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# GENERALIZATION OF PAWLAK APPROXIMATION SPACE

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***ABSTRACT***

*This paper can be viewed as a generalization of Pawlak approximation space using general topological structure. Our approach depends on a general topology generated by binary relation. The introduced technique is useful because the concepts and the properties of the generated topology are applied on rough set theory and this open the way for more topological applications in rough context. Several properties and examples are provided.*

## 1. INTRODUCTION

Since Z. Pawlak [9-11] introduced the concept of approximation space in 1982, many authors have been introduced several generalizations to Pawlak space in order to destroying the constraints of the equivalence relation (see: [1], [3-8], [12-22]). Our approach depends on the concept of after and fore set. Let the non empty set $U$ be a finite set and $R$ be a binary relation on $U$ [2],then the after (resp. the fore) set of element $x \in U$ is the class $xR = \{y \in U : xRy\}$ (resp. $Rx = \{y \in U : yRx\}$). The pair $\mathscr{A} = (U, R)$, where $R$ is an equivalence relation, is called Pawlak approximation space [9] in briefly "PAS". If $R$ be a binary general relation, then the pair $\mathscr{A} = (U, R)$ is called "a generalized approximation space" in briefly "GAS".

## 2. GENERALIZATION OF PAS

**Definition: 2.1** Let $\mathscr{A} = (U, R)$ be a GAS and $X \subseteq U$. Then $X$ is called "after composed" (resp. "after-*c* composed") set if $X$ contains all after (resp. fore) sets for all elements of its i.e., $\forall\ x \in X,\ xR \subseteq X$ (resp. $Rx \subseteq X$).

The class of all after (resp. after-*c*) composed sets in $\mathscr{A}$ is defined by the class
$\tau_R = \{X \subseteq U : \forall\ x \in X,\ xR \subseteq X\}$ (resp $\tau_R^* = \{X \subseteq U : \forall\ x \in X,\ Rx \subseteq X\}$).

**Remark: 2.1** In the GAS $\mathscr{A} = (U, R)$ if $R$ is an equivalence relation, then the GAS becomes Pawlak approximation space. Moreover, the class $\tau_R$ in this case is coinciding with the class of composed sets in PAS. Thus PAS can be considered as a special case of a GAS which given in Definition 2.1.

**Proposition: 2.1** Let $\mathscr{A} = (U, R)$ be a GAS. Then the class $\tau_R$ (resp. $\tau_R^*$) in $\mathscr{A}$ forms a topology on $U$.

**Proof:** We shall prove that $\tau_R$ is a topology on $U$ and similarly for $\tau_R^*$.

Clearly $U$ and $\phi$ are after-composed sets, then $U, \phi \in \tau_R$.

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Let $A, B \in \tau_R$, and let $x \in (A \cap B)$. Then $x \in A$ and $x \in B$, which implies that $xR \subseteq A$ and $xR \subseteq B$. Thus $x\,R \subseteq A \cap B$, and then $A \cap B \in \tau_R$.

Now, let $A_i \in \tau_R$, $\forall\ i \in I$. Then $x \in \bigcup_{i \in I} A_i$ imply that $\exists\ i_0 \in I$ such that $x \in A_{i_0} \in \bigcup_{i \in I} A_i$, and hence $xR \subseteq A_{i_0} \subseteq \bigcup_{i \in I} A_i$, that is $\bigcup_{i \in I} A_i \in \tau_R$.

Thus $\tau_R$ is a topology on *U*.

**Theorem: 2.1** Let $\mathscr{A} = (U, R)$ be a GAS. Then $\tau_R$ is the complement topology of $\tau_R^*$ and vice versa.

**Proof:** We must prove that: For any $X \subseteq U$, $X \in \tau_R$ iff $X^c \in \tau_R^*$.

First, let $X \in \tau_R$. Then $\forall\ x \in X$, $xR \subseteq X$ (1)

Now, let $z \in X^c$, then the fore set of $z$ is given by:

$$Rz = \{a \in U\ :\ aRz\}$$

Then there are two different cases are:

**Case: 1** If $Rz \cap X \neq \phi$. Then $\exists b \in X$ and $b \in Rz$ which implies that $\exists b \in X$ and $bRz$ such that $z \in X^c$. Thus $\exists b \in X$ and $z \in bR$ such that $z \notin X$ which is a contradiction to assumption (1). Thus the following case is true:

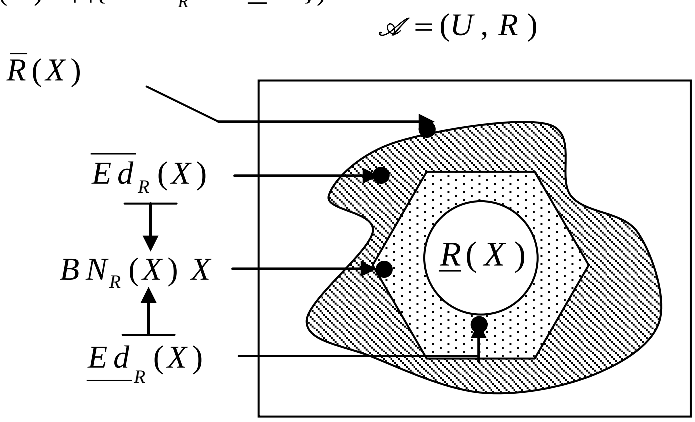
**Case: 2** If $Rz \subseteq X^c$, then $X^c \in \tau_R^*$.

By the same way we can prove that: If $X^c \in \tau_R^*$, then $X \in \tau_R$.

Thus $\tau_R$ is the complement topology of $\tau_R^*$.

**Definition: 2.2** Let $\mathscr{A} = (U, R)$ be a GAS and $X \subseteq U$. Then *X* is called "*R*-definable" (exact) set in $\mathscr{A}$ if *X* and $X^c$ are after composed set. Otherwise, *X* is called "*R*-undefinable" (rough) set.

The lower (resp. the upper) approximation of any subset $X \subseteq U$ is given by
$\underline{R}(X) = \bigcup\{G \in \tau_R : G \subseteq X\}$ (resp. $\overline{R}(X) = \bigcap\{H \in \tau_R^* :\ X \subseteq H\}$).




The boundary set of *X* is given by:
$BN_R(X) = \overline{R}(X) - \underline{R}(X)$.
The internal edge of *X* is given by:
$\underline{Ed}_R(X) = X - \underline{R}(X)$
The external edge of *X* is given by:
$\overline{Ed}_R(X) = \overline{R}(X) - X$.

**Remarks: 2.2**

(i) It is easy to notice that the lower $\underline{R}(X)$ (resp. the upper $\overline{R}(X)$) approximation of a subset $X$ in GAS $\mathscr{A} = (U, R)$ is exactly the interior $\operatorname{int}(X)$ (resp. the closure $cl(X)$) of $X$ in the topology $\tau_R$ .

(ii) It is clear that $BN_R(X) = \underline{Ed}_R(X) \bigcup \overline{Ed}_R(X)$.

(iii) The best lower (resp. upper) approximation of any subset is given when the internal (resp. the external) edge of its tends to empty set i. e., if $\underline{Ed}_R(X) = \varphi$ (resp $\overline{E.d}_R(X) = \varphi$).

(iv) $\underline{R}(X)$ (resp. $\overline{R}(X)$) is the largest after (resp. smallest after-*c*) composed set contained in $X$ (resp. contain $X$).

(v) $X$ is after (resp. after-*c*) composed set iff $\underline{R}(X) = X$ (resp $\overline{R}(X) = X$).

**Proposition: 2.2** Let $\mathscr{A} = (U, R)$ be a GAS and $X \subseteq U$. Then:

(i) $X$ is exact set if and only if $\underline{R}(X) = \overline{R}(X) = X$ .

(ii) $X$ is rough set if and only if $\underline{R}(X) \neq \overline{R}(X) \neq X$ .

**Proof:** Obvious.

**Corollary: 2.1** Let $\mathscr{A} = (U, R)$ be a GAS and $X \subseteq U$ . Then:

(i) $X$ is exact set if and only if $BN_R(X) = \varphi$ .

(ii) $X$ is rough set if and only if $BN_R(X) \neq \varphi$ .

## 3. PROPERTIES OF APPROXIMATIONS

**Proposition: 3.1** Let $\mathscr{A} = (U, R)$ be a GAS and $X, Y \subseteq U$ . Then:

(i) $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$.

(ii) $\underline{R}(U) = \overline{R}(U) = U$ and $\underline{R}(\varphi) = \overline{R}(\varphi) = \varphi$ .

(iii) If $X \subseteq Y$ , then $\underline{R}(X) \subseteq \underline{R}(Y)$ and $\overline{R}(X) \subseteq \overline{R}(Y)$ .

**Proof:**

(i) Obvious.

(ii) Obvious, since $U$ and $\phi$ are exact sets.

(iii) Let $X \subseteq Y$ and $x \in \underline{R}(X)$, then there exist $G \in \tau_R$ such that $x \in G \subseteq X$ . Hence $x \in G \subseteq Y$ such that $G \in \tau_R$ and then $x \in \underline{R}(Y)$ , which implies that $\underline{R}(X) \subseteq \underline{R}(Y)$ .

Similarly $\overline{R}(X) \subseteq \overline{R}(Y)$ .

**Proposition: 3.2** Let $\mathscr{A} = (U, R)$ be a GAS and $X \subseteq U$ . Then:

(i) $\underline{R}(-X) = -\overline{R}(X)$ and $\overline{R}(-X) = -\underline{R}(X)$ , where $-X$ is the complement of $X$.

(ii) $\underline{R}(\underline{R}(X)) = \underline{R}(X)$ and $\overline{R}(\overline{R}(X)) = \overline{R}(X)$ .

(iii) $\underline{R}(\underline{R}(X)) \subseteq \overline{R}(\underline{R}(X))$ and $\underline{R}(\overline{R}(X)) \subseteq \overline{R}(X)$ .

**Proof:**

(i) By Definition 2.2 we have: $\underline{R}(-X) = \bigcup\{G \in \tau_R : G \subseteq (-X)\}$, and since $G \in \tau_R$ , then $U - G \in \tau_R^*$ . Thus

$\underline{R}(-X) = \bigcup\{(U-(U-G)) \in \tau_R : (U-(U-G)) \subseteq (U-X)\}$

$= U - \bigcap\{(U-G) \in \tau_R^* : X \subseteq (U-G)\}$

$= U - \overline{R}(X) = -\overline{R}(X)$ .

Similarly $\overline{R}(-X) = -\underline{R}(X)$ .

(ii) Since $\underline{R}(X)$ is after composed set, then $\underline{R}(\underline{R}(X)) = \underline{R}(X)$.
Similarly $\overline{R}(\overline{R}(X)) = \overline{R}(X)$.

(iii) By Proposition 3.1, $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)\ \forall X \subseteq U$ , then
$\underline{R}(\underline{R}(X)) \subseteq \underline{R}(X) \subseteq \overline{R}(\underline{R}(X))$, and
$\underline{R}(\overline{R}(X)) \subseteq \overline{R}(X) \subseteq \overline{R}(\overline{R}(X))$.

**Proposition: 3.3** Let $\mathscr{A} = (U, R)$ be a GAS and $X, Y \subseteq U$. Then:

(i) $\underline{R}(X) \cap \underline{R}(Y) = \underline{R}(X \cap Y)$.

(ii) $\overline{R}(X) \cup \overline{R}(Y) = \overline{R}(X \cup Y)$.

(iii) $\underline{R}(X) \cup \underline{R}(Y) \subseteq \underline{R}(X \cup Y)$.

(iv) $\overline{R}(X) \cap \overline{R}(Y) \supseteq \overline{R}(X \cap Y)$.

**Proof:**

(i) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, then
$\underline{R}(X \cap Y) \subseteq \underline{R}(X)$ and $\underline{R}(X \cap Y) \subseteq \underline{R}(Y)$, which implies that

$$\underline{R}(X \cap Y) \subseteq \underline{R}(X) \cap \underline{R}(Y) \tag{1}$$

Now, since $\underline{R}(X)\ and\ \underline{R}(Y) \in \tau_R$, then $\underline{R}(X) \cap \underline{R}(Y) \in \tau_R$, and hence $\underline{R}(X) \cap \underline{R}(Y)$ is an after composed set contained in $X \cap Y$ .Thus

$$\underline{R}(X) \cap \underline{R}(Y) \subseteq \underline{R}(X \cap Y) \tag{2}$$

(1), (2) implies that $\underline{R}(X) \cap \underline{R}(Y) = \underline{R}(X \cap Y)$.

(ii) By the same way as in (i).

(iii) Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, then $\underline{R}(X) \subseteq \underline{R}(X \cup Y)$ and $\underline{R}(Y) \subseteq \underline{R}(X \cup Y)$ .Thus $\underline{R}(X) \cup \underline{R}(Y) \subseteq \underline{R}(X \cup Y)$.

(iv) By the same way as in (iii).

**Proposition: 3.4** Let $\mathscr{A} = (U, R)$ be a GAS and $X, Y \subseteq U$. Then the approximations satisfies the following properties:

(i) $-(\underline{R}(X) \cup \underline{R}(Y)) = \overline{R}(-X) \cap \overline{R}(-Y)$.

(ii) $-(\underline{R}(X) \cup \overline{R}(Y) = \overline{R}(-X) \cap \underline{R}(-Y)$.

(iii) $-(\overline{R}(X) \cup \underline{R}(Y)) = \underline{R}(-X) \cap \overline{R}(-Y)$.

(iv) $-(\overline{R}(X) \cup \overline{R}(Y)) = \underline{R}(-X) \cap \underline{R}(-Y)$.

(v) $-(\underline{R}(X) \cap \underline{R}(Y)) = \overline{R}(-X) \cup \overline{R}(-Y)$.

(vi) $-(\underline{R}(X) \cap \overline{R}(Y)) = \overline{R}(-X) \cup \underline{R}(-Y)$.

(vii) $-(\overline{R}(X) \cap \underline{R}(Y)) = \underline{R}(-X) \cup \overline{R}(-Y)$.

(viii) $-(\overline{R}(X) \cap \overline{R}(Y)) = \underline{R}(-X) \cup \underline{R}(-Y)$.

**Proof:** By using the properties of the approximations, the proof is obvious.

**Proposition: 3.5** Let $\mathscr{A} = (U, R)$ be a GAS and $X \subseteq U$. Then:

(i) $\overline{R}(X) \cup \overline{R}(-X) = U$.

(ii) $\overline{R}(X) \cup \underline{R}(-X) = U$.

(iii) $\underline{R}(X) \cup \overline{R}(-X) = U$.

(iv) $\underline{R}(X) \cup \underline{R}(-X) = -BN_R(X)$.

(v) $\overline{R}(X) \cap \overline{R}(-X) = BN_R(X)$.

(vi) $\overline{R}(X)\bigcap \underline{R}(-X)=\varphi$.

(vii) $\underline{R}(X)\bigcap \overline{R}(-X)=\varphi$ .

(viii) $\underline{R}(X)\bigcap \underline{R}(-X)=\varphi$.

**Proof:** By using the properties of the approximations, the proof is obvious.

**Remark: 3.1** The above propositions can be considered as a one of the differences between our generalization and the others generalizations such as Yao space [18, 19], supra approximation space [3] and ([1], [4-8], [12-22]).

Although they used general binary relation but they added some conditions to satisfy the properties of Pawlak space.

**Definition: 3.1** Let $\mathscr{A}=(U,\ R)$ be a GAS. Then the subset $X\subseteq U$ is said to be:

(i) Totally-definable or "*R*-definable" (exact) set if $X=\underline{R}(X)=\overline{R}(X)$ (i.e., $BN_R(X)=\phi$).

(ii) Internally-definable set if $X=\underline{R}(X)$, such that $\underline{Ed}_R(X)=\phi$.

(iii) Externally-definable set if $X=\overline{R}(X)$, such that $\overline{Ed}_R(X)=\phi$.

(iv) Undefinable or "*R*-undefinable" (rough) set if $X\neq\underline{R}(X)\neq\overline{R}(X)$ (i.e., $BN_R(X)\neq\varphi$).

**Remark: 3.2** In the above definition $BN_R(X)\neq\varphi$ in cases (ii), (iii).

**Lemma: 3.1** Let $\mathscr{A}=(U,\ R)$ be a GAS and $X\subseteq U.$ Then:
(i) *X* is internally definable set if and only if it is after composed set.
(ii) *X* is externally definable set if and only if it is after-*c* composed set.

**Proposition: 3.6** Let $\mathscr{A}=(U,\ R)$ be a GAS and $X\subseteq U.$ Then:
(i) *X* is exact set if and only if *X* is internally and externally definable set.
(ii) *X* is rough set if and only if *X* is neither internally nor externally definable set.

**Proof:** By Lemma 3.1, the proof is obvious.

**Remark: 3.3** From the above proposition and lemma we have:
(i) *X* is exact iff it is after and after-*c* composed set.
(ii) *X* is rough iff it is neither after composed nor after-*c* composed set.

By considering $\tau_R$ , of a GAS $\mathscr{A}=(U,\ R)$, forms a topology on *U*, then we can reformulate Definition 3.1 by a topological view as follow:

**Definition: 3.2** Let $\mathscr{A}=(U,\ R)$ be a GAS with a topology $\tau_R$ on *U*. Then for every $X\subseteq U$ :

(i) *X* is said to be internally (resp. externally, totally) definable set if *X* is open (resp. closed, *cl* open) set in $\tau_R$ .

(ii) *X* is said to be *R*-undefinable (rough) set if *X* neither open nor closed set in $\tau_R$ .

**Remark: 3.4** According to Definition 3.2 we have seen how the topology represents the magic box for definability of sets. Thus the collection of open and closed sets represents the golden tools to measure exactness or the roughness of sets.

## 4. DIFFERENCES BETWEEN ROUGH SET THEORY AND ORDINARY SET THEORY

In this section we will give the basic deviations between rough set theory "RST" and ordinary set theory "OST”. The space in "RST" represents a space equipped with relation, this relation represents the basic and necessary concept to define the rough set.

We will give the deviation to four concepts "membership relation, equality, inclusion and power set".
**Definition: 4.1** Let $\mathscr{A}=(U,\ R)$ be a GAS and $X\subseteq U$ . Then we say that:

(i) *x* is "surely" belongs to *X*, written $x\,\underline{\in}\,X$ , if $x\in\underline{R}(X)$.

(ii) *x* is "possibly" belongs to *X*, written $x\,\overline{\in}\,X$ , if $x\in\overline{R}(X)$.

These two membership relations $\underline{\in}$ and $\overline{\in}$ are called "strong" and "weak" membership relations respectively and it is clear that:

If $x \underline{\in} X$ implies to $x \in X$ and if $x \in X$ implies to $x \overline{\in} X$.

The converse is not true in general as the following example illustrated:

**Example: 4.1** Consider $U = \{a, b, c, d\}$ and $R$ is a binary relation on $U$ such that:
$aR = \{a\}, bR = \{b\}, cR = \{b, c, d\}$ and $dR = \{a\}$, then
$\tau_R = \{U, \phi, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}$, and
$\tau_R^* = \{U, \phi, \{c\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Let $X = \{a, c\}$, then $\underline{R}(X) = \{a\}$ and $\overline{R}(X) = \{a, c, d\}$. It is clear that $c \in X$ but $c \notin \underline{R}(X)$ i.e., non $c \underline{\in} X$ and also $d \in \overline{R}(X)$ i.e., $d \overline{\in} X$ but $d \notin X$.

**Proposition: 4.1** Let $\mathscr{A} = (U, R)$ be a GAS and $X, Y \subseteq U$. Then by using the properties of approximations we can prove the following properties:

(i) If $X \subseteq Y$, then ( $x \underline{\in} X$ implies to $x \underline{\in} Y$ and $x \overline{\in} X$ implies to $x \overline{\in} Y$ ).

(ii) $x \overline{\in} (X \bigcup Y)$ if and only if $x \overline{\in} X$ or $x \overline{\in} Y$.

(iii) $x \overline{\in} (X \bigcap Y)$ if and only if $x \overline{\in} X$ and $x \overline{\in} Y$.

(iv) If $x \underline{\in} X$ or $x \underline{\in} Y$, then $x \underline{\in} (X \bigcup Y)$.

(v) If $x \underline{\in} (X \bigcap Y)$, then $x \underline{\in} X$ and $x \underline{\in} Y$.

(vi) $x \underline{\in} (-X)$ if and only if non $x \overline{\in} X$.

(vii) $x \overline{\in} (-X)$ if and only if non $x \underline{\in} X$.

**Remarks: 4.1**

(i) In the case of $R$ is an equality relation, all these memberships relations $\underline{\in}$ and $\overline{\in}$ are the same and coincides with ordinary membership relation $\in$ as in "OST".

(ii) We can redefine the approximations by using $\underline{\in}$ and $\overline{\in}$ as follow:

For any $X \subseteq U$, $\underline{R}(X) = \{x \in U : x \underline{\in} X\}$ and $\overline{R}(X) = \{x \in U : x \overline{\in} X\}$.

**Definition: 4.2** Let $\mathscr{A} = (U, R)$ be a GAS. Then the two subsets $X, Y \subseteq U$ are called:

(i) Roughly bottom-equal in $\mathscr{A}$, written $X \eqsim Y$, if $\underline{R}(X) = \underline{R}(Y)$.

(ii) Roughly top-equal in $\mathscr{A}$, written $X \simeq Y$, if $\overline{R}(X) = \overline{R}(Y)$.

(iii) Roughly equal in $\mathscr{A}$, written $X \approx Y$, if $X \eqsim Y$ and $X \simeq Y$.

**Definition: 4.3** Let $\mathscr{A} = (U, R)$ be a GAS. Then the subset $X \subseteq U$ is said to be:

(i) Dense in $\mathscr{A}$ if $X \simeq U$.

(ii) Co-dense in $\mathscr{A}$ if $X \eqsim \phi$.

(iii) Dispersed in $\mathscr{A}$ if $X \simeq U$ and $X \eqsim \phi$.

**Remark: 4.2** Two different sets which are not equal in "OST", can be equal (approximately) in "RST" as the following example illustrated:

**Example: 4.2** Consider $U = \{a, b, c, d, e\}$ and $R$ be a binary relation on $U$ where
$aR = \{a\}, bR = \{c, d\}, cR = \{e, a\}, dR = \{d, a\}$ and $eR = \{e\}$. Then
$\tau_R = \{U, \phi, \{a\}, \{e\}, \{a, d\}, \{a, e\}, \{a, c, e\}, \{a, d, e\}, \{a, c, d, e\}\}$,

and $\tau_R^* = \{U, \phi, \{b\}, \{b, c\}, \{b, d\}, \{b, c, d\}, \{b, c, e\}, \{a, b, c, d\}, \{b, c, d, e\}\}$.

Let $X_1 = \{a, c, d\}, Y_1 = \{a, b, d\}, X_2 = \{b, c, d\}$ and $Y_2 = \{a, b, d, e\}$.

Then $\underline{R}(X_1) = \{a, d\} = \underline{R}(Y_1)$ i.e., $X_1 \eqsim Y_1$ and $\overline{R}(X_1) = \{a, b, c, d\} = \overline{R}(Y_1)$ i.e., $X_1 \simeq Y_1$. Thus $X_1 \approx Y_1$ although $X_1 \neq Y_1$.

Also $\underline{R}(X_2) = \varphi$ and $\overline{R}(Y_2) = U$, then $X_2 \eqsim \phi$ and $Y_2 \simeq U$ that is $Y_2$ is dense and $X_2$ is co-dense in $\mathscr{A}$.

**Proposition: 4.2** Let $\mathscr{A} = (U, R)$ be a GAS and $X, Y, X', Y' \subseteq U$. Then:

(i) If $X \simeq Y$, then $(X \bigcup Y) \simeq X \simeq Y$.

(ii) If $X \eqsim Y$, then $(X \bigcap Y) \eqsim X \eqsim Y$.

(iii) If $X \simeq X'$ and $Y \simeq Y'$, then $(X \bigcup Y) \simeq (X' \bigcup Y')$.

(iv) If $X \eqsim X'$ and $Y \eqsim Y'$, then $(X \bigcap Y) \eqsim (X' \bigcap Y')$.

**Proof:**

(i) Let $X \simeq Y$, then $\overline{R}(X) = \overline{R}(Y)$. But $\overline{R}(X \bigcup Y) = \overline{R}(X) \bigcup \overline{R}(Y)$,
then $\overline{R}(X \bigcup Y) = \overline{R}(X) \bigcup \overline{R}(X) = \overline{R}(X)$
and $\overline{R}(X \bigcup Y) = \overline{R}(Y) \bigcup \overline{R}(Y) = \overline{R}(Y)$. Hence $(X \bigcup Y) \simeq X \simeq Y$.

(ii) By similar way as in (i).

(iii) Let $X \simeq X'$ and $Y \simeq Y'$, then $\overline{R}(X) = \overline{R}(X')$ and $\overline{R}(Y) = \overline{R}(Y')$. Thus
$\overline{R}(X) \bigcup \overline{R}(Y) = \overline{R}(X') \bigcup \overline{R}(Y')$,
which implies that $\overline{R}(X \bigcup Y) = \overline{R}(X' \bigcup Y')$. Thus $(X \bigcup Y) \simeq (X' \bigcup Y')$.

(iv) By similar way as in (iii).

**Proposition: 4.3** Let $\mathscr{A} = (U, R)$ be a GAS and $X \subseteq U$. Then:

(i) *X* is dense set if and only if $(-X)$ is co-dense.

(ii) *X* is dispersed set if and only if $(-X)$ is dispersed.

(iii) Any superset of dense set is also dense.

(iv) Any subset of co-dense set is also co-dense.

**Proof:**

(i) *X* is dense in $\mathscr{A}$ iff $X \simeq U$ iff $\overline{R}(X) = \overline{R}(U)$. But $\overline{R}(X) = -\underline{R}(-X)$,
Hence *X* is dense iff $-\underline{R}(-X) = -\underline{R}(-U) = -\underline{R}(\varphi)$ iff $\underline{R}(-X) = \underline{R}(\phi)$ iff $(-X)$ is co-dense.

(ii) *X* is dispersed in $\mathscr{A}$ iff *X* is dense and co-dense
iff $(-X)$ is co-dense and dense iff $(-X)$ is dispersed

(iii) Let *Y* is a superset of *X* and *X* is dense, then $X \subseteq Y$ and $X \simeq U$.
Hence $\overline{R}(X) \subseteq \overline{R}(Y)$ and $\overline{R}(X) = \overline{R}(U) = U$,
which means that $U \subseteq \overline{R}(Y)$.
But $\overline{R}(Y) \subseteq U$, then $\overline{R}(Y) = U = \overline{R}(U)$, that is $Y \simeq U$ and then *Y* is dense set.

(iv) By similar way as in (iii).

**Proposition: 4.4** Let $\mathscr{A} = (U, R)$ be a GAS, and then the lower (resp. the upper) approximation of any subset $X \subseteq U$ can be defined by: $\underline{R}(X)$ (resp. $\overline{R}(X)$) is the intersection (resp. the union) of all sets *Y* such that $X \eqsim Y$ (resp. $X \simeq Y$).

**Proof:** We prove the proposition in the case of $\underline{R}(X)$ and the case of $\overline{R}(X)$ by similar way:

First, let $Y \subseteq U$ such that $X \eqsim Y$, then $\underline{R}(X) = \underline{R}(Y)$.

But $\underline{R}(Y) \subseteq Y,\ \forall\ Y \subseteq U$, hence $\underline{R}(X) \subseteq \bigcap\{Y \subseteq U:\ X \eqsim Y\}$ (1)

Now, since $\underline{R}(\underline{R}(X)) = \underline{R}(X)$, then $\underline{R}(X) \eqsim Y$. Thus
$\bigcap\{Y \subseteq U\ :\ X \eqsim Y\} \subseteq \underline{R}(X)$ (2)

By (1) and (2); $\underline{R}(X) = \bigcap\{Y \subseteq U : X \eqsim Y\}$

**Proposition: 4.5** Let $\mathscr{A} = (U\,,\ R\ )$ be a GAS and $X, Y \subseteq U$. Then:

(i) $X \simeq Y$ if and only if $(-X) \eqsim (-Y)$.

(ii) If $X \eqsim \phi$ or $Y \eqsim \phi$, then $(X \bigcap Y) \eqsim \varphi$.

(iii) If $X \simeq U$ or $Y \simeq U$, then $(X \bigcup Y) \simeq U$.

**Proof:**

(i) Obvious.

(ii) Let $X \eqsim \phi$ or $Y \eqsim \phi$, then $\underline{R}(X) = \underline{R}(\phi) = \phi$ or $\underline{R}(Y) = \underline{R}(\phi) = \phi$.

Then $\underline{R}(X \bigcap Y) = \underline{R}(X) \bigcap \underline{R}(Y) = \varphi = \underline{R}(\varphi)$, that is $(X \bigcap Y) \eqsim \phi$.

(iii) By similar way as in (ii).

**Remark: 4.3** The rough equalities $\eqsim, \simeq$ *and* $\approx$ are equivalence relations on the power set $P(U)$ in a GAS $\mathscr{A} = (U\,,\ R\ )$.

**Remarks: 4.4**

(i) If *R* in a GAS $\mathscr{A} = (U\,,\ R\ )$ is an equality relation, then all rough equalities of sets are coincides with the classical set theoretical equality of sets in "OST".

(ii) Rough equality of sets does not imply the equality in general which illustrated by Example 4.2.

(iii) According to Remark 4.3, any GAS $\mathscr{A} = (U\,,\ R\ )$ with a binary general relation generate the following three different Pawlak approximation spaces:

$\underline{\mathscr{A}_*} = (P(U),\ \eqsim)$, $\overline{\mathscr{A}_*} = (P(U),\ \simeq)$ and $\mathscr{A}_* = (P(U),\ \approx)$ which are called "the lower, the upper and the rough" approximation spaces of a GAS $\mathscr{A} = (U,\ R)$ respectively.

Moreover, these spaces form three topologies on $P(U)$ which are quasi-discrete topologies and they are given by $T_{\underline{\mathscr{A}_*}} = (P(U),\ \tau_R\,(\underline{\mathscr{A}_*}))$, $T_{\overline{\mathscr{A}_*}} = (P(U),\ \tau_R\,(\overline{\mathscr{A}_*}))$ and $T_{\mathscr{A}_*} = (P(U),\ \tau_R\,(\mathscr{A}_*))$.

**Definition: 4.4** Let $\mathscr{A} = (U\,,\ R\ )$ be a GAS and $X, Y \subseteq U$. We say that:

(i) *X* is "roughly bottom-included" in *Y*, written $X \underset{\sim}{\subseteq} Y$, if $\underline{R}(X) \subseteq \underline{R}(Y)$.

(ii) *X* is "roughly top-included" in *Y*, written $X \tilde{\subseteq} Y$, if $\overline{R}(X) \subseteq \overline{R}(Y)$.

(iii) *X* is "roughly included" in *Y*, written $X \underset{\sim}{\tilde{\subseteq}} Y$, if $X \tilde{\subseteq} Y$ and $X \underset{\sim}{\subseteq} Y$.

We call *X* in the above cases (i), (ii) and (iii) by the following notations: *X* is "rough lower, rough upper and rough" subset in *Y* respectively.

The rough inclusion of sets does not imply the inclusion of sets as the following example illustrated:

**Example: 4.3** Consider $U = \{a, b, c, d\}$ and *R* is a binary relation on *U*, where:

$aR = \{a\},\ bR = \{b\},\ cR = \{b, c\}$ and $dR = \{a\}$. Then

$\tau_R = \{U, \phi, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ and

$\tau_R^* = \{U, \phi, \{c\}, \{d\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$

Let $X=\{b, c\}$ and $Y=\{a, b, d\}$, clearly $X \not\subseteq Y$ and we have $\underline{R}(X)=\{b\}$, $\underline{R}(Y)=\{a, b, d\}$, $\overline{R}(X)=\{b, c\}$ and $\overline{R}(Y)=U$. Then $X \underset{\sim}{\subset} Y$ and $X \tilde{\subset} Y$ which implies that $X \underset{\sim}{\tilde{\subset}} Y$ although $X \not\subseteq Y$.

**Proposition: 4.6** Let $\mathscr{A}=(U, R)$ be a GAS and $X, Y \subseteq U$. Then

(i) If $X \subseteq Y$, then $X \underset{\sim}{\subset} Y$, $X \tilde{\subset} Y$ and $X \underset{\sim}{\tilde{\subset}} Y$.
(ii) $X \underset{\sim}{\subset} Y$ and $Y \underset{\sim}{\subset} X$ if and only if $X \eqsim Y$.
(iii) $X \tilde{\subset} Y$ and $Y \tilde{\subset} X$ if and only if $X \simeq Y$.
(iv) $X \underset{\sim}{\tilde{\subset}} Y$ and $Y \underset{\sim}{\tilde{\subset}} X$ if and only if $X \approx Y$.

**Proof:**
(i) Obvious.
(ii) $X \underset{\sim}{\subset} Y$ and $Y \underset{\sim}{\subset} X$ iff $\underline{R}(X) \subseteq \underline{R}(Y)$ and $\underline{R}(Y) \subseteq \underline{R}(X)$ iff $\underline{R}(X)=\underline{R}(Y)$ iff $X \eqsim Y$.
(iii) and (iv) by similar way as in (ii).

**Proposition: 4.7** Let $\mathscr{A}=(U, R)$ be a GAS and $X, Y, X', Y' \subseteq U$. Then:

(i) $X \underset{\sim}{\subset} Y$ if and only if $(X \cap Y) \eqsim X$.
(ii) $X \tilde{\subset} Y$ if and only if $(X \cup Y) \simeq Y$.
(iii) $(X \cap Y) \underset{\sim}{\subset} X \tilde{\subset} (X \cup Y)$.
(iv) If $X \subseteq Y$, $X \eqsim X'$ and $Y \eqsim Y'$, then $X' \underset{\sim}{\subset} Y'$.
(v) If $X \subseteq Y$, $X \simeq X'$ and $Y \simeq Y'$, then $X' \tilde{\subset} Y'$.
(vi) If $X \subseteq Y$, $X \approx X'$ and $Y \approx Y'$, then $X' \underset{\sim}{\tilde{\subset}} Y'$.
(vii) If $X' \tilde{\subset} X$ and $Y' \tilde{\subset} Y$, then $(X' \cup Y') \tilde{\subset} (X \cup Y)$.
(viii) If $X' \underset{\sim}{\subset} X$ and $Y' \underset{\sim}{\subset} Y$, then $(X' \cap Y') \underset{\sim}{\subset} (X \cap Y)$.

**Proof:**
(i) $X \underset{\sim}{\subset} Y$ iff $\underline{R}(X) \subseteq \underline{R}(Y)$ iff $\underline{R}(X) \cap \underline{R}(Y)=\underline{R}(X)$ iff $\underline{R}(X \cap Y)=\underline{R}(X)$ iff $(X \cap Y) \eqsim X$.
(ii) By similar way as in (i).
(iii) Since $\underline{R}(X \cap Y)=\underline{R}(X) \cap \underline{R}(Y)$ and $\overline{R}(X \cup Y)=\overline{R}(X) \cup \overline{R}(Y)$. Then $\underline{R}(X \cap Y) \subseteq \underline{R}(X)$ and $\overline{R}(X) \subseteq \overline{R}(X \cup Y)$, and hence $(X \cap Y) \underset{\sim}{\subset} X \tilde{\subset} (X \cup Y)$.
(iv) Let $X \subseteq Y$, $X \eqsim X'$ and $Y \eqsim Y'$, then $\underline{R}(X) \subseteq \underline{R}(Y)$, $\underline{R}(X)=\underline{R}(X')$ and $\underline{R}(Y)=\underline{R}(Y')$. Thus $\underline{R}(X) \subseteq \underline{R}(Y')$ and then $X' \underset{\sim}{\subset} Y'$.
(v) and (vi) by similar way as in (iv).
(vi) Let $X' \tilde{\subset} X$ and $Y' \tilde{\subset} Y$, then $\overline{R}(X') \subseteq \overline{R}(X)$ and $\overline{R}(Y') \subseteq \overline{R}(Y)$. Hence $\overline{R}(X') \cup \overline{R}(Y') \subseteq \overline{R}(X) \cup \overline{R}(Y)$, and then $\overline{R}(X' \cup Y') \subseteq \overline{R}(X \cup Y)$. That is $(X' \cup Y') \tilde{\subset} (X \cup Y)$.
(viii) By similar way as in (vii).

**Proposition: 4.8** Let $\mathscr{A}=(U, R)$ be a GAS and $X, Y, Z \subseteq U$. Then

(i) If $X \underset{\sim}{\subset} Y$ and $X \eqsim Z$, then $Z \underset{\sim}{\subset} Y$.
(ii) If $X \tilde{\subset} Y$ and $X \simeq Z$, then $Z \tilde{\subset} Y$.
(iii) If $X \underset{\sim}{\tilde{\subset}} Y$ and $X \approx Z$, then $Z \underset{\sim}{\tilde{\subset}} Y$.

**Proof:** Obvious.

**Remark: 4.5** The rough inclusions $\underset{\sim}{\subset}$, $\tilde{\subset}$ and $\underset{\sim}{\tilde{\subset}}$ represent ordering relations on $P(U)$ in the GAS $\mathscr{A}=(U, R)$.

**Definition: 4.5** Let $\mathscr{A}=(U,R)$ be a GAS and $X\subseteq U$. Then the family of all lower (resp. upper, rough) subsets of $X$ in $\mathscr{A}$, denoted by $P_{\underline{\mathscr{A}}}(X)$ (resp. $P_{\overline{\mathscr{A}}}(X), P_{\mathscr{A}}(X)$), is given by $P_{\underline{\mathscr{A}}}(X)=\{Y\subseteq U: Y \underset{\sim}{\subseteq} X\}$ (resp. $P_{\overline{\mathscr{A}}}(X)=\{Y\subseteq U: Y \tilde{\subset} X\}$, $P_{\mathscr{A}}(X)=\{Y\subseteq U: Y \underset{\sim}{\tilde{\subseteq}} X\}$).

**Example: 4.4** Consider the GAS in Example 4.3 and let $X=\{a,c\}$. Then $\underline{R}(X)=\{a\}$ and $\overline{R}(X)=\{a,c,d\}$, and hence $P(X)=\{X,\phi,\{a\},\{c\}\}$,

$P_{\underline{\mathscr{A}}}(X)=\{X,\phi,\{a\},\{c\},\{d\},\{c,d\}\{a,c,d\}\}$,

$P_{\overline{\mathscr{A}}}(X)=\{X,\phi,\{a\},\{c\},\{d\},\{a,d\},\{c,d\},\{a,c,d\}\}$ and

$P_{\mathscr{A}}(X)=\{X,\phi,\{a\},\{c\},\{d\},\{c,d\},\{a,c,d\}\}$.

Clearly $P_{\mathscr{A}}(X)=P_{\overline{\mathscr{A}}}(X)\cap P_{\underline{\mathscr{A}}}(X)$.

**Remark: 4.6** The concept of power set $P(X)$ in "OST" differs from the concept of rough power set in "RST", for instance, in Example 4.4, it is clear that $\{d\},\{c,d\},\{a,c,d\}\in P_{\mathscr{A}}(X)$, but $\{d\},\{c,d\},\{a,c,d\}\notin P(X)$.

The relation between ordinary power set and rough power set will give in the following proposition.

**Proposition: 4.9** Let $\mathscr{A}=(U,R)$ be a GAS and $X\subseteq U$. Then $P(X)\subseteq P_{\underline{\mathscr{A}}}(X)$, $P(X)\subseteq P_{\overline{\mathscr{A}}}(X)$ and $P(X)\subseteq P_{\mathscr{A}}(X)$.

**Proof:** Let $Y\in P(X)$, then $Y\subseteq X$ and hence $\underline{R}(Y)\subseteq\underline{R}(X)$ and $\overline{R}(Y)\subseteq\overline{R}(X)$. Thus $Y\subset X$, $Y \tilde{\subset} X$ and $Y \underset{\sim}{\tilde{\subseteq}} X$, which implies that $Y\in P_{\underline{\mathscr{A}}}(X)$, $Y\in P_{\overline{\mathscr{A}}}(X)$ and $Y\in P_{\mathscr{A}}(X)$. Thus $P(X)\subseteq P_{\underline{\mathscr{A}}}(X)$, $P(X)\subseteq P_{\overline{\mathscr{A}}}(X)$ and $P(X)\subseteq P_{\mathscr{A}}(X)$.

**Proposition: 4.10** Let $\mathscr{A}=(U,R)$ be a GAS and $X,Y\subseteq U$, Then:

(i) If $X \underset{\sim}{\subseteq} Y$ then $P_{\underline{\mathscr{A}}}(X)\subseteq P_{\underline{\mathscr{A}}}(Y)$.

(ii) If $X \tilde{\subset} Y$ then $P_{\overline{\mathscr{A}}}(X)\subseteq P_{\overline{\mathscr{A}}}(Y)$.

(iii) If $X \underset{\sim}{\tilde{\subseteq}} Y$ then $P_{\mathscr{A}}(X)\subseteq P_{\mathscr{A}}(Y)$.

(iv) $X \eqsim Y$ if and only if $P_{\underline{\mathscr{A}}}(X)=P_{\underline{\mathscr{A}}}(Y)$.

(v) $X \simeq Y$ if and only if $P_{\overline{\mathscr{A}}}(X)=P_{\overline{\mathscr{A}}}(Y)$.

(vi) $X\approx Y$ if and only if $P_{\mathscr{A}}(X)=P_{\mathscr{A}}(Y)$.

**Proof:**

(i) Let $X \underset{\sim}{\subseteq} Y$, then $\underline{R}(X)\subseteq\underline{R}(Y)$ (1)

Now let $Z\in P_{\underline{\mathscr{A}}}(X)$, then $Z \underset{\sim}{\subseteq} X$, that is $\underline{R}(Z)\subseteq\underline{R}(X)$. Thus by (1), $\underline{R}(Z)\subseteq\underline{R}(Y)$ and then $Z \underset{\sim}{\subseteq} Y$, hence $Z\in P_{\underline{\mathscr{A}}}(Y)$ and then $P_{\underline{\mathscr{A}}}(X)\subseteq P_{\underline{\mathscr{A}}}(Y)$.

(ii) and (iii) by the same way as in (i).

(iv) $X \eqsim Y$ iff $X \underset{\sim}{\subseteq} Y$ and $Y \underset{\sim}{\subseteq} X$ iff $P_{\underline{\mathscr{A}}}(X)\subseteq P_{\underline{\mathscr{A}}}(Y)$ and $P_{\underline{\mathscr{A}}}(Y)\subseteq P_{\underline{\mathscr{A}}}(X)$ iff $P_{\underline{\mathscr{A}}}(X)=P_{\underline{\mathscr{A}}}(Y)$.

(iv) and (vi) by the same way as in (iv).

**Proposition: 4.11** Let $\mathscr{A}=(U,R)$ be a GAS and $X,Y\subseteq U$. Then:

(i) $X\in P_{\underline{\mathscr{A}}}(X)$, $X\in P_{\overline{\mathscr{A}}}(X)$ and $X\in P_{\mathscr{A}}(X)$.

(ii) If $X\subseteq Y$, then $P_{\underline{\mathscr{A}}}(X)\subseteq P_{\underline{\mathscr{A}}}(Y)$, $P_{\overline{\mathscr{A}}}(X)\subseteq P_{\overline{\mathscr{A}}}(Y)$ and $P_{\mathscr{A}}(X)\subseteq P_{\mathscr{A}}(Y)$.

**Proof:**

(i) Since $\underset{\sim}{\subseteq}$, $\tilde{\subset}$ and $\underset{\sim}{\tilde{\subseteq}}$ are ordering relations. Then

$X \underset{\sim}{\subseteq} X$ , $X \tilde{\subset} X$ and $X \underset{\sim}{\tilde{\subseteq}} X$ , and hence

$X \in P_{\underline{\mathscr{A}}}(X)$ , $X \in P_{\overline{\mathscr{A}}}(X)$ and $X \in P_{\mathscr{A}}(X)$ .

(ii) Let $X \subseteq Y$ , then $X \underset{\sim}{\subseteq} Y$ , $X \tilde{\subset} Y$ and $X \underset{\sim}{\tilde{\subseteq}} Y$ . Hence

$P_{\underline{\mathscr{A}}}(X) \subseteq P_{\underline{\mathscr{A}}}(Y)$, $P_{\overline{\mathscr{A}}}(X) \subseteq P_{\overline{\mathscr{A}}}(Y)$ and $P_{\mathscr{A}}(X) \subseteq P_{\mathscr{A}}(Y)$ .

## 5. CONCLUSION

Although many authors have been introduced sorts to generalize Pawlak approximation space, but most of them could not applied and satisfied the all properties of Pawlak approximations (see: [1], [3-8], [12-22]). In our approach we did not added any conditions and we initiated a topological structure based on a binary relation to generalize Pawlak space which open the way to more topological applications in rough set context and help in formalizing many applications from real-life data.

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