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SOME RESULTS RELATED TO CAUCHY'S PROPER BOUND FOR THE ZEROS OF ENTIRE FUNCTIONS OF ORDER ZERO

Sanjib Kumar Datta[1*] & Manab Biswas[2]

[1]Department of Mathematics, University of Kalyani, Kalyani, Dist-Nadia, Pin-741235, West Bengal, India.

[2]Barabilla High School, P.O. Haptiagach, Dist-Uttar Dinajpur, Pin-733202, West Bengal, India.

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ABSTRACT

A single valued function of one complex variable which is analytic in the finite complex plane is called an entire function. In this paper we would like to establish the bounds for the moduli of zeros of entire functions of order zero. Some examples are provided to clear the notions.

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1. INTRODUCTION, DEFINITIONS AND NOTATIONS

Let $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots + a_{n-1} z^{n-1} + a_n z^n$; $|a_n| \neq 0$ be a polynomial of degree n. Datt and Govil [2]; Govil and Rahaman [5]; Marden [9]; Mohammad [10]; Chattopadhyay, Das, Jain and Konwar [1]; Joyal, Labelle and Rahaman [6]; Jain {[7], [8]}; Sun and Hsieh [11]; Zilovic, Roytman, Combettes and Swamy [13]; Das and Datta [4] etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some sharper results concerning the theory of distribution of zeros of entire functions of order zero.

The following definitions are well known:

Definition: 1 The order ρ and lower order λ of a meromorphic function f are defined as

$$\rho = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho = \limsup_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r},$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, \ldots\ldots$ and $\log^{[0]} x = x$.

If $\rho < \infty$ then f is of finite order. Also $\rho = 0$ means that f is of order zero. In this connection Datta and Biswas [3] gave the following definition:

Definition: 2 Let f be a meromorphic function of order zero. Then the quantities ρ^* and λ^* of f are defined by:

$$\rho^* = \limsup_{r \to \infty} \frac{T(r,f)}{\log r} \quad \text{and} \quad \lambda^* = \liminf_{r \to \infty} \frac{T(r,f)}{\log r}.$$

Corresponding author: Sanjib Kumar Datta[1]*
[1]Department of Mathematics, University of Kalyani, Kalyani, Dist-Nadia, Pin-741235, West Bengal, India.

If f is an entire function then clearly

$$\rho^* = \limsup_{r\to\infty}\frac{\log M(r,f)}{\log r} \quad\text{and}\quad \lambda^* = \liminf_{r\to\infty}\frac{\log M(r,f)}{\log r}.$$

2. LEMMAS

In this section we present a lemma which will be needed in the sequel.

Lemma: 1 If $f(z)$ is an entire function of order $\rho = 0$, then for every $\varepsilon > 0$ the inequality $N(r) \le (\log r)^{\rho^*+\varepsilon}$ holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \le \log r$.

Proof: Let us suppose that $f(z) = 1$. This supposition can be made without loss of generality because if $f(z)$ has a zero of order $'m'$ at the origin then we may consider $g(z) = c.\frac{f(z)}{z^m}$ where c is so chosen that $g(0) = 1$. Since the function $g(z)$ and $f(z)$ have the same order therefore it will be unimportant for our investigations that the number of zeros of $g(z)$ and $f(z)$ differ by m.

We further assume that $f(z)$ has no zeros on $|z| = \log 2r$ and the zeros z_i's of $f(z)$ in $|z| < \log r$ are in non decreasing order of their moduli so that $|z_i| \le |z_{i+1}|$. Also let ρ^* suppose to be finite where $\rho = 0$ is the zero of order of $f(z)$.

Now we shall make use of Jenson's formula as state below

$$\log|f(0)| = -\sum_{i=1}^{n}\log\frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(R\,e^{i\phi})\right|d\phi\ . \tag{1}$$

Let us replace R by $2r$ and n by $N(2r)$ in (1).

$$\therefore\ \log|f(0)| = -\sum_{i=1}^{N(2r)}\log\frac{2r}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(2r\,e^{i\phi})\right|d\phi.$$

Since $f(0) = 1$, $\therefore$ $\log|f(0)| = \log 1 = 0$.

$$\therefore\ \sum_{i=1}^{N(2r)}\log\frac{2r}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(2r\,e^{i\phi})\right|d\phi \tag{2}$$

$$\text{L.H.S.} = \sum_{i=1}^{N(2r)}\log\frac{2r}{|z_i|} \ge \sum_{i=1}^{N(r)}\log\frac{2r}{|z_i|} \ge N(r)\log 2 \tag{3}$$

because for large values of r,

$$\log\frac{2r}{|z_i|} \ge \log 2\ .$$

$$\begin{aligned}\text{R.H.S.} &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(2r\,e^{i\phi})\right|d\phi\\ &\le \frac{1}{2\pi}\int_0^{2\pi}\log M(2r)\,d\phi = \log M(2r).\end{aligned} \tag{4}$$

Again by definition of order ρ^* of $f(z)$ we have for every $\varepsilon > 0$,

$$\log M(2r) \le \{\log(2r)\}^{\rho^*+\varepsilon/2}\ . \tag{5}$$

Hence from (2) by the help of (3), (4) and (5) we have

$N(r)\log 2 \le (\log 2r)^{\rho^*+\varepsilon/2}$

i.e., $N(r) \leq \frac{(\log 2)^{\rho^*+\varepsilon/2}}{\log 2} \cdot \frac{(\log r)^{\rho^*+\varepsilon}}{(\log r)^{\varepsilon/2}} \leq (\log r)^{\rho^*+\varepsilon}$.

This proves the lemma.

3. THEOREMS

In this section we present the main results of the paper.

Theorem: 1 Let $P(z)$ be an entire function defined by

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n + \ldots\ldots$$

whose order $\rho = 0$. Also for all sufficiently large r in the disc $|z| \leq \log r$, $|a_{N(r)}| \neq 0$, $|a_0| \neq 0$ and also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where t_0 is the greatest positive root of

$$g(t) \equiv |a_{N(r)}| t^{N(r)+1} - \left(|a_{N(r)}| + M\right) t^{N(r)} + M = 0$$

and t_0' is the greatest positive root of

$$f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0$$

where $M = \max\{|a_0|, |a_1|, \ldots\ldots\ldots, |a_{N(r)-1}|\}$ and $M' = \max\{|a_1|, |a_2|, \ldots\ldots\ldots, |a_{N(r)}|\}$.

Proof: Now, $P(z) \approx a_0 + a_1 z + a_2 z^2 + \ldots..+ a_{N(r)} z^{N(r)}$ because $N(r)$ exists for $|z| \leq \log r$; r is sufficiently large and $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region given in Theorem 1 which we are to prove.

Now

$$|P(z)| \approx |a_0 + a_1 z + a_2 z^2 + \cdots\cdots + a_{N(r)} z^{N(r)}|$$

$$\geq |a_{N(r)}||z|^{N(r)} - |a_0 + a_1 z + + a_2 z^2 + \cdots\cdots + a_{N(r)-1} z^{N(r)-1}|.$$

Also

$$\begin{aligned} |a_0 + a_1 z + + a_2 z^2 + \cdots\cdots + a_{N(r)-1} z^{N(r)-1}| &\leq |a_0| + \ldots\ldots + |a_{N(r)-1}||z|^{N(r)-1} \\ &\leq M\left(1 + |z| + \cdots + |z|^{N(r)-1}\right) \\ &= M \frac{|z|^{N(r)} - 1}{|z| - 1} \text{ if } |z| \neq 1. \end{aligned} \tag{6}$$

Therefore using (6) we obtain that

$$\begin{aligned} |P(z)| &\geq |a_{N(r)}||z|^{N(r)} - |a_0 + a_1 z + + a_2 z^2 + \cdots\cdots + a_{N(r)-1} z^{N(r)-1}| \\ &\geq |a_{N(r)}||z|^{N(r)} - M \frac{|z|^{N(r)} - 1}{|z| - 1}. \end{aligned}$$

Hence

$|P(z)| \geq 0$ if $|a_{N(r)}||z|^{N(r)} - M \frac{|z|^{N(r)} - 1}{|z| - 1} > 0$

i.e., if $|a_{N(r)}||z|^{N(r)} > M \frac{|z|^{N(r)} - 1}{|z| - 1}$

i.e., if $|a_{N(r)}||z|^{N(r)+1} - |a_{N(r)}||z|^{N(r)} > M\left(|z|^{N(r)} - 1\right)$

i.e., if $|a_{N(r)}||z|^{N(r)+1} - |a_{N(r)}||z|^{N(r)} - M|z|^{N(r)} + M > 0$

i.e., if $|a_{N(r)}||z|^{N(r)+1} - \left(|a_{N(r)}| + M\right)|z|^{N(r)} + M > 0.$

Therefore on $|z| \neq 1$,

$|P(z)| \geq 0$ if $|a_{N(r)}||z|^{N(r)+1} - (|a_{N(r)}| + M)|z|^{N(r)} + M > 0$.

Now let us consider

$$g(t) \equiv |a_{N(r)}|t^{N(r)+1} - (|a_{N(r)}| + M)t^{N(r)} + M = 0. \tag{7}$$

Clearly the maximum number of changes in sign in (7) is two. So the maximum number of positive roots of $g(t) = 0$ is two and by Descartes' rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of (7). So $g(t) = 0$ must have another positive root t_1(say). Let us take $t_0 = \max\{1, t_1\}$. Clearly for $t > t_0$, $g(t) > 0$. If not, for some $t = t_2 > t_0$, $g(t_2) < 0$.

Now $g(t_2) < 0$ and $g(\infty) > 0$ imply that $g(t) = 0$ has another positive root in (t_2, ∞) which gives a contradiction.

Therefore for $t > t_0$, $g(t) > 0$ and so $t_0 > 1$.

Hence $|P(z)| \geq 0$ for $|z| > t_0$.

Therefore all the zeros of $P(z)$ lie in the disc $|z| \leq t_0$. (8)

Again let us consider

$$\begin{aligned} Q(z) &= z^{N(r)} P\left(\frac{1}{z}\right) \\ &\approx z^{N(r)}\left\{a_0 + \frac{a_1}{z} + \cdots + \frac{a_{N(r)}}{z^{N(r)}}\right\} \\ &= a_0 z^{N(r)} + a_1 z^{N(r)-1} + \ldots\ldots + a_{N(r)} \end{aligned}$$

i.e., $|Q(z)| \geq |a_0||z|^{N(r)} - |a_1 z^{N(r)-1} + \cdots + a_{N(r)}|$ for $|z| \neq 1$.

Now

$$\begin{aligned} |a_1 z^{N(r)-1} + \cdots + a_{N(r)}| &\leq |a_1||z|^{N(r)-1} + \cdots + |a_{N(r)}| \\ &\leq M'(|z|^{N(r)-1} + \cdots + 1) \\ &= M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) \text{ for } |z| \neq 1. \end{aligned} \tag{9}$$

Using (9) we get that

$$\begin{aligned} |Q(z)| &\geq |a_0||z|^{N(r)} - |a_1 z^{N(r)-1} + \cdots + a_{N(r)}| \\ &\geq |a_0||z|^{N(r)} - M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) \text{ for } |z| \neq 1. \end{aligned}$$

Therefore for $|z| \neq 1$,

$|Q(z)| \geq 0$ if $|a_0||z|^{N(r)} - M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) > 0$

i.e., if $|a_0||z|^{N(r)} > M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right)$

i.e., if $|a_0||z|^{N(r)+1} - |a_0||z|^{N(r)} - M'|z|^{N(r)} + M' > 0$

i.e., if $|a_0||z|^{N(r)+1} - (|a_0| + M')|z|^{N(r)} + M' > 0$.

So for $|z| \neq 1$,

$|Q(z)| \geq 0$ if $|a_0||z|^{N(r)+1} - (|a_0| + M')|z|^{N(r)} + M' > 0$.

Let us consider

$$f(t) \equiv |a_0|t^{N(r)+1} - (|a_0| + M')t^{N(r)} + M' = 0.$$

Since the maximum number of changes in sign in $f(t)$ is two, the maximum number of positive roots of $f(t)=0$ is two and by Descartes' rule of sign if it is less, less by two. Clearly $t=1$ is one positive root of $f(t)=0$. So $f(t)=0$ must have another positive root t_2 (say). Let us take $t_0^{'}=\max\{1,t_2\}$. Clearly for $t>t_0^{'}$, $f(t)>0$. If not, for some $t_3>t_0^{'}$, $f(t_3)<0$.

Now $f(t_3)<0$ and $f(\infty)>0$ implies that $f(t)=0$ has another positive root in (t_3,∞) which is a contradiction.

Therefore for $t>t_0^{'}$, $f(t)>0$.

Also $t_0^{'}\geq 1$. So $|Q(z)|\geq 0$ for $|z|>t_0^{'}$.

Therefore $Q(z)$ does not vanish in $|z|>t_0^{'}$.

Hence all the zeros of $Q(z)$ lie in $|z|\leq t_0^{'}$.

Let $z=z_0$ be a zero of $P(z)$. Therefore $P(z_0)=0$. Clearly $z_0\neq 0$ as $a_0\neq 0$. Putting $z=\frac{1}{z_0}$ in $Q(z)$ we get that
$Q\left(\frac{1}{z_0}\right)=\left(\frac{1}{z_0}\right)^n.P(z_0)=\left(\frac{1}{z_0}\right)^n\cdot 0=0.$

Therefore $Q\left(\frac{1}{z_0}\right)=0$. So $z=\frac{1}{z_0}$ is a root of $Q(z)=0$. Hence $\left|\frac{1}{z_0}\right|\leq t_0^{'}$ implies that $|z_0|\geq\frac{1}{t_0^{'}}\cdot$

As z_0 is an arbitrary root of $P(z)=0$.

Therefore all the zeros of $P(z)$ lie in $|z|\geq\frac{1}{t_0^{'}}\cdot$ (10)

From (8) and (10) we get that all the zeros of $P(z)$ lie in the proper ring shaped region

$\frac{1}{t_0^{'}}\leq|z|\leq t_0$

where t_0 and $t_0^{'}$ are the greatest positive roots of the equations

$g(t)\equiv\left|a_{N(r)}\right|t^{N(r)+1}-\left(\left|a_{N(r)}\right|+M\right)t^{N(r)}+M=0$
and
$f(t)\equiv|a_0|t^{N(r)+1}-(|a_0|+M^{'})t^{N(r)}+M^{'}=0$

where M and $M^{'}$ are given in the statement of Theorem 1.

This proves the theorem.

Remark: 1 The limit in Theorem 1 is attained by $P(z)=z^2-z-1$. Here $\rho=0,\rho^*=2$ and $N(r)=2\leq(\log r)^{2+\varepsilon}$. For $\varepsilon>0$ and sufficiently large r, all $a_n=0,n\geq 2$. Also $a_0=-1$, $a_1=-1$, $a_2=1$.

Therefore
$M=\max\{|a_0|,|a_1|\}=1$ and $M^{'}=\max\{|a_1|,|a_2|\}=1$
and
$g(t)\equiv|a_2|t^3-(|a_2|+M)t^2+M=0$

i.e., $g(t)\equiv t^3-(1+1)t^2+1=0$

i.e., $g(t)\equiv t^3-2t^2+1=0.$

Again
$f(t)\equiv|a_0|t^3-(|a_0|+M^{'})t^2+M^{'}=0$

i.e., $f(t)\equiv 1.t^3-(1+1)t^2+1=0$

i.e., $f(t)\equiv t^3-2t^2+1=0.$

So $f(t) = 0$ and $g(t) = 0$ represent the same equation. Maximum number of positive roots of $f(t) = 0$ and $g(t) = 0$ are same. Now

$g(t) = 0$

implies that $t^3 - 2t^2 + 1 = 0.$

i.e., $(t-1)\ \ (t^2 - t - 1) = 0.$

Therefore
$t = 1$ and $t = \frac{1 \pm \sqrt{(-1)^2 - 4.1.(-1)}}{2.1} = \frac{1 \pm \sqrt{5}}{2}\cdot$

Hence the positive roots of $g(t) = 0$ are 1 and $\frac{1+\sqrt{5}}{2}\cdot$ So
$t_0 = \max\left\{1, \frac{1+\sqrt{5}}{2}\right\} = \frac{1+\sqrt{5}}{2}\cdot$

Also the maximum positive root of $f(t) = 0$ is
$t_0' = \max\left\{1, \frac{1+\sqrt{5}}{2}\right\} = \frac{1+\sqrt{5}}{2}\cdot$

So in view of Theorem 1 all the zeros of $P(z)$ lie in
$$\frac{1}{t_0'} \leq |z| \leq t_0$$

i.e., $\frac{1}{\frac{1+\sqrt{5}}{2}} \leq |z| \leq \frac{1+\sqrt{5}}{2}$

i.e., $\frac{\sqrt{5}-1}{2} \leq |z| \leq \frac{1+\sqrt{5}}{2}\cdot$

Now the zeros of $P(z)$ are given by solving $z^2 - z - 1 = 0$. Therefore $z = \frac{1+\sqrt{5}}{2}\cdot$ Let us denote the zeros of $P(z)$ by $z_1 = \frac{1+\sqrt{5}}{2}$ and $z_2 = \frac{1-\sqrt{5}}{2} = -\frac{\sqrt{5}-1}{2}\cdot$ Clearly z_1 lies on the upper boundary and z_2 lies on the lower boundary. So the best result is given by $P(z) = z^2 - z - 1$.

Theorem: 2 Let $P(z)$ be an entire function defined by $P(z) = a_0 + a_1 z + a_2 z^2 + \dots\dots\dots + a_n z^n + \dots\dots\dots$with order $\rho = 0$, $a_{N(r)} \neq 0$, $a_0 \neq 0$ and also $a_n \to 0$ for $n > N(r)$ for the disc $|z| \leq \log r$, when r is sufficiently large. Further for some $\rho^* > 0$, $|a_0|(\rho^*)^{N(r)} \geq |a_1|(\rho^*)^{N(r)-1} \geq \dots\dots \geq |a_{N(r)-1}|\rho^* \geq |a_{N(r)}|$.

Then all the zeros of $P(z)$ lie in the ring shaped region
$$\frac{1}{\rho^*\left(1 + \frac{|a_1|}{|a_0|\rho^*}\right)} < |z| < \frac{1}{\rho^*}\left(1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^*)^{N(r)}\right).$$

Proof: For the given entire function $P(z) = a_0 + a_1 z + a_2 z^2 + \dots\dots\dots + a_n z^n + \dots\dots\dots$with $a_n \to 0$ as $n > N(r)$, where r is sufficiently large, $N(r)$ exists and $N(r) \leq (\log r)^{\rho^* + \varepsilon}\cdot$

Therefore
$P(z) \approx a_0 + a_1 z + a_2 z^2 + \dots\dots + a_{N(r)} z^{N(r)}$
as $a_0 \neq 0$, $a_{N(r)} \neq 0$ and $a_n \to 0$ for $n > N(r)$.

Let us consider
$$\begin{aligned} R(z) &= (\rho^*)^{N(r)} P\left(\frac{z}{\rho^*}\right) \\ &\approx (\rho^*)^{N(r)}\left(a_0 + a_1 \frac{z}{\rho^*} + a_2 \frac{z^2}{(\rho^*)^2} + \cdots\cdots + a_{N(r)} \frac{z^{N(r)}}{(\rho^*)^{N(r)}}\right) \\ &= a_0(\rho^*)^{N(r)} + a_1(\rho^*)^{N(r)-1} z + \dots\dots\dots + a_{N(r)} z^{N(r)}. \end{aligned}$$

Therefore

$$|R(z)| \geq |a_{N(r)}||z|^{N(r)} - |a_0(\rho^*)^{N(r)} + a_1(\rho^*)^{N(r)-1} z + \dots\dots + a_{N(r)-1}\rho^* z^{N(r)-1}|. \quad (11)$$

Now by the given condition $|a_0|(\rho^*)^{N(r)} \geq |a_1|(\rho^*)^{N(r)-1} \geq \ldots\ldots \geq |a_{N(r)-1}|\rho^* \geq |a_{N(r)}|$ provided $|z| \neq 0$, we obtain that

$$\left|a_0(\rho^*)^{N(r)} + a_1(\rho^*)^{N(r)-1}z + \ldots\ldots + a_{N(r)-1}\rho^* z^{N(r)-1}\right| \leq |a_0|(\rho^*)^{N(r)} + \ldots\ldots + |a_{N(r)-1}|\,\rho^*\,|z|^{N(r)-1}$$
$$\leq |a_0|(\rho^*)^{N(r)}|z|^{N(r)}\left(\frac{1}{|z|} + \cdots\cdots + \frac{1}{|z|^{N(r)}}\right).$$

Therefore on $|z| \neq 0$,

$$-\left|a_0(\rho^*)^{N(r)} + a_1(\rho^*)^{N(r)-1}z + \ldots\ldots + a_{N(r)-1}\rho^* z^{N(r)-1}\right| \geq -|a_0|(\rho^*)^{N(r)}|z|^{N(r)}\left(\frac{1}{|z|} + \cdots\cdots + \frac{1}{|z|^{N(r)}}\right). \quad (12)$$

Therefore using (12) we get from (11) that

$$|R(z)| \geq |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}|z|^{N(r)}\left(\frac{1}{|z|} + \cdots\cdots + \frac{1}{|z|^{N(r)}}\right)$$
$$\geq |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}|z|^{N(r)}\left(\frac{1}{|z|} + \cdots\cdots + \frac{1}{|z|^{N(r)}} + \cdots\cdots\right)$$
$$= |z|^{N(r)}\left[|a_{N(r)}| - |a_0|(\rho^*)^{N(r)}\sum_{k=1}^{\infty}\frac{1}{|z|^k}\right].$$

Clearly $\sum_{k=1}^{\infty}\frac{1}{|z|^k}$ is a geometric series which is convergent for $\frac{1}{|z|} < 1$ i.e., for $|z| > 1$ and converges to

$$\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}.$$

Therefore

$$\sum_{k=1}^{\infty}\frac{1}{|z|^k} = \frac{1}{|z|-1} \text{ if } |z| > 1.$$

Hence we get from above that for $|z| > 1$

$$|R(z)| > |z|^{N(r)}\left(|a_{N(r)}| - (\rho^*)^{N(r)}|a_0|\frac{1}{|z|-1}\right).$$

Now for $|z| > 1$,

$|R(z)| > 0$ if $|z|^{N(r)}\left(|a_{N(r)}| - (\rho^*)^{N(r)}|a_0|\frac{1}{|z|-1}\right) \geq 0$

i.e., if $|a_{N(r)}| - (\rho^*)^{N(r)}|a_0|\frac{1}{|z|-1} \geq 0$

i.e., if $|a_{N(r)}| \geq (\rho^*)^{N(r)}\frac{|a_0|}{|z|-1}$

i.e., if $|z| - 1 \geq (\rho^*)^{N(r)}\frac{|a_0|}{|a_{N(r)}|}$

i.e., if $|z| \geq 1 + (\rho^*)^{N(r)}\frac{|a_0|}{|a_{N(r)}|} > 1.$

Therefore

$$|R(z)| > 0 \text{ if } |z| \geq 1 + (\rho^*)^{N(r)}\frac{|a_0|}{|a_{N(r)}|}.$$

So all the zeros of $R(z)$ lie in

$$|z| < 1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^*)^{N(r)}.$$

Let z_0 be an arbitrary zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^* z_0$ in $R(z)$ we have

$$R(\rho^* z_0) = (\rho^*)^{N(r)} \cdot P(z_0) = (\rho^*)^{N(r)}.0 = 0.$$

Hence $z = \rho^* z_0$ is a zero of $R(z)$. Therefore

$$|\rho^* z_0| < 1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^*)^{N(r)}$$

i.e., $|z_0| < \frac{1}{\rho^*}\left(1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^*)^{N(r)}\right)$.

Since z_0 is any zero of $P(z)$ therefore all the zeros of $P(z)$ lie in

$$|z| < \frac{1}{\rho^*}\left(1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^*)^{N(r)}\right). \tag{13}$$

Again let us consider

$$F(z) = (\rho^*)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^* z}\right).$$

Now

$$\begin{aligned} F(z) &= (\rho^*)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^* z}\right) \\ &\approx (\rho^*)^{N(r)} z^{N(r)} \left\{a_0 + \frac{a_1}{\rho^* z} + \cdots\cdots + \frac{a_{N(r)}}{(\rho^* z)^{N(r)}}\right\} \\ &= a_0 (\rho^*)^{N(r)} z^{N(r)} + a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \cdots\cdots + a_{N(r)}. \end{aligned}$$

Therefore

$$|F(z)| \geq |a_0| (\rho^*)^{N(r)} |z|^{N(r)} - \left|a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \cdots\cdots + a_{N(r)}\right|.$$

Again

$$\begin{aligned} \left|a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \cdots\cdots + a_{N(r)}\right| &\leq |a_1| (\rho^*)^{N(r)-1} |z|^{N(r)-1} + \cdots\cdots + \left|a_{N(r)}\right| \\ &\leq |a_1| (\rho^*)^{N(r)-1} \left(|z|^{N(r)-1} + \cdots\cdots + |z| + 1\right) \end{aligned}$$

provided $|z| \neq 0$. So

$$\left|a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \cdots\cdots + a_{N(r)}\right| \leq |a_1| (\rho^*)^{N(r)-1} |z|^{N(r)} \left(\frac{1}{|z|} + \cdots\cdots + \frac{1}{|z|^{N(r)}}\right).$$

So for $|z| \neq 0$,

$$\begin{aligned} |F(z)| &\geq |a_0| (\rho^*)^{N(r)} |z|^{N(r)} - |a_1| (\rho^*)^{N(r)-1} |z|^{N(r)} \left(\frac{1}{|z|} + \cdots\cdots + \frac{1}{|z|^{N(r)}}\right) \\ &= (\rho^*)^{N(r)-1} |z|^{N(r)} \left[|a_0|\rho^* - |a_1| \left(\frac{1}{|z|} + \cdots\cdots + \frac{1}{|z|^{N(r)}}\right)\right]. \end{aligned}$$

Therefore for $|z| \neq 0$,

$$|F(z)| > (\rho^*)^{N(r)-1} |z|^{N(r)} \left[|a_0|\rho^* - |a_1| \sum_{k=1}^{\infty} \frac{1}{|z|^k}\right]. \tag{14}$$

The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\frac{1}{|z|} < 1$$

i.e., for $|z| > 1$

and converges to

$$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z|-1} \text{ if } |z| > 1. \tag{15}$$

Using (14) and (15) we have for $|z| > 1$,

$$|F(z)| > (\rho^*)^{N(r)-1}|z|^{N(r)}\left[|a_0|\rho^* - \frac{|a_1|}{|z|-1}\right].$$

Hence for $|z| > 1$,

$$|F(z)| > 0 \ \text{ if } \ (\rho^*)^{N(r)-1}|z|^{N(r)}\left[|a_0|\rho^* - \frac{|a_1|}{|z|-1}\right] \geq 0$$

i.e., if $|a_0|\rho^* - \frac{|a_1|}{|z|-1} \geq 0$

i.e., if $|a_0|\rho^* \geq \frac{|a_1|}{|z|-1}$

i.e., if $|z| \geq 1 + \frac{|a_1|}{|a_0|\rho^*} > 1$.

Therefore

$|F(z)| > 0$ for $|z| \geq 1 + \frac{|a_1|}{|a_0|\rho^*}\cdot$

So $F(z)$ does not vanish in

$$|z| \geq 1 + \frac{|a_1|}{|a_0|\rho^*}\cdot$$

Equivalently all the zeros of $F(z)$ lie in

$$|z| < 1 + \frac{|a_1|}{|a_0|\rho^*}\cdot$$

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $a_0 \neq 0$ and $z_0 \neq 0$.

Now let us put $z = \frac{1}{\rho^* z_0}$ in $F(z)$. So we have

$$\begin{aligned} F\left(\frac{1}{\rho^* z_0}\right) &= (\rho^*)^{N(r)}\left(\frac{1}{\rho^* z_0}\right)^{N(r)}.P(z_0) \\ &= \left(\frac{1}{z_0}\right)^{N(r)}.0 = 0 \end{aligned}$$

Therefore $z = \frac{1}{\rho^* z_0}$ is a root of $F(z)$.

Hence

$$\left|\frac{1}{\rho^* z_0}\right| < 1 + \frac{|a_1|}{|a_0|\rho^*}$$

i.e., $\left|\frac{1}{z_0}\right| < \rho^*\left(1 + \frac{|a_1|}{|a_0|\rho^*}\right)$,

i.e., $|z_0| > \frac{1}{\rho^*\left(1+\frac{|a_1|}{|a_0|\rho^*}\right)}$,

As z_0 is an arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie on

$$|z| > \frac{1}{\rho^*\left(1+\frac{|a_1|}{|a_0|\rho^*}\right)}\cdot \qquad (16)$$

From (13) and (16) we get that all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho^*\left(1 + \frac{|a_1|}{|a_0|\rho^*}\right)} < |z| < \frac{1}{\rho^*}\left(1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^*)^{N(r)}\right)$$

where

$|a_0|(\rho^*)^{N(r)} \geq |a_1|(\rho^*)^{N(r)-1} \geq \ldots\ldots \geq |a_{N(r)}|$

for some $\rho^* > 0$.

This proves the theorem.

Corollary: 1 From Theorem 2 we can easily conclude that all the zeros of $P(z) = a_0 + a_1 z + \ldots\ldots + a_n z^n$ of degree n, $|a_n| \neq 0$ with the property $|a_0| \geq |a_1| \geq \ldots\ldots \geq |a_n|$ lie in the proper ring shaped region

$$\frac{1}{\left(1+\frac{|a_1|}{|a_0|}\right)} < |z| < \left(1+\frac{|a_0|}{|a_n|}\right)$$

just on putting $\rho^* = 1$.

Theorem: 3 Let $P(z)$ be an entire function with order $\rho = 0$. For sufficiently large values of r in the disc $|z| \leq \log r$, the Taylor's series expansion of $P(z)$

$P(z) = a_0 + a_{p_1} z^{p_1} + a_{p_2} z^{p_2} + \ldots\ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}$, $a_0 \neq 0$ be such that $1 \leq p_1 < p_2 < \ldots\ldots < p_m \leq N(r) - 1$, p_i's are integers and for some $\rho^* > 0$,

$|a_0|(\rho^*)^{N(r)} \geq |a_{p_1}|(\rho^*)^{N(r)-p_1} \geq \ldots\ldots \geq |a_{p_m}|(\rho^*)^{N(r)-p_m} \geq |a_{N(r)}|$.

Then all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho^* t_0'} < |z| < \frac{1}{\rho^*} t_0$$

where t_0 and t_0' are the unique positive roots of the equations

$$g(t) \equiv |a_{N(r)}| t^{N(r)-p_m} - |a_{N(r)}| t^{N(r)-p_m-1} - |a_0|(\rho^*)^{N(r)} = 0$$

and

$$f(t) \equiv |a_0|(\rho^*)^{p_1} t^{p_1} - |a_0|(\rho^*)^{p_1} t^{p_1-1} - |a_{p_1}| = 0$$

respectively.

Proof: Let

$$P(z) = a_0 + a_{p_1} z^{p_1} + a_{p_2} z^{p_2} + \ldots\ldots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)},\ |a_{N(r)}| \neq 0. \tag{17}$$

Also for some $\rho^* > 0$,

$|a_0|(\rho^*)^{N(r)} \geq |a_{p_1}|(\rho^*)^{N(r)-p_1} \geq \ldots\ldots \geq |a_{p_m}|(\rho^*)^{N(r)-p_m} \geq |a_{N(r)}|$.

Let us consider

$$\begin{aligned} R(z) &= (\rho^*)^{N(r)} P\left(\frac{z}{\rho^*}\right) \\ &= (\rho^*)^{N(r)} \left\{a_0 + a_{p_1} \frac{z^{p_1}}{(\rho^*)^{p_1}} + \cdots\cdots + a_{p_m} \frac{z^{p_m}}{(\rho^*)^{p_m}} + a_{N(r)} \frac{z^{N(r)}}{(\rho^*)^{N(r)}}\right\} \\ &= a_0(\rho^*)^{N(r)} + a_{p_1}(\rho^*)^{N(r)-p_1} z^{p_1} + \ldots\ldots + a_{p_m}(\rho^*)^{N(r)-p_m} z^{p_m} + a_{N(r)} z^{N(r)}. \end{aligned}$$

Therefore

$$|R(z)| \geq |a_{N(r)} z^{N(r)}| - |a_0(\rho^*)^{N(r)} + a_{p_1}(\rho^*)^{N(r)-p_1} z^{p_1} + \ldots. + a_{p_m}(\rho^*)^{N(r)-p_m} z^{p_m}|. \tag{18}$$

Now for $|z| \neq 0$,

$$\begin{aligned} &|a_0(\rho^*)^{N(r)} + a_{p_1}(\rho^*)^{N(r)-p_1} z^{p_1} + \ldots.. + a_{p_m}(\rho^*)^{N(r)-p_m} z^{p_m}| \\ &\quad \leq |a_0|(\rho^*)^{N(r)} + |a_{p_1}|(\rho^*)^{N(r)-p_1}|z|^{p_1} + \cdots\cdots + |a_{p_m}|(\rho^*)^{N(r)-p_m}|z|^{p_m} \\ &\quad \leq |a_0|(\rho^*)^{N(r)}(1 + |z|^{p_1} + \cdots\cdots + |z|^{p_m}) \\ &\quad = |a_0|(\rho^*)^{N(r)}|z|^{p_m+1}\left(\frac{1}{|z|} + \cdots\cdots + \frac{1}{|z|^{p_m+1-p_2}} + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}}\right). \end{aligned} \tag{19}$$

Using (18) and (19), we have for $|z| \neq 0$

$$
\begin{aligned}
|R(z)| &\geq |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}|z|^{p_m+1}\left(\frac{1}{|z|} + \cdots + \frac{1}{|z|^{p_m+1-p_2}} + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}}\right)\\
&> |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}|z|^{p_m+1}\left(\frac{1}{|z|} + \cdots + \frac{1}{|z|^{p_m+1-p_2}} + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} + \cdots\right)\\
&= |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}|z|^{p_m+1}\sum_{k=1}^{\infty}\frac{1}{|z|^k}. \qquad (20)
\end{aligned}
$$

The geometric series $\sum_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for

$\frac{1}{|z|} < 1$

i.e., for $|z| > 1$

and converges to

$$\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}.$$

Therefore

$$\sum_{k=1}^{\infty}\frac{1}{|z|^k} = \frac{1}{|z|-1} \text{ for } |z| > 1.$$

So on $|z| > 1$,

$|R(z)| > 0$ if $|a_{N(r)}||z|^{N(r)} - \frac{|a_0|(\rho^*)^{N(r)}|z|^{p_m+1}}{|z|-1} \geq 0$

i.e., if $|a_{N(r)}||z|^{N(r)} \geq \frac{|a_0|(\rho^*)^{N(r)}|z|^{p_m+1}}{|z|-1}$

i.e., if $|a_{N(r)}||z|^{N(r)+1} - |a_{N(r)}||z|^{N(r)} \geq |a_0|(\rho^*)^{N(r)}|z|^{p_m+1}$

i.e., if $|z|^{p_m+1}\left(|a_{N(r)}||z|^{N(r)-p_m} - |a_{N(r)}||z|^{N(r)-p_m-1} - |a_0|(\rho^*)^{N(r)}\right) \geq 0.$

Let us consider

$$g(t) \equiv |a_{N(r)}|t^{N(r)-p_m} - |a_{N(r)}|t^{N(r)-p_m-1} - |a_0|(\rho^*)^{N(r)} = 0.$$

Clearly, $g(t) = 0$ has one positive root because the maximum number of changes in sign in $g(t)$ is one and $g(0) = -|a_0|\rho^{N(r)}$ is $-ve$, $g(\infty)$ is $+ve$. Let t_0 be the positive root of $g(t) = 0$ and $t_0 > 1$. Clearly for $t > t_0$, $g(t) \geq 0$. If not, for some $t_1 > t_0$, $g(t_1) < 0$. Then $g(t_1) < 0$ and $g(\infty) > 0$. Therefore $g(t) = 0$ must have another positive root in (t_1, ∞) which gives a contradiction.

Hence for $t \geq t_0$, $g(t) \geq 0$ and $t_0 > 1$. So $|R(z)| > 0$ for $|z| \geq t_0$.

Thus $R(z)$ does not vanish in $|z| \geq t_0$.

Hence all the zeros of $R(z)$ lie in $|z| < t_0$.

Let $z = z_0$ be any zero of $P(z)$. So $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^* z_0$ in $R(z)$ we have

$$R(\rho^* z_0) = (\rho^*)^{N(r)} \cdot P(z_0) = (\rho^*)^{N(r)}.0 = 0.$$

Therefore $R(\rho^* z_0) = 0$ and so $z = \rho^* z_0$ is a zero of $R(z)$ and consequently $|\rho^* z_0| < t_0$ which implies $|z_0| < \frac{t_0}{\rho^*}$.

As z_0 is an arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie in $|z| < \frac{t_0}{\rho^*}\cdot$ (21)

Again let us consider

$$F(z) = (\rho^*)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^* z}\right)\cdot$$

Now

$$F(z) = (\rho^*)^{N(r)} z^{N(r)} \left\{a_0 + a_{p_1}\frac{1}{(\rho^*)^{p_1} z^{p_1}} + \cdots + a_{p_m}\frac{1}{(\rho^*)^{p_m} z^{p_m}} + a_{N(r)}\frac{1}{(\rho^*)^{N(r)} z^{N(r)}}\right\}$$

$$= a_0(\rho^*)^{N(r)} z^{N(r)} + a_{p_1}(\rho^*)^{N(r)-p_1} z^{N(r)-p_1} + \cdots + a_{p_m}(\rho^*)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}.$$

Also

$$\left|a_{p_1}(\rho^*)^{N(r)-p_1} z^{N(r)-p_1} + \cdots + a_{p_m}(\rho^*)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}\right|$$
$$\leq \left|a_{p_1}\right|(\rho^*)^{N(r)-p_1}|z|^{N(r)-p_1} + \cdots + \left|a_{p_m}\right|(\rho^*)^{N(r)-p_m}|z|^{N(r)-p_m} + \left|a_{N(r)}\right|$$
$$\leq \left|a_{p_1}\right|(\rho^*)^{N(r)-p_1}\left(|z|^{N(r)-p_1} + |z|^{N(r)-p_2} + \cdots + |z|^{N(r)-p_m} + 1\right).$$

So for $|z| \neq 0$,

$$|F(z)| \geq |a_0|(\rho^*)^{N(r)}|z|^{N(r)} - \left|a_{p_1}(\rho^*)^{N(r)-p_1} z^{N(r)-p_1} + \cdots + a_{p_m}(\rho^*)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}\right|$$

$$\geq |a_0|(\rho^*)^{N(r)}|z|^{N(r)} - \left|a_{p_1}\right|(\rho^*)^{N(r)-p_1}\left(|z|^{N(r)-p_1} + |z|^{N(r)-p_2} + \cdots + |z|^{N(r)-p_m} + 1\right)$$

$$= |a_0|(\rho^*)^{N(r)}|z|^{N(r)} - \left|a_{p_1}\right|(\rho^*)^{N(r)-p_1}|z|^{N(r)-p_1+1}\left(\frac{1}{|z|} + \frac{1}{|z|^{p_2-p_1+1}} + \cdots + \frac{1}{|z|^{N(r)-p_1+1}}\right)$$

i.e., on $|z| \neq 0$,

$$|F(z)| > |a_0|(\rho^*)^{N(r)}|z|^{N(r)} - \left|a_{p_1}\right|(\rho^*)^{N(r)-p_1}|z|^{N(r)-p_1+1}\left(\sum_{k=1}^{\infty}\frac{1}{|z|^k}\right).$$

The geometric series $\sum_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for

$$\frac{1}{|z|} < 1$$

i.e., for $|z| > 1$

and converges to

$$\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}\cdot$$

Therefore

$$\sum_{k=1}^{\infty}\frac{1}{|z|^k} = \frac{1}{|z|-1} \text{ for } |z| > 1.$$

Therefore for $|z| > 1$

$$|F(z)| > |a_0|(\rho^*)^{N(r)}|z|^{N(r)} - \left|a_{p_1}\right|(\rho^*)^{N(r)-p_1}|z|^{N(r)-p_1+1}\left(\frac{1}{|z|-1}\right)$$
$$= (\rho^*)^{N(r)-p_1}\left((\rho^*)^{p_1}|a_0||z|^{N(r)} - \left|a_{p_1}\right|\frac{|z|^{N(r)-p_1+1}}{|z|-1}\right)$$
$$= (\rho^*)^{N(r)-p_1}|z|^{N(r)-p_1+1}\left(|a_0|(\rho^*)^{p_1}|z|^{p_1-1} - \frac{\left|a_{p_1}\right|}{|z|-1}\right).$$

For $|z| > 1$,

$|F(z)| > 0$ if $|a_0|(\rho^*)^{p_1}|z|^{p_1-1} - \frac{|a_{p_1}|}{|z|-1} \geq 0$

i.e., if $|a_0|(\rho^*)^{p_1}|z|^{p_1-1} \geq \frac{|a_{p_1}|}{|z|-1}$

i.e., if $|a_0|(\rho^*)^{p_1}|z|^{p_1} - |a_0|(\rho^*)^{p_1}|z|^{p_1-1} - |a_{p_1}| \geq 0.$ (22)

Therefore on $|z| > 1$, $|F(z)| > 0$ if (22) holds.

Let us consider
$f(t) \equiv |a_0|(\rho^*)^{p_1}t^{p_1} - |a_0|(\rho^*)^{p_1}t^{p_1-1} - |a_{p_1}| = 0.$

Clearly $f(t) = 0$ has exactly one positive root and is greater than one. Let $t_0^{'}$ be the positive root of $f(t) = 0$.

Therefore $t_0^{'} > 1$. Obiviously if $t \geq t_0^{'}$ then $f(t) \geq 0$. So for $|F(z)| > 0$, $|z| \geq t_0^{'}$.

Therefore $F(z)$ does not vanish in $|z| \geq t_0^{'}$.

Hence all the zeros of $F(z)$ lie in $|z| < t_0^{'}$.

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Now putting $z = \frac{1}{\rho^* z_0}$ in $F(z)$ we obtain that

$$F\left(\frac{1}{\rho^* z_0}\right) = (\rho^*)^{N(r)}\left(\frac{1}{\rho^* z_0}\right)^{N(r)} \cdot P(z_0)$$
$$= \left(\frac{1}{z_0}\right)^{N(r)} . P(z_0)$$
$$= \left(\frac{1}{z_0}\right)^{N(r)} . 0$$
$$= 0.$$

Therefore $z = \frac{1}{\rho^* z_0}$ is a zero of $F(z)$. Now
$\left|\frac{1}{\rho^* z_0}\right| < t_0^{'}$

i.e., $\left|\frac{1}{z_0}\right| < \rho^* t_0^{'}$

i.e., $|z_0| > \frac{1}{\rho^* t_0^{'}}\cdot$

As z_0 is an arbitrary zero of $P(z)$ therefore we obtain that all the zeros of $P(z)$ lie in $|z| > \frac{1}{\rho^* t_0^{'}}\cdot$ (23)

Using (21) and (23) we get that all the zeros of $P(z)$ lie in the ring shaped region
$$\frac{1}{\rho^* t_0^{'}} < |z| < \frac{t_0}{\rho^*}$$

where t_0, $t_0^{'}$ are the unique positive roots of the equations $g(t) = 0$ and $f(t) = 0$ respectively whose form is given in the statement of Theorem 3.

This proves the theorem.

Corollary: 2 In view of Theorem 3 we may state that all the zeros of the polynomial

$P(z) = a_0 + a_{p_1}z^{p_1} + \ldots\ldots + a_{p_m}z^{p_m} + a_n z^n$ of degree n with $1 \leq p_1 < p_2 \ldots\ldots < p_m \leq n-1$, p_i's are integers such that $|a_0| \geq |a_{p_1}| \geq \ldots\ldots \geq |a_n|$ lie in the ring shaped region $\frac{1}{t_0^{'}} < |z| < t_0$
where t_0 , $t_0^{'}$ are the unique positive roots of the equations

$g(t) \equiv |a_n|t^{n-p_m} - |a_n|t^{n-p_m-1} - |a_0| = 0$
and
$f(t) \equiv |a_0|t^{p_1} - |a_0|t^{p_1-1} - |a_{p_1}| = 0$

respectively just substituting $\rho^* = 1$.

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