrow x = y = z$
(c) $G(x,y,z) \le G(w,y,z) + G(x,w,z) + G(x,y,w)$

Here $(X,G)_q$ will denote a G-quasi-metric space.

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Definition: 2 Let X be a non-empty set, and $G: X \times X \times X \to R_+$ be a distance function . If for all $x, y, z, w \in X$

(a) $G(x, y, z) \geq 0$

(b) $G(x, y, z) = G(y, x, z) = \ldots = 0 \Rightarrow x = y = z$

(c) $G(x, y, z) \leq G(w, y, z) + G(x, w, z) + G(x, y, w)$

Here $(X, G)_{dq}$ will denote a G-dislocated quasi-metric space.

Definition: 3 Let X be a non-empty set, and let $G: X \times X \times X \to R_+$ be a distance function. If for all $x, y, z, w \in X$

(a) $G(x, y, z) \geq 0$

(b) $G(x, y, z) = G(y, x, z) = \ldots = 0 \Rightarrow x = y = z$

(c) $G(x, y, z) = G(y, x, z) = \ldots$

(d) $G(x, y, z) \leq G(w, y, z) + G(x, w, z) + G(x, y, w)$

Here $(X, G)_d$ will denote G-dislocated metric space

Definition: 4 A sequence $\{x_n\}$ in $(X, G)_{dq}$ will be called a G-dq-Cauchy sequence if for a given $\varepsilon > 0$, there exist $n_0 \in N$ such that $G(x_m, x_l, x_n) < \varepsilon$, or $G(x_l, x_m, x_n) < \varepsilon$, or $\ldots$, that is, $\min\{G(x_m, x_n, x_l), G(x_n, x_m, x_l), \ldots\} < \varepsilon$ for all $m, n, l \geq n_0$.

Definition: 5 A sequence $\{x_n\}$ in the G-metric space (X, G) will be called G-convergent to some $x \in X$ provided that $\lim G(x_n, x_m, x) = \lim G(x_m, x_n, x) = \ldots = 0$. We will call x, the G-limit of $\{x_n\}$.

Definition: 6 We will say the G-metric space (X, G) is G-complete if every G-Cauchy sequence in it converges with respect to $x \in X$.

Lemma 7: Every convergent sequence in (X, G) is a Cauchy sequence

Proof: Let $\{x_n\}$ be a sequence which converges to some $x \in X$. Suppose $\varepsilon > 0$ is arbitrary, then there exists $n_0 \in N$ with $G(x_m, x_n, x) < \frac{\varepsilon}{3}$ for all $n, m \geq n_0$. So for $n, m, l \geq 0$,

We obtain $G(x_m, x_n, x) + G(x_m, x, x_l) + G(x, x_n, x_l) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Hence $\{x_n\}$ is Cauchy.

Lemma: 8 Limit in $(X, G)_d$ are unique.

Proof: Suppose x, y, z are limits of the sequence $\{x_n\}$. Then $x_n \to x, x_n \to y, x_n \to z$ as $n \to \infty$. By triangle inequality,

$G(x, y, z) \leq G(x, x_n, x_n) + G(x_n, y, x_n) + G(x_n, x_n, z)$. If we take the limit as $n \to \infty$, this implies that

$G(x, y, z) \leq G(x, x, x) + G(y, y, y) + G(z, z, z)$. Since G is symmetric in the variables we see that

$\ldots = G(x, y, z) = G(x, y, z) \leq G(x, x, x) + G(y, y, y) + G(z, z, z)$.

Obviously,

$|G(y, x, z) - G(x, y, z)| \leq 0, |G(z, y, x) - G(x, y, z)| \leq 0, \ldots$, etc. So

$\ldots = G(z, y, x) = G(y, x, z) = G(x, y, z) \leq G(x, x, x) + G(y, y, y) + G(z, z, z)$. Also if we go in the limit of

$G(x,y,z) \le G(x,x_n,x_n) + G(x_n,y,x_n) + G(x_n,x_n,z)$, we see that $G(x,y,z) = 0$. So obviously,

$\ldots = G(z,y,x) = G(y,x,z) = G(x,y,z) = 0$. In particular $x = y = z$.

Example: 9 Let $X = R_+$. Define $G(x,y,z) = |x-y| + |x-z| + |y-z| = d(x,y) + d(x,z) + d(y,z)$, then $(X,G)_d$ is a metric space. If $\{x_n\}$ is an arbitrary sequence in X; suppose there exists a positive integer N, such that $k > N$ gives $|x_k - a| < \frac{\varepsilon}{6}$, then for any $m,n,l > N$, we see that

$$G(x_n,x_m,x_l) = d(x_n,x_m) + d(x_n,x_l) + d(x_m,x_l) \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Since $d(x_n,x_m) = |x_n - x_m| \le |x_n - a| + |x_m - a| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}$. So $\{x_n\}$ is a Cauchy sequence in X. Also as $n \to \infty$, $x_n \to a \in X$. Hence every Cauchy sequence in X is convergent with respect to G. It follows that $(X,G)_d$ is a complete metric space.

III. MAIN RESULTS

Before stating the main result, we introduce some definitions and a lemma that will be useful in the sequel.

Definition: 10 [R. Chen, Fixed Point Theory and Applications, National Defense Industry press, 2012]: There exist $\phi(t)$ that satisfy condition ϕ', if one lets $\phi:[0,\infty) \to [0,\infty)$ be a nondecreasing and nonnegative, then $\lim \phi_n(t) = 0$ for a given $t > 0$

Lemma: 11[R. Chen, Fixed Point Theory and Applications, National Defense Industry press, 2012]: If ϕ satisfy ϕ', then $\phi(t) < t$, for a given $t > 0$

In analogy to Xia *et.al* [Common fixed points for two self-mappings on symmetric sets and complete metric spaces, Advances in Mathematics, vol. 36, no. 4, pp. 415–420, 2007] we have the following lemma

Lemma: 12 Let $F: R_+^4 \to R_+$ be a mapping, and suppose it satisfies the condition ϕ', for all $u,v \ge 0$, such that, $u \le F(v,v,v,u)$, or $u \le F(v,v,u,v)$, or $u \le F(v,u,v,v)$, or $u \le F(u,v,v,v)$, then $u \le \phi(v)$

Our main result is as follows.

Theorem: 13 Let $(X,G)_d$ be a complete metric space, and let $f,g,h: X \times X \to R_+$ be mappings such that (i) either f,g,h is continuous, and (ii) there exist F satisfying the condition ϕ', for all $x,y,z,u,v,w \in X$, such that $G(f(x,u),g(y,v),h(z,w)) \le F(G(x,y,z),G(x,u,f(x,u)),G(y,v,g(y,v)),G(z,w,h(z,w)))$, then f,g,h have a unique fixed point.

Proof:
Put $x_n = u_n = (fgh)^n(x_0,u_0) = fgh(x_{n-2},u_{n-2})$, $y_n = v_n = g(fgh)^{n-1}(x_0,u_0)$, $z_n = w_n = h(fgh)^{n-2}(x_0,u_0)$, for $n = 2,3,4,5,\ldots$

Obviously,
$y_n = g(x_{n-1},u_{n-1})$, $z_n = h(x_{n-2},u_{n-2})$, $fg(z_n,w_n) = y_n$, $fg(y_n,v_n) = x_n$,

$z_{n+2} = h(x_n,u_n) = hfg(y_n,v_n)$, $y_{n+1} = g(x_n,u_n) = gfg(y_n,v_n)$.

Let $x = u = gh(x_n, u_n)$, $y = z = x_n$, $v = w = u_n$, then by the condition, we have

$$G(x_{n+2}, y_{n+1}, z_{n+2}) \le F\begin{bmatrix} G(gh(x_n,u_n), x_n, x_n), G(gh(x_n,u_n), gh(x_n,u_n), fgh(x_n,u_n), \\ G(x_n,u_n,g(x_n,u_n)), G(x_n,u_n,h(x_n,u_n)) \end{bmatrix}$$

By Lemma 12,

$$G(x_{n+2}, y_{n+1}, z_{n+2}) \le \phi\big(G(x_n, x_n, gh(x_n,u_n)\big)$$

Also we notice that

$$G(x_n, x_n, gh(x_n,u_n)) \le \phi\big(G(x_n, x_n, y_n)\big)$$

By Lemma 11

$$G(x_{n+2}, y_{n+1}, z_{n+2}) \le \phi^2\big(G(x_n, x_n, y_n)\big)$$

By induction, we notice that

$$G(x_{n+2}, y_{n+1}, z_{n+2}) \le \varphi^{2n}\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right)$$

Similarly, we obtain

$$G(y_{n+1}, x_{n+1}, z_{n+2}) \le \varphi^{2n-1}\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right)$$
$$G(y_{n+1}, z_{n+2}, x_n) \le \varphi^{2n-2}\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right)$$

If $n \ge 2$, we obtain

$$\begin{aligned} G(x_{n+2}, x_{n+1}, x_n) &\le G(x_{n+2}, y_{n+1}, z_{n+2}) + G(y_{n+1}, x_{n+1}, z_{n+2}) + G(y_{n+1}, z_{n+2}, x_n) \\ &\le \varphi^{2n}\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right) + \varphi^{2n-1}\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right) \\ &+\varphi^{2n-2}\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right) \\ &\le 3\varphi^{2n-2}\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right) \end{aligned}$$

Now we observe by the condition ϕ' for $n, m, l \in N$ such that $l > m > n$, we have

$$\begin{aligned} G(x_n, x_m, x_{n+m+l}) &\le G(x_{n+2}, x_{n+1}, x_n) + G(x_{n+3}, x_{n+2}, x_{n+1}) + \ldots + G(x_{n+m+l-2}, x_{n+m+l-1}, x_{n+m+l}) \\ &\le 3\varphi^{2n-2}\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right) + 3\varphi^{2(n+3)-2}\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right) + \ldots \\ &+3\varphi^{2(n+m+l-2\_-2}\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right) \\ &\le 3\sum_{i=2n-2}^{2(n+m+l-2)-2} \varphi^i\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right) \\ &\le 3\sum_{i=2n-2}^{\infty} \varphi^i\left(G(x_2, g(x_0,u_0), h(x_0,u_0)\right) \to 0 \end{aligned}$$

It follows that $G(x_n, x_m, x_l) \to 0$ as $m, n, l \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in X.

Since $(X,G)_d$ is complete, $\{x_n\}$ converges to some $x_* \in X$. In a similar way, $\{u_n\}$ converges to $u_* = x_* \in X$, say. If h, g are both continuous, the by their continuity, we see that as. $n \to \infty$ $z_{n+2} = z_* = h(x_*, u_*)$, and $y_{n+1} = y_* = g(x_*, u_*)$. So as $n \to \infty$, $G(x_{n+2}, y_{n+1}, z_{n+2}) \le G(x_*, y_*, z_*) \le 0$.
So, (x_*, u_*) is a fixed point of h, g.

By the given condition (note : $u_* = x_* \in X$, say), then,

$$G(f(x_*,u_*), g(x_*,u_*), h(x_*,u_*)) \le F(0, G(x_*, u_*, f(x_*,u_*), 0, 0$$

It follows that $G(x_*,u_*, f(x_*,u_*)) \le \phi(0) = 0$, thus, $f(x_*,u_*) = x_* = u_*$. So (x_*,u_*) is a common fixed point of f,g,h.

Regarding uniqueness, if y_* is another common fixed point of f,g,h, then by the given condition,

$$\begin{aligned} G(x_*,x_*,y_*) &= G(f(x_*,x_*), g(x_*,x_*), h(y_*,y_*)) \\ &\le F(G(x_*,x_*,y_*), G(x_*,x_*, f(x_*,x_*)), G(x_*,x_*, g(x_*,x_*)), G(y_*,y_*, f(y_*,y_*))) \\ &= F(G(x_*,x_*,y_*),0,0,0) \end{aligned}$$

Since $G(x_*,x_*,y_*) \le \phi(0) = 0$, it follows that $x_* = y_*$, and uniqueness follows, completing the proof.

IV. CONCLUDING REMARKS

Matthews [Metric domains for completeness. Technical Report 76 [Ph.D. thesis], Department of Computer Science, University of Warwick, Coventry, UK, 1986] generalized Banach contraction mapping theorem in dislocated metric space that is a wider space than metric space. In this paper, we established common fixed point theorems for a class of contractive mappings in the setting of G-dislocated metric spaces.

Let $\{x_n\}$ be a sequence of points, we will say that x_* is the condensation point of $\{x_n\}$, if there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_*$.

By way of this definition, we have the following theorem

Theorem 14: Let $(X,G)_d$ be a complete metric space, and let $f,g,h: X \times X \to R_+$ be continuous mappings, if

(a) There exists F satisfying the condition ϕ', for all $x,y,z,u,v,w \in X$ such that
$$G(f(x,u), g(y,v), h(z,w)) \le F(G(x,y,z), G(x,u, f(x,u)), G(y,v, g(y,v)), G(z,w,h(z,w)))$$

(b) There exists $(x_0,y_0) \in X \times X$, such that $\{(fgh)^n(x_0,y_0)\}$ have a condensation point.

Then f,g,h have a unique common fixed point

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