NUMERICAL SOLUTION FOR THE INTEGRO-DIFFERENTIAL EQUATIONS USING SINGLE TERM HAAR WAVELET SERIES METHOD

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ABSTRACT

In this paper presents numerical solutions of Integro-Differential Equations (IDE) using single-term Haar wavelet series (STHWS) method. The obtained discrete results were compared with exact solution of the IDE and Local Polynomial Regression (LPR) method [7] to highlight the efficiency of the STHWS method. The numerical solution shows that this method is powerful in solving integro-differential equations. The method will be tested on three model problems in order to demonstrate its usefulness and accuracy.

Mathematics Subject Classification: 41A45, 41A46, 41A58.

Keywords: Haar wavelet; single-term Haar wavelet series (STHWS), Integro-Differential Equations, Local Polynomial Regression, Kernel functions.

1. INTRODUCTION

In recent years, there has been a growing interest in the Integro-Differential Equations (IDEs) which are a combination of differential and Fredholm-Volterra integral equations. IDEs play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electrostatics. The mentioned integro-differential equations are usually difficult to solve analytically, so a numerical method is required. Many different methods are used to obtain the solution of the linear and non-linear IDEs such as the successive approximations, Adomain decomposition, Homotopy perturbation method, Chebyshev and Taylor collocation, Haar Wavelet, Tau and Walsh series methods [1 - 8, 10]. Recently, the authors [2] have used local polynomial regression method for the numerical solution of linear and non-linear Fredholm and Volterra integral equations.

In this article we developed numerical methods for IDEs to get discrete solutions via STHW method which was studied by S. Sekar and team of his researchers [11 - 17]. The subject of this paper is to try to find numerical solutions of integro-differential equations using STHWS method and compare the discrete results with the local polynomial regression method which is presented firstly by Hikmat Caglar [2]. Finally, we show the method to achieve the desired accuracy. Details of the structure of the present method are explained in sections. We apply STHWS and LPR methods for IDEs. In Section 3, it’s proved the efficiency of the STHWS method. Finally, Section 4 contains some conclusions and directions for future expectations and researches.

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2. PRELIMINARIES

The term integral equation was apparently first used by Du Bois-Reymond in 1883. An integral equation is an equation in which the function to be determined appears under the integral sign. If we consider linear integral equations, that is, equations in which no nonlinear functions of the unknown function are involved. Consider the linear integro-differential equation of the form

\[ y'(x) = p(x)y(x) + g(x) + \lambda \int_a^x K(x,t)y(t)dt, \]

\[ y(a) = \alpha \]

where the upper limit of the integral is constant or variable, \( \lambda, \alpha, a \) are constants, \( g(x), p(x) \) and the kernel \( K(x,t) \) are given functions, whereas \( y(x) \) needs to be determined.

3. SINGLE TERM HAAR WAVELET SERIES METHOD

The orthogonal set of Haar wavelets \( h(t) \) is a group of square waves with magnitude of ±1 in some intervals and zeros elsewhere [12]. In general,

\[
h_n(t) = h(2^j t - k), n = 2^j + k, \\
j \geq 0, 0 \leq k < 2^j, n, j, k \in \mathbb{Z} \\
h_i(t) = \begin{cases} 1,0 \leq t < \frac{1}{2} \\
-1, \frac{1}{2} \leq t < 1 
\end{cases}
\]

Namely, each Haar wavelet contains one and just one square wave, and is zero elsewhere. Just these zeros make Haar wavelets to be local and very useful in solving stiff systems. Any function \( y(t) \), which is square integrable in the interval \([0,1)\). Can be expanded in a Haar series with an infinite number of terms

\[ y(t) = \sum_{i=0}^{\infty} c_i h_i(t), i = 2^j + k, \\
j \geq 0, 0 \leq k < 2^j, n, j, t \in [0,1] \]  \hspace{1cm} (1)

where the Haar coefficients

\[ c_i = 2^j \int_0^1 y(t)h_i(t)dt \]

are determined such that the following integral square error \( \varepsilon \) is minimized:

\[ \varepsilon = \int_0^1 \left[ y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt, \]

where \( m = 2^j, j \in \{0\} \cup \mathbb{N} \)

Usually, the series expansion Equation (1) contains an infinite number of terms for a smooth \( y(t) \). If \( y(t) \) is a piecewise constant or may be approximated as a piecewise constant, then the sum in Eq. (1) will be terminated after \( m \) terms, that is

\[ y(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) = c^T_m h_m(t), t \in [0,1] \]

\[ c_m(t) = [c_1, c_2, \ldots, c_{m-1}]^T, \]

\[ h_m(t) = [h_1(t), h_2(t), \ldots, h_{m-1}(t)]^T, \]  \hspace{1cm} (2)

where “\(^T\)” indicates transposition, the subscript \( m \) in the parantheses denotes their dimensions. The integration of Haar wavelets can be expandable into Haar series with Haar coefficient matrix \( P \) [3].
\[
\int h_m(\tau)d\tau \approx P_{(m \times m)}h_m(t), t \in [0,1]
\]

where the m-square matrix \(P\) is called the operational matrix of integration and single-term \(P_{(1 \times 1)} = \frac{1}{2}\). Let us define [12]
\[
h_m(t)h_m^r(t) \approx M_{(m \times m)}(t),
\]
and \(M_{(1 \times 1)}(t) = h_0(t)\). Equation (3) satisfies
\[
M_{(m \times m)}(t)c_m = C_{(m \times m)}h_m(t),
\]
where \(c_m\) is defined in Equation (2) and \(C_{(1 \times 1)} = c_0\).

4. NUMERICAL EXAMPLE

In this section, we consider the following IDEs. [7 - 8]

**Example: 1** Consider the linear integro-differential equation [7]
\[
y''(x) = 3e^{3x} - \frac{1}{3}(2e^{3} + 1)x + \int_0^x 3xy(t)dt,
\]
\[
y(0) = 1
\]
For which the exact solution is \(y(x) = e^{3x}\).

**Example: 2** [7] Consider the linear integro-differential equation [7]
\[
y''(x) = 1 + 2x - y(x) + \int_0^x x(1 + 2x)e^{(t-x)}y(t)dt, 0 \leq x \leq 1
\]
\[
y(0) = 1
\]
For which the exact solution is \(y(x) = e^{x^2}\).

**Example: 3** Consider the system of integro-differential equation [8]
\[
y_1'(t) + \int_0^t e^{-\tau}y_1(s)ds - \int_0^t e^{(t-\tau)}y_2'(s)ds = 2e^t + \frac{e^t - 1}{t+1}
\]
\[
y_2'(t) + \int_0^t e^{\tau(t-\tau)}y_2(s)ds = e^t + \frac{e^t - 1}{t+1}
\]
\[
y_1(0) = 1, \quad y_2(0) = 1.
\]
For which the exact solutions \(y_1(t) = e^t, \quad y_2(t) = e^{-t}\).

**Example: 4** Consider the system of integro-differential equation [8]
\[
2\pi y_1(t) - \int_0^t \cos(2\pi s) \sin(4\pi t) y_1'(s)ds + \int_0^t \sin(4\pi t + 2\pi s)y_1'(s)ds = 2\pi \cos(2\pi t)(1 + \sin(2\pi t)),
\]
\[
y_2'(t) + \int_0^t \cos(4\pi t) \sin(2\pi s)y_1(s)ds + \int_0^t \cos(4\pi t + 2\pi s)y_2(s)ds = \cos(2\pi t)(2\pi - \sin(2\pi t)),
\]
\[
y_1(0) = 1, \quad y_2(0) = 0
\]
For which the exact solutions \(y_1(t) = \cos(2\pi t), \quad y_2(t) = \sin(2\pi t)\).
Table 1: Exact Solutions and Error calculation of Example 1

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<tr>
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<th>STHWS Error</th>
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Table 2: Exact Solutions and Error calculation of Example 2

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Table 3: Exact Solutions and Error calculation of Example 3

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Table 4: Exact Solutions and Error calculation of Example 4

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Figure: 1. Error estimation of Example - 1

Figure: 2. Error estimation of Example - 2

Figure: 3. Error estimation of Example - 3 at $y_i$
Figure: 4. Error estimation of Example - 3 at $y_2$

Figure: 5. Error estimation of Example - 4 at $y_1$

Figure: 6. Error estimation of Example - 4 at $y_2$
The above examples 1 to 4 has been solved numerically using the LPR method and STHWS method. The obtained results (with step size time = 0.1) along with exact solutions of the examples 1 to 4 and absolute errors between them are calculated and are presented in Table 1 to 4. A graphical representation is given for the IDEs in Figures 1 to 6, using three-dimensional effect to highlight the efficiency of the STHWS method.

5. CONCLUSIONS

Most integro-differential equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, the STHWS Method presented in this paper can be applied. We make use of STHWS method to solve the linear integro-differential equations. From the Tables 1 to 4 and Figures 1 to 6 showed that this method is very convergent for solving linear integro-differential equations. Moreover, the numerical results approximate the exact solution very well. STHWS method can also solve integro-differential equations which can be researched and resolved.

REFERENCES


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