USING TAYLOR EXPANSION TO PARTIAL FRACTIONS DECOMPOSITION

Mircea Ion Cîrnu*
Polytechnic University of Bucharest, Romania.

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ABSTRACT

The automatic Taylor development, based on discrete convolution and deconvolution, is used to partial fractions decomposition.

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1. INTRODUCTION

Using the automatic Taylor expansion, we give some new numerical methods to partial fractions decomposition of rational functions. Firstly, we consider the Taylor differential transform that associates to a function the sequence of coefficients of its Taylor series or only a finite number of them. The Taylor transform of a product, respective quotient of functions is computed by discrete convolution, respective deconvolution, of the corresponding sequences of coefficients. This eliminates the algebraic calculation and the Taylor development becomes automatic.

As is well known, partial fractions decomposition of rational functions applied to the integration of these functions. Also it is used to determine the inverse Laplace transforms of such functions and thus to physical system theory. So far, there are numerous attempts to find a simple numerical method for this decomposition. In the paper [7] are presented nine such different methods. Using the automatic Taylor expansion, we give a new method for partial fractions decomposition of rational functions. It extend so called “particular values method”. These particular values, used to obtain the numerators of the decomposition, are even the poles of the rational function. When the poles are complex numbers, must work in complex, but there are cases in which it is possible to work only with real numbers. See Example 8.3 below. Our method of partial fractions decomposition is simpler than the already known methods, covers all cases and can be applied even to difficult examples. We give two theorems, one for a single pole of arbitrary multiplicity and the second, for a pair of poles of the same multiplicity. The second theorem can be used in the case of two complex or irrational conjugate poles of the same multiplicity. In the latest example 8.6, we do the partial fractions decomposition by both theorems. Based on the convenient decomposition, we compute the indefinite integral and the inverse Laplace transform of the considered function in that example.

Other applications of the discrete convolution and deconvolution were given by the Author in the works [1]-[5].

2. TAYLOR DIFFERENTIAL TRANSFORM

We call Taylor differential transform of an indefinite differentiable function $f(x)$, the sequence of functions

$$ T\left(f(x)\right) = \left\{ \frac{1}{n!} f^{(n)}(x); n = 0,1,2,\ldots \right\} $$

and Taylor transform in a point $x = x_0$, the sequence of numbers

$$ T_{x_0}\left(f(x)\right) = \left\{ c_n = \frac{1}{n!} f^{(n)}(x_0), n = 0,1,\ldots \right\} $$

For a natural number $m$, we also will denote

$$ T^m\left(f(x)\right) = \left\{ \frac{1}{n!} f^{(n)}(x); n = 0,1,\ldots,m \right\}, \ T^m_{x_0}\left(f(x)\right) = \left\{ c_n : n = 0,1,\ldots,m \right\} $$

and

$$ P^m_{x_0}\left(f(x)\right) = P^m_{x_0}\left(c_n\right) = \sum_{n=0}^{m} c_n (x-x_0)^n $$

the Taylor polynomial of degree m.
If we denote \(a \cdot b = (a_n b_n)\), the usual product of two sequences \(a = (a_n)\) and \(b = (b_n)\), the derivatives of the function \(f(x)\) are given by the formula \(f^{(n)}(x) = (n!) T(f(x))\) and there values in the point \(x = x_0\) by the formula \(f^{(n)}(x_0) = (n!) T_{x_0} (f(x))\).

### 3. EXAMPLES OF TAYLOR TRANSFORMS

Some examples of Taylor transforms in an arbitrary point and in origin, are:

\[
T(x^m) = \left\{ x^m, mx^{m-1}, \ldots, \left(\frac{m}{n}\right)x^{m-n}, \ldots, mx, 1, 0, 0, \ldots \right\}, \text{ for } m = 1, 2, \ldots
\]

Particularly, \(T(x) = (x, 1, 0, 0, \ldots)\), and \(T(x^2) = (x^2, 2x, 1, 0, 0, \ldots)\).

\[
T\left(\frac{1}{x}\right) = \left(\frac{1}{x}, -\frac{1}{x^2}, \frac{1}{x^3}, -\frac{1}{x^4}, \ldots, (-1)^n x^{-n+1}, \ldots \right), \text{ for } x \neq 0,
\]

\[
T(e^x) = \left(1, \frac{1}{1!}, \frac{1}{2!}, \ldots, \frac{1}{n!}, \ldots \right)
\]

\[
T(\ln x) = \left(\ln x, -\frac{1}{x^2}, \frac{1}{2x^3}, -\frac{1}{3x^4}, \ldots, (-1)^{n-1} x^{-n}, \ldots \right), \text{ for } x > 0.
\]

\[
T(\sin x) = \left(\sin x, \cos x, -\frac{1}{2!} \sin x, -\frac{1}{3!} \cos x, -\frac{1}{4!} \sin x, \ldots \right)
\]

\[
T(\cos x) = \left(\cos x, -\sin x, -\frac{1}{2} \cos x, -\frac{1}{3!} \sin x, \frac{1}{4!} \cos x, \ldots \right).
\]

Particularly, \(T_0 (a) = (a, 0, 0, \ldots)\), \(T_0 (x) = (0, 1, 0, 0, \ldots)\), \(T_0 (x^2) = (0, 0, 1, 0, 0, \ldots)\),

\[
T_0 (x^m) = \left(0, \ldots, 0, 1, 0, 0, \ldots \right), \quad T_0 (e^x) = \left(1, \frac{1}{1!}, \frac{1}{2!}, \ldots, \frac{1}{n!}, \ldots \right),
\]

\[
T_0 (\sin x) = \left(0, 1, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, \ldots \right), \quad T_0 (\cos x) = \left(1, 0, -\frac{1}{2}, 0, \frac{1}{4!}, \ldots \right).
\]

**Remark:** These formulas was given in the author book [1], pg. 84. However, unfortunately, in [1] the two formulas for \(T\left(\frac{1}{x}\right)\) and \(T(\ln x)\) contain mistakes.

### 4. OPERATIONS WITH SEQUENCES

We consider (see [1] and [6]) the Cauchy product or (truncated or short, linear discrete) convolution \(c = (c_n) = a * b = \left(\sum_{k=0}^{n} a_k b_{n-k}\right)\) of the sequences \(a = (a_n)\) and \(b = (b_n)\).

The convolution can be computed by the multiplication algorithm

\[
\begin{array}{cccccc}
& a_0 & a_1 & a_2 & \cdots \\
& b_0 & b_1 & b_2 & \cdots \\
\hline
a_0 b_0 & a_1 b_0 & a_2 b_0 & \cdots \\
a_0 b_1 & a_1 b_1 & a_2 b_1 & \cdots \\
a_0 b_2 & a_1 b_2 & a_2 b_2 & \cdots \\
\end{array}
\]

\[
c_0 = a_0 b_0, \quad c_1 = a_1 b_0 + a_0 b_1, \quad c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2, \ldots
\]
For $m$ natural numbers one denotes $a^{*m} = a \ast a \ast \cdots \ast a$.

If the sequences $a = (a_n)$ with $a_0 \neq 0$ and $c = (c_n)$ are given, then the sequence $b = (b_n)$ such that $c = a \ast b$ can be determined by the deconvolution formula (see [1] and [6])

$$b = c / a = \frac{1}{a_0} \left( c_0, c_n - \sum_{k=1}^{n} a_k b_{n-k} : n = 1, 2, \ldots \right).$$

The deconvolution can be computed by the division algorithm

$$\begin{array}{ccc}
    c_0 & c_1 & \cdots & c_n & \cdots \\
    a_0 & a_1 & \cdots & a_n & \cdots \\
    \hline
    b_0 & b_1 & \cdots & b_n & \cdots
\end{array}$$

$$\begin{array}{ccc}
    c_0 \cdot a_0 & a_1 - a_0 b_0 & \cdots & c_n - a_0 b_n & \cdots \\
    c_0 = c_1 - a_1 b_0 & \cdots & a_n b_1 & \cdots
\end{array}$$

On the base of the usual (long) convolution and deconvolution, which are used to compute the product and the quotient (with rest) of two polynomials, introduced in MATLAB by the instructions `conv` and `deconv`, it is possible also to consider the truncated convolution and deconvolution by the instructions `tconv` and `tdeconv`, in the following manner:

```matlab
function [l,c] = size(a);
res = zeros(1,c);
for i = 1:c
    l = size(a);
    temp = conv(a,b);
    res(i) = deconv(a,b(1:1:c-i+1));
end
end
```

In the examples of partial fractions decomposition given below, we present several algorithms of convolution and deconvolution, the others being omitted.

5. TAYLOR TRANSFORMS OF PRODUCTS AND QUOTIENS OF FUNCTIONS

If a function $f(x)$ contains products and quotients of other functions, then to determine its Taylor transform we can use the discrete convolution and deconvolution, as can see from the following theorem and its consequence.

**Theorem 1:** If $f(x)$ and $g(x)$ are indefinite differentiable functions, then

$$T(f(x)g(x)) = T(f(x)) * T(g(x)).$$

**Proof:** Using Leibniz formula for derivatives of a product of functions,

$$T(f(x)g(x)) = \frac{1}{n!}(f(x)g(x))^{(n)} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x)$$

$$= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) \frac{1}{(n-k)!} g^{(n-k)}(x) = \frac{1}{n!} f^{(n)}(x) * \frac{1}{n!} g^{(n)}(x) = T(f(x)) * T(g(x)).$$

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Corollary: If \( f(x) \) and \( g(x) \neq 0 \) are indefinite differentiable functions, then
\[
T(f(x)/g(x)) = T(f(x))/T(g(x)).
\] (2)

Proof: From theorem, \( T(f(x)) = T((f(x)/g(x))g(x)) = T(f(x)/g(x))\cdot T(g(x)) \), and therefore it results the above formula.

Remarks: 1) Obviously, the formulas from above Theorem and its Corollary, are also true for Taylor transformation \( T_{x_0} \) Concentrated in a point \( x = x_0 \).
2) If we want to calculate only derivatives up to order \( m \), then are used the Taylor transforms \( f(x)^m \) and \( g(x)^m \), sequences of the same finite length \( m + 1 \).

6. PARTIAL FRACTIONS DEVELOPMENT

We will use the automatic Taylor development, based on discrete convolution and deconvolution, to get a numerical automatic method for partial fractions development of the rational functions. It is based on the following two theorems.

Theorem: 2 If \( f(x) \) is a function of the form
\[
f(x) = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + g(x) = \sum_{k=0}^{m-1} A_k (x-a)^k \frac{1}{(x-a)^m} + g(x),
\] (3)

where \( m \neq 0 \) is a natural number, \( a \) and \( A_k \), for \( k = 0,1,\ldots,m-1 \), are complex numbers and \( g(x) \) is a function with derivative of \( m-1 \) order in \( a \), then the coefficients \( A_k \) can be computed by the relation
\[
A_k = \frac{1}{k!} \lim_{x \to a} \frac{d^k}{dx^k} \left[(x-a)^m f(x)\right] = c_k, \quad k = 0,1,\ldots,m-1,
\] (4)
hence the numbers \( A_k \) are the coefficients \( c_k \) of the Taylor development of the function \( (x-a)^m f(x) \) in the point \( x = a \), namely \( \sum_{k=0}^{m-1} A_k (x-a)^k = P_{a}^{m-1}((x-a)^m f(x)) \).

Proof: In conformity with (3), we have
\[
c_n = \frac{1}{n!} \lim_{x \to a} \frac{d^n}{dx^n} \left[(x-a)^m f(x)\right] = \frac{1}{n!} \lim_{x \to a} \frac{d^n}{dx^n} \left[\sum_{k=0}^{m-1} A_k (x-a)^k + (x-a)^m g(x)\right].
\]
For \( n = 0 \) results \( c_0 = A_0 \). For \( 1 \leq n \leq m-1 \), it results
\[
c_n = \frac{1}{n!} \lim_{x \to a} \sum_{k=0}^{m-1} A_k k(k-1)\cdots(k-n+1)(x-a)^{k-n} + \frac{1}{n!} \lim_{x \to a} \sum_{j=0}^{m-1} \frac{n!(m-1)\cdots(m-j+1)(x-a)^{m-j}}{j!} \frac{d^{m-j}}{dx^{m-j}} g(x) = A_n.
\]
Remark: As is well known, if \( f(x) \) is a rational function, its nominator has degree less than the denominator and \( a \) is a pole of multiplicity \( m \) of \( f(x) \), then the function has the form (3), therefore the Theorem 2 work.

Corollary: If \( f(x) \) is a rational function with degree of the numerator smaller than the denominator, having the distinct poles \( x_1,\ldots,x_p \) of multiplicities \( m_1,\ldots,m_p \), then \( f(x) \) has the partial fractions development
\[
f(x) = \sum_{j=1}^{p} P_{x_j}^{m_j-1}[(x-x_j)^m f(x)] \frac{1}{(x-x_j)^{m_j}},
\] (5)
where \( P_{x_j}^{m_j-1}[(x-x_j)f(x)] \) denotes, for \( j = 1,\ldots,p \), the Taylor polynomial of degree \( m_j-1 \) in the point \( x = x_j \).
Theorem: If \( f(x) \) is a function of the form
\[
f(x) = \sum_{k=0}^{m-1} \frac{A_k x^k + B_k}{(x-a)^{m-k} (x-b)^{m-k}} + g(x),
\]
where \( m \neq 0 \) is a natural number, \( a, b, A_k \) and \( B_k \), for \( k = 0, 1, \ldots, m-1 \), are complex numbers and \( g(x) \) is a function with derivative of \( m-1 \) order in \( x \), then the coefficients \( A_k \) and \( B_k \) can be computed recursively by the relations
\[
c_0 = A_0 a + B_0, \quad (7)
\]
\[
c_1 = (A_1 a + B_1)(a-b) + A_0, \quad (8)
\]
\[
c_2 = (A_2 a + B_2)(a-b)^2 + A_1 a + B_1 + A_0(a-b), \quad (9)
\]
\[
c_3 = (A_3 a + B_3)(a-b)^3 + 2(A_2 a + B_2)(a-b) + A_2(a-b)^2 + A_1, \quad (10)
\]
\[
c_n = (A_n a + B_n)(a-b)^n + (n-1)(A_{n-1} a + B_{n-1})(a-b)^{n-1} + A_{n-1}(a-b)^{n-1} + \sum_{k=0}^{n-1} \frac{k(k-1)\cdots(2k-n+1)}{(n-k)!} (A_k a + B_k)(a-b)^{2k-n} + \sum_{k=0}^{n-1} \frac{k(k-1)\cdots(2k-n+2)}{(n-k-1)!} A_k(a-b)^{2k-n+1},
\]
where \( (c_k : k = 0, 1, \ldots, m-1) = T_a^{m-1}((x-a)^m (x-b)^m f(x)) \) are the coefficients of the Taylor polynomial of degree \( m-1 \) of the function \( (x-a)^m (x-b)^m f(x) \) in the point \( x = a \).

Proof: In conformity with (6), we have
\[
c_n = \frac{1}{n!} \lim_{x \to a} \frac{d^n}{dx^n} \left[ (x-a)^m (x-b)^m f(x) \right]
\]
\[
= \frac{1}{n!} \lim_{x \to a} \frac{d^n}{dx^n} \left[ \sum_{k=0}^{m-1} A_k(x-a)^k(x-b)^k + (x-a)^m(x-b)^m g(x) \right].
\]
For \( n = 0 \), it results (7). For \( 1 \leq n \leq m-1 \), it results
\[
c_n = \frac{1}{n!} \lim_{x \to a} \sum_{j=0}^{m-1} \sum_{k=0}^{n} \binom{n}{j} \frac{d^{n-j}}{dx^{n-j}} \left[ (x-a)^{k-j}(x-b)^{j-k} \right] \left[ (A_k a + B_k)(x-b)^k \right]
\]
\[
+ \frac{1}{n!} \lim_{x \to a} \sum_{j=0}^{m} \binom{n}{j} m(m-1)\cdots(m-j+1)(x-a)^{m-j} \frac{d^{n-j}}{dx^{n-j}} \left[ (x-b)^m g(x) \right]
\]
\[
= \frac{1}{n!} \lim_{x \to a} \sum_{k=0}^{n} \binom{n}{k} k! \frac{d^{n-k}}{dx^{n-k}} \left[ (A_k a + B_k)(x-b)^k \right] = (A_n a + B_n)(a-b)^n
\]
\[
+ \sum_{k=0}^{n-1} \frac{1}{(n-k)!} \lim_{x \to a} \frac{d^{n-k-1}}{dx^{n-k-1}} \left[ (x-a)^{n-k}(x-b)^k \right] + (n-k)A_k \frac{d^{n-k-1}}{dx^{n-k-1}} \left[ (x-b)^k \right],
\]
from which results the formulas (8)-(11).

Particular cases: From (11), if \( m \geq 5 \), it results the formulas
\[
c_4 = (A_4 a + B_4)(a-b)^4 + 3(A_4 a + B_4)(a-b)^2 + A_3(a-b)^3 + A_2 a + B_2 + 2A_2(a-b), \quad (12)
\]
\[
c_5 = (A_5 a + B_5)(a-b)^5 + 4(A_4 a + B_4)(a-b)^3 + A_4(a-b)^4 + 3(A_3 a + B_3)(a-b) + 3A_3(a-b)^2 + A_2 \quad (13)
\]

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Remark: As is well known, if \( f(x) \) is a rational function, its nominator has degree less than the denominator, the last being a polynomial with real coefficients and \( a \) is a complex pole of multiplicity \( m \) of \( f(x) \), then the function has the form (6) with \( b = \overline{a} \) the complex conjugate of \( a \), therefore the Theorem 3 work. See Example 8.5. Same situation when the denominator has rational numbers as coefficients and \( a \) is an irrational pole of the function of multiplicity \( m \). Then \( b \) is the irrational conjugate of \( a \) and the Theorem 3 also work. See Example 8.6.

8. EXAMPLES OF PARTIAL FRACTIONS DEVELOPMENT

8.1. \( f(x) = \frac{x}{(x+1)^2(x-1)^3(x-2)^3} = P^{-1}_1 \left( \frac{x}{(x-1)^3(x-2)^3} \right) \frac{1}{(x+1)^2} \)

\[ + P^2_1 \left( \frac{x}{(x+1)^2(x-2)^5} \right) \frac{1}{(x-1)^3} + P^4_2 \left( \frac{x}{(x+1)^2(x-1)^5} \right) \frac{1}{(x-2)^5} \]

\[ = P^1_1 \left( \frac{(-1,1)}{(-2,1)^3 * (-3,1)^5} \right) \frac{1}{(x+1)^2} + P^2_2 \left( \frac{(1,1,0)}{(2,1,0)^2 * (-1,1,0)^5} \right) \frac{1}{(x-1)^3} \]

\[ + P^3_1 \left( \frac{(2,1,0,0,0)}{(3,1,0,0,0)^2 * (1,1,0,0,0)^5} \right) \frac{1}{(x-2)^5} = P^1_1 \left( \frac{(-1,1)}{(1944, -6156)} \right) \frac{1}{(x+1)^2} \]

\[ + P^2_1 \left( \frac{(1,1,0)}{(-4,16, -21)} \right) \frac{1}{(x-1)^3} + P^4_2 \left( \frac{(2,1,0,0,0)}{(9,33,46,30,9)} \right) \frac{1}{(x-2)^5} \]

\[ = P^1_1 \left( \frac{-1}{1944}, -\frac{13}{11664} \right) \frac{1}{(x+1)^2} + P^2_2 \left( \frac{-1}{4}, -\frac{5}{4}, \frac{59}{16} \right) \frac{1}{(x-1)^3} \]

\[ + P^3_1 \left( \frac{2}{9}, -\frac{19}{27}, \frac{13}{9}, -\frac{593}{243}, \frac{2689}{729} \right) \frac{1}{(x-2)^5} = \left[ -\frac{1}{1944}, -\frac{13}{11664} \right] \frac{1}{(x+1)^2} \]

\[ + \left[ \frac{-1}{4}, \frac{5}{4} (x+1) \right] \frac{1}{(x-1)^3} + \left[ \frac{2}{9}, \frac{19}{22} (x-2) + \frac{13}{9} (x-2)^2 \right] \frac{593}{243} (x-2)^3 \]

\[ + \frac{2689}{729} (x-2)^4 = \frac{1}{1944} \frac{1}{(x+1)^2} - \frac{13}{11664} \frac{1}{(x+1)^2} - \frac{1}{4} \frac{1}{(x-1)^3} - \frac{5}{4} \frac{1}{(x-1)^2} \]

\[ + \frac{59}{16} \frac{1}{(x+1)^2} + \frac{2}{9} \frac{19}{27} (x-2)^4 + \frac{13}{9} \frac{593}{243} (x-2)^3 + \frac{2689}{729} (x-2)^2 \]

For the Taylor polynomial of four degree in \( x = 2 \) have been performed the convolutions

| 1 1 0 0 0 | 3 1 0 0 0 | 1 3 3 1 0 |
| 1 1 0 0 0 | 3 1 0 0 0 | 9 6 1 0 0 |
| 1 1 0 0 0 | 9 3 0 0 0 | 9 27 27 9 0 |
| 1 1 0 0 0 | 3 1 0 0 0 | 6 18 18 6 |
| 1 2 1 0 0 | 9 6 1 0 0 | 1 3 3 |
| 1 1 0 0 0 | 9 33 46 30 9 |

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From which it results
\[(1,1,0,0,0) \ast^3 = (1,3,3,1,0), (3,1,0,0,0) \ast^2 = (9,6,1,0,0)\]
and
\[(3,1,0,0,0) \ast^2 \ast (1,1,0,0,0) \ast^3 = (9,33,46,30,9)\]
and the deconvolution

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\[
\begin{array}{c|cccc}
0 & \frac{19}{3} & \frac{92}{9} & \frac{-20}{3} & -2 \\
\frac{-19}{3} & \frac{209}{9} & \frac{874}{27} & \frac{-190}{9} \\
\frac{13}{9} & \frac{694}{27} & \frac{172}{9} & \\
\frac{13}{3} & \frac{143}{9} & \frac{598}{9} & \\
\frac{-593}{27} & \frac{-142}{3} & \frac{-6523}{27} & \frac{-81}{}
\end{array}
\]

\[
\begin{array}{c|c}
\frac{-2689}{81} & 2689 \\
\frac{2689}{81} & 81 \\
\end{array}
\]
Hence
\[
\frac{(2,1,0,0,0)}{(3,1,0,0,0) \ast^2 \ast (1,1,0,0,0) \ast^3} = \left(\frac{2}{9}, \frac{-19}{27}, \frac{13}{9}, \frac{-593}{243}, \frac{2689}{729}\right).
\]

8.2. \( f(x) = \frac{15625}{(x-2)^3(x^2+1)^6} = \sum_{j=0}^{2} \frac{\vec{c}_j}{(x-2)^{2-j}} + \sum_{k=0}^{5} A_k x + B_k \). We have

\[
(\vec{c}_0, \vec{c}_1, \vec{c}_2) = T_2^2 \left(\frac{15625}{(x^2+1)^6}\right) = \frac{15625(1,0,0)}{(5,4,1)^6} = \frac{(15625,0,0)}{(15625,75000,168750)} = (1,-4.8,12.24).
\]

\[
(\vec{c}_0, \ldots, \vec{c}_5) = T_1^5 \left(\frac{15625}{(x-2)^3}\right) = \frac{(15625 \cdot 1,0,0,0,0)}{(-2 + i,1,0,0,0,0)^3} = \frac{(156250,0,0,0,0)}{(-2+11i,9-12i,-6+3i,1,0,0)}
\]

\[
= (-250-1375i,525-1800i,1140-1230i,1170-440i,834+87i,442.68+282.24i).
\]
Hence \( c_0 = A_0 i + B_0 = -250-1375i, A_0 = -1375, B_0 = -250. \)
\begin{align*}
c_1 &= 2i(A_i + B_i) - 1375 = 525 - 1800i, \quad A_i = -950, \quad B_i = -900, \\
c_2 &= -4(A_i + B_i) - 1900i - 950i - 900 = 1140 - 1230i, \quad A_i = -405, \quad B_i = -510, \\
c_3 &= -8i(A_i + B_i) + 1620 + 4i(-405i - 510) - 950 = 1170 - 440i, \quad A_i = -140, \\
B_i &= -200, \quad c_4 = 16(A_i + B_i) + 1132i + 12(140i + 200) - 405i - 510 - 1620i = 834 + 87i, \\
A_i &= -43, \\
B_i &= -66, \quad c_5 = 32i(A_i + B_i) - 688 - 32i(-43i - 66) + 2112i + 840 - 1200i + 1680 - 405 \\
&= 442.68 + 282.24i, \\
A_i &= -12.24, \quad B_i = -19.68.
\end{align*}

Therefore, 
\[
f(x) = \frac{1}{(x - 2)^3} - \frac{4.8}{(x - 2)^2} + \frac{12.24}{x - 2} \frac{1375x + 250}{(x^2 + 1)^6} \frac{950x + 900}{(x^2 + 1)^5} \frac{405x + 510}{(x^2 + 1)^4} + \frac{140x + 200}{(x^2 + 1)^3} \frac{43x + 66}{(x^2 + 1)^2} + \frac{12.24x + 19.68}{x^2 + 1}.
\]

8.3. \( f(x) = \frac{x^6 + x^5 + x^3 + 3x^2 + x + 4}{(x^2 + 1)^5(x^2 + 3)^2} \).

**Solution: 1 (complex)** We will obtain \( f(x) = \sum_{j=0}^{4} \frac{A_j x + B_j}{(x^2 + 3)^{4-j}} + \sum_{k=0}^{4} A_k x + B_k \) for \( f(x) \). We have

\[
(c_0, c_1) = T_i \left( \frac{x^6 + x^5 + x^3 + 3x^2 + x + 4}{(x^2 + 1)^5} \right) = \left( \frac{-32 + 7i\sqrt{3}, 37 + 60i\sqrt{3}}{-2, 2i\sqrt{3}} \right) \]

\[
= \frac{-32 + 7i\sqrt{3}, 37 + 60i\sqrt{3}}{32(-1, -5i\sqrt{3})} = \frac{1}{32} \left( -32 + 7i\sqrt{3}, -68 - 100i\sqrt{3} \right).
\]

Hence

\[
\tilde{c}_0 = \tilde{A}_0 i\sqrt{3} + \tilde{B}_0 = 1 - \frac{7}{32} i\sqrt{3}, \quad \tilde{A}_0 = -\frac{7}{32}, \quad \tilde{B}_0 = 1,
\]

\[
\tilde{c}_1 = 2i\sqrt{3} \left( \frac{\tilde{A}_0 i\sqrt{3} + \tilde{B}_0}{-732} \right) = \frac{68}{32} + \frac{100}{32} i\sqrt{3}, \quad \tilde{A}_1 = -\frac{25}{64}, \quad \tilde{B}_1 = \frac{25}{16},
\]

\[
(c_0, \ldots, c_4) = T_i \left( \frac{x^6 + x^5 + x^3 + 3x^2 + x + 4}{(x^2 + 3)^2} \right) = \left( \frac{i + 3 + 12i, 18 - 7i, -9 - 20i, -15 + 5i}{(4i, 0, 4i, 1)} \right) = \left( \frac{i, 5 + 3i, 21}{4, 2, 4}, -\frac{17}{4}, -\frac{21}{2}, -26i, -\frac{211}{4}, -\frac{335}{16} \right),
\]
\[ c_0 = A_0 i + B_0 = \frac{i}{4}, \quad A_0 = \frac{1}{4}, \quad B_0 = 0; \quad c_1 = 2i(A_0 i + B_0) + \frac{5}{4} + 3i, \quad A_1 = -\frac{1}{2}, \quad B_1 = \frac{3}{2}; \]
\[ c_2 = -4(A_1 i + B_1) - \frac{i}{2} + \frac{3}{2} = \frac{21 - 17}{4} i, \quad A_2 = \frac{11}{16}, \quad B_2 = -\frac{9}{4}; \]
\[ c_3 = -8i(A_1 i + B_1) + 4i \left( \frac{11i}{4} - \frac{9}{16} \right) - \frac{11}{4} i = \frac{21 - 26i}{2}, \quad A_3 = -\frac{9}{16}, \quad B_3 = \frac{17}{8}; \]
\[ c_4 = 16(A_1 i + B_1) - 12 \left( -\frac{9i}{16} + \frac{17}{8} \right) + \frac{9i}{2} = \frac{211 - 335i}{4}, \quad A_4 = \frac{25}{64}, \quad B_4 = -\frac{25}{16}. \]

Therefore
\[
f(x) = \frac{x}{4(x^2 + 1)^3} + \frac{3 - x}{2(x^2 + 1)^4} + \frac{11x - 36}{16(x^2 + 1)^4} + \frac{34 - 9x}{64(x^2 + 1)^2} + \frac{25(x - 4)}{32(x^2 + 3)^2} + \frac{32 - 7x}{64(x^2 + 3)}.
\]

Solution: 2 (real) \( f(x) = f_1(t) + x f_2(t), \) where \( t = x^2 \)

\[
f_1(t) = \frac{t^3 + 3t + 4}{(t + 1)^2 (t + 3)^2} = P_4 \left( \frac{t^3 + 3t + 4}{(t + 1)^2} \right) \frac{1}{(t + 3)^2} + P_3 \left( \frac{t^3 + 3t + 4}{(t + 1)^2} \right) \frac{1}{(t + 3)^2} - 4 \left[ \frac{1}{(t + 1)^2} \right] \frac{1}{(t + 3)^2} - 4 \left[ \frac{1}{(t + 1)^2} \right] \frac{1}{(t + 3)^2} = \left[ \frac{3}{2} (t + 1)^2 - \frac{9}{4} (t + 1)^2 + \frac{17}{8} (t + 1)^3 - \frac{25}{16} (t + 1)^4 \right] \frac{1}{(t + 1)^2} + \left[ 1 - \frac{25}{16} (t + 3) \right] \frac{1}{(t + 3)^2}
\]

\[
f_2(t) = \frac{t^2 + t + 1}{(t + 1)^5 (t + 3)^2} = P_4 \left( \frac{t^2 + t + 1}{(t + 1)^2} \right) \frac{1}{(t + 3)^2} + P_3 \left( \frac{t^2 + t + 1}{(t + 1)^2} \right) \frac{1}{(t + 3)^2} - 4 \left[ \frac{1}{(t + 1)^2} \right] \frac{1}{(t + 3)^2} - 4 \left[ \frac{1}{(t + 1)^2} \right] \frac{1}{(t + 3)^2} = \left[ \frac{1}{4} - \frac{1}{2} (t + 1)^2 + \frac{11}{16} (t + 1)^2 + \frac{9}{16} (t + 1)^3 \right] \frac{1}{(t + 1)^2} + \left[ 1 - \frac{25}{64} (t + 3) \right] \frac{1}{(t + 3)^2}.
\]
\[
-\left[ \frac{7}{32} + \frac{25}{64} (t+3) \right] \frac{1}{(t+3)^2} = \frac{1}{4(t+1)^3} - \frac{1}{2(t+1)^4} + \frac{11}{1(t+1)^5} - \frac{9}{1(t+1)^6}
\]
\[
+ \frac{25}{64(t+1)} - \frac{7}{32(t+3)^2} - \frac{25}{64(t+3)}
\]

It results that \( f(x) \) has the same form as that obtained in the previous solution.

**8.4.** \( f(x) = \frac{x^6 - 5x^5 + 10x^4 - 9x^3 + 5x^2 - 3x + 2}{(x^2 - 2x + 2)^3(x^2 - 2x + 2)^2} \).

Performing the change of variables \( x = y + 1 \) we obtain the function \( f(y) = \frac{y^6 + y^5 + y^3 + 3y^2 + y + 4}{(y^2 + 1)^3(y^2 + 3)^2} \), hence the example 8.4 is reduced to example 8.3.

**Remark:** All the rational functions whose denominator is a product of trinomials whose canonical forms have the same binomial, can be developed in partial fractions by the real method presented in the second solution of the Example 8.3, after a convenient change of variable.

**8.5** \( f(x) = \frac{1}{(x^2 + 1)^2(x^2 + 2x + 2)^2} = \sum A_jx^j + \sum B_kx^k \).

\[ (c_0, c_1) = T_i\left( \frac{1}{(x^2 + 2x + 2)^3} \right) = \frac{(1,0)}{(1+2i,2+2i)^3} = \frac{(1,0)}{(-11-2i,-42+6i)} = \]
\[ = (-0.088 + 0.016i, 0.2974 - 0.1632i) \), hence

\[ \tilde{c}_0 = \tilde{A}_0i + \tilde{B}_0 = -0.088 + 0.016i, \quad \tilde{A}_0 = 0.016, \quad \tilde{B}_0 = -0.088; \]

\[ \tilde{c}_1 = 2i(\tilde{A}_1 + \tilde{B}_1) + 0.016 = 0.2974 - 0.1632i, \quad \tilde{A}_1 = -0.1408, \quad \tilde{B}_1 = -0.0816; \]

\[ (c_0, c_1, c_2) = T_{2+i}^2\left( \frac{1}{(x^2 + 1)^2} \right) = \frac{(1,0,0)}{(1-2i, -2+2i)^2} = \frac{(1,0,0)}{(-3-4i, 4+12i, 2-12i)} = \]
\[ = (-0.12 + 0.16i, -0.416 + 0.288i, -0.7456 + 0.3008i) \), hence

\[ c_0 = A_0(-1+i) + B_0 = -0.12 + 0.16i, \quad A_0 = 0.16, \quad B_0 = 0.04; \]

\[ c_1 = 2i[A_1(-1+i) + B_1] + 0.16 = -0.416 + 0.288i, \quad A_1 = 0.288, \quad B_1 = 0.432; \]

\[ c_2 = -4[A_2(-1+i) + B_2] + 0.288(-1+i) + 0.432 + 0.576i = -0.7456 + 0.3005i, \]

\[ A_2 = 0.1408, \quad B_1 = 0.3632. \) Therefore

\[ f(x) = \frac{0.016x - 0.088}{(x^2 + 1)^2} - \frac{0.1408x + 0.0816}{x^2 + 1} + \frac{0.16x + 0.04}{(x^2 + 2x + 2)^3} + \frac{0.288x + 0.432}{(x^2 + 2x + 2)^2} + \frac{0.1408x + 0.3632}{x^2 + 2x + 2}. \]
8.6. \( f(x) = \frac{4}{(x + 1)^3(x^2 - 2x - 1)^2} \).

Solution: 1 (Based on Theorem 3)

\[
f(x) = \sum_{k=0}^{2} \frac{c_k}{(x + 1)^2} + \sum_{n=0}^{1} \frac{A_n x + B_n}{(x^2 - 2x - 1)^{n+1}}.
\]

We have

\[
(\bar{c}_0, \bar{c}_1, \bar{c}_2) = T_{12}^1 \left( \frac{4}{(x^2 - 2x - 1)^2} \right) = \frac{(4,0,0)}{(2,-4,1)^2} = (1,0,0) = (1,4,11) \quad \text{and}
\]

\[
(c_0, c_1) = T_{12}^{12} \left( \frac{4}{(x+1)^3} \right) = \frac{(4,0)}{(2+\sqrt{2},1)^3} = \left( \frac{20 + 14\sqrt{2},18 + 12\sqrt{2}}{2} \right) = \left( 10 - 7\sqrt{2}, -51 + 36\sqrt{2} \right),
\]

in conformity with the algorithms

\[
\begin{array}{ccc|cc}
2 + \sqrt{2} & 1 & 4 & 0 & 20 + 14\sqrt{2} & 18 + 12\sqrt{2} \\
2 + \sqrt{2} & 1 & 4 & 12 - 6\sqrt{2} & 10 - 7\sqrt{2} & -51 + 36\sqrt{2} \\
\hline
6 + 4\sqrt{2} & 2 + \sqrt{2} & / & -12 + 6\sqrt{2} \\
2 + \sqrt{2} & / & -12 + 6\sqrt{2}
\end{array}
\]

\[
\begin{array}{ccc|cc}
6 + 4\sqrt{2} & 4 + 2\sqrt{2} & / \\
2 + \sqrt{2} & 1 \\
\hline
20 + 14\sqrt{2} & 12 + 8\sqrt{2} & 6 + 4\sqrt{2} \\
\hline
20 + 14\sqrt{2} & 18 + 12\sqrt{2}
\end{array}
\]

So, \( c_0 = A_0 \left( 1 + \sqrt{2} \right) + B_0 = 10 - 7\sqrt{2}, A_0 = -7, B_0 = 17 \) and

\( c_1 = 2\sqrt{2} \left[ A_1 \left( 1 + \sqrt{2} \right) + B_1 \right] - 7 = -51 + 36\sqrt{2}, A_1 = -11, B_1 = 29 \).

Therefore the decomposition in partial fractions is

\[
f(x) = \frac{1}{(x + 1)^3} + \frac{4}{(x + 1)^2} + \frac{11}{x + 1} + \frac{17 - 7x}{(x^2 - 2x - 1)^2} + \frac{29 - 11x}{x^2 - 2x - 1}.
\]

Solution: 2 (Based on Theorem 2)

\[
f(x) = \frac{4}{(x + 1)^3 \left( x - 1 - \sqrt{2} \right)^2 \left( x - 1 + \sqrt{2} \right)^2}
\]

\[
= P_{21}^2 \left( \frac{4}{(x^2 - 2x - 1)^2} \right) + \frac{1}{(x + 1)^3} + P_{12}^1 \left( \frac{4}{(x + 1)^3 \left( x - 1 + \sqrt{2} \right)^2} \right) \frac{1}{(x - 1 - \sqrt{2})^2}
\]

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\[ P_1^i \left( \frac{4}{(x+1)^2 \left(x - 1 + \sqrt{2}\right)^2} \right) \left( \frac{1}{x - 1 + \sqrt{2}} \right) \]

\[ = P_1^i \left( \frac{(4,0,0)}{(2,-4,1)^2} \right) \frac{1}{(x+1)^2} + P_1^i \left( \frac{4(1,0)}{(2 + \sqrt{2},1)^3 \cdot (2 \sqrt{2},1)^3} \right) \frac{1}{(x - 1 + \sqrt{2})^2} \]

\[ + P_1^i \left( \frac{4(1,0,0)}{(2 - \sqrt{2},1)^3 \cdot (-2 \sqrt{2},1)^3} \right) \frac{1}{(x + 1 + \sqrt{2})^2} = P_1^i \left( \frac{(1,0,0)}{(1,-4,5)} \right) \frac{1}{(x+1)^3} \]

\[ + P_1^i \left( \frac{(1,0,0)}{(40 + 28 \sqrt{2},64 + 44 \sqrt{2})} \right) \frac{1}{(x - 1 - \sqrt{2})^2} \]

\[ + P_1^i \left( \frac{(1,0,0)}{(40 - 28 \sqrt{2},64 - 44 \sqrt{2})} \right) \frac{1}{(x - 1 + \sqrt{2})^2} = P_1^i \left( 1,4,11 \right) \frac{1}{(x+1)^3} \]

\[ + \frac{1}{8} P_1^i \left( 10 - 7 \sqrt{2},-44 + 37 \sqrt{2} \right) \frac{1}{(x - 1 - \sqrt{2})^2} \]

\[ + \frac{1}{8} P_1^i \left( 10 + 7 \sqrt{2},-44 - 37 \sqrt{2} \right) \frac{1}{(x - 1 + \sqrt{2})^2} = \frac{1}{(x+1)^3} + \frac{4}{(x+1)^2} + \frac{11}{x+1} \]

\[ + \frac{10 - 7 \sqrt{2}}{8(x - 1 - \sqrt{2})^2} + \frac{-44 + 31 \sqrt{2}}{8(x - 1 - \sqrt{2})} + \frac{10 + 7 \sqrt{2}}{8(x - 1 + \sqrt{2})^2} + \frac{-44 - 7 \sqrt{2}}{8(x - 1 + \sqrt{2})}, \quad (15) \]

Remark: A little algebraic calculus shows that the two developments of \( f(x) \) in partial fractions given by the formulas (14) and (15) are equal. However, for integration and for calculus of the inverse Laplace transform, more convenient is the development (15). Indeed, it gives

\[ \int f(x) \, dx = -\frac{1}{2(x+1)^2} - \frac{4}{x+1} + 11 \ln |x+1| - \frac{10 - 7 \sqrt{2}}{8(x - 1 - \sqrt{2})} - \frac{44 - 31 \sqrt{2}}{8} \ln |x - 1 - \sqrt{2}| \]

\[ - \frac{10 + 7 \sqrt{2}}{8(x - 1 + \sqrt{2})} - \frac{44 + 31 \sqrt{2}}{8} \ln |x - 1 + \sqrt{2}| + C, \]

where \( C \) is an arbitrary constant. If \( x \) is a complex variable with \( \text{Re}(x) > 1 + \sqrt{2} \), from formula (15) it results the inverse Laplace transform

\[ L^{-1} \left( f(x) \right) = \frac{1}{2} t^2 e^{-t} + 4te^{-t} + 11e^{-t} + \frac{10 - 7 \sqrt{2}}{8} te^{\left(1 - i \sqrt{2}\right)t} - \frac{44 - 31 \sqrt{2}}{8} e^{\left(1 - i \sqrt{2}\right)t} \]

\[ + \frac{10 + 7 \sqrt{2}}{8} te^{\left(1 + i \sqrt{2}\right)t} - \frac{44 + 31 \sqrt{2}}{8} e^{\left(1 + i \sqrt{2}\right)t}, \quad t \geq 0. \]

9. CONCLUSIONS

Besides the well known applications of the Taylor series development – the approximation of the functions by polynomials and power series method for solving different types of equations, the last being now named Taylor differential transformation method, we added other two, the automatic high order differentiation in [1] and the partial fractions decomposition in this paper. As can see by the above examples, the new methods given in the present article can be applied to cases that are difficult to solve by the methods known so far.
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