

A RELATED FIXED POINT THEOREM OF INTEGRAL TYPE ON TWO SEMI - METRIC SPACES

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ABSTRACT

A related fixed point theorem for two pairs of mappings on two semi-metric spaces satisfying integral type inequality is obtained. The result extends a result of R.K. Namdeo, N.K. Tiwari, B. Fisher and K. Tas [6].

Keywords: Symmetric space, semi-metric space, fixed point, related fixed point, integral type inequality.

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1. INTRODUCTION

Many mathematicians have studied and developed a number of generalizations of a metric space which play a variety of roles in the study of fixed point theories. One of the generalizations is semi-metric space. Reference and background are given in [8]. Related fixed point theories are studied in [1- 6] and many others.

2. PRELIMINARIES

We recall some basic concepts.

Definition: 2.1 A symmetric function on a set X is a non-negative real valued function d on $X \times X$ such that for $x, y \in X$,

- (i) $d(x, y) = 0$ if and only if $x = y$.
- (ii) $d(x, y) = d(y, x)$.

Let d be a symmetric on set X . For $r > 0$ and $x \in X$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. A topology $\tau(d)$ on X is defined as $U \in \tau(d)$ if and only if for each $x \in U$, $B(x, r) \subseteq U$. A subset S of X is a neighbourhood of $x \in X$ if there exists $U \in \tau(d)$ such that $x \in U \subseteq S$.

Definition: 2.2 A symmetric d is semi-metric if for each $x \in X$ and for each $r > 0$, $B(x, r)$ is a neighbourhood of x in the topology $\tau(d)$.

We note that for every $\{x_n\} \subseteq X$ and $x \in X$, $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ if and only if $x_n \rightarrow x$ in the topology $\tau(d)$.

Now, some axioms are stated as follows:

Let (X, d) be a semi-metric space. Then,

(W₃) Given $\{x_n\}$, x in X , $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, y) = 0$ imply $x = y$.

(W₄) Given $\{x_n\}$, $\{y_n\}$ and x in X , $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ imply $\lim_{n \rightarrow \infty} d(y_n, x) = 0$

(1C) A symmetric d on a set X is said to be 1-continuous if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ implies $\lim_{n \rightarrow \infty} d(x_n, y) = d(x, y) = 0$ for all $y \in X$.

(W₃), (W₄) and (1C) are respectively found in [8], [8] and [7].

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Definition: 2.3 A sequence in X is d -Cauchy if it satisfies the usual metric condition with respect to d .

* (X, d) is (Σ) d -complete if for every sequence $\{x_n\}$, $\sum_{n=1}^{+\infty} d(x_n, x_{n+1}) < \infty$ implies that $\{x_n\}$ is convergent in the topology $\tau(d)$.

* (X, d) is d -Cauchy complete if for every d -Cauchy sequence $\{x_n\}$, there exists $x \in X$ with $x_n \rightarrow x$ in the topology $\tau(d)$.

* (X, d) is S -complete if for every d -Cauchy sequence $\{x_n\}$, there exists $x \in X$ with $\lim_{n \rightarrow \infty} d(x_n, x) = 0$.

The following was proved by R.K. Namdeo, N.K. Tiwari, B. Fisher and K. Tas in [6].

Theorem: 2.4 Let (X, d) and (Y, ρ) be complete metric spaces. Let T be a mapping of X into Y and S be a mapping of Y into X satisfying the inequalities

$$d(Sy, Sy') d(STx, STx') \leq c \max\{d(Sy, Sy') \rho(Tx, Tx'), d(x', Sy) \rho(y', Tx), \\ d(x, x') d(Sy, Sy'), d(Sy, STx) d(Sy', STx')\}$$

$$\rho(Tx, Tx') \rho(TSy, TSy') \leq c \max\{d(Sy, Sy') \rho(Tx, Tx'), d(x', Sy) \rho(y', Tx), \\ \rho(y, y') \rho(Tx, Tx'), \rho(Tx, TSy) \rho(Tx', TSy')\}$$

for all x, x' in X and y, y' in Y , where $0 \leq c < 1$. If either S or T is continuous, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y . Further, $Tz = w$ and $Sw = z$.

Now, theorem 2.4 is extended to two pairs of mappings on semi-metric spaces in integral setting as follows.

3. MAIN RESULT

Theorem: 3.1 Let (X, d) and (Y, ν) be two 1- continuous semi-metric spaces. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities

$$\int_0^1 d(Sy, Ty') d(SAx, TBx') \varphi(t) dt \leq \int_0^1 c \max\{d(Sy, Ty') \nu(Ax, Bx'), d(x', Sy) \nu(y', Ax), \\ d(x, x') d(Sy, Ty'), d(Sy, SAx) d(Ty', TBx')\} \varphi(t) dt \quad (1)$$

$$\int_0^1 \nu(Ax, Bx') \nu(BSy, ATy') \varphi(t) dt \leq \int_0^1 c \max\{d(Sy, Ty') \nu(Ax, Bx'), d(x', Sy) \nu(y', Ax), \\ \nu(y, y') \nu(Ax, Bx'), \nu(Ax, BSy) \nu(Bx', ATy')\} \varphi(t) dt \quad (2)$$

for all x, x' in X and y, y' in Y , where $0 \leq c < 1$. If either X is (Σ) d - complete and Y satisfies (W_4) or Y is $(\Sigma) \nu$ - complete and X satisfies (W_4) and one of the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y . Further, $Az = Bz = w$ and $Sw = Tw = z$.

Proof: Let y_1 be any arbitrary point in X . We define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively as follows:

$$S y_{2n-1} = x_{2n-1}, \quad Bx_{2n-1} = y_{2n}, \quad Ty_{2n} = x_{2n}, \quad Ax_{2n} = y_{2n-1}, \quad \text{for } n = 1, 2, 3, \dots$$

Applying inequality (1), we get

$$\int_0^1 d(Sy_{2n-1}, Ty_{2n}) d(SAx_{2n}, TBx_{2n-1}) \phi(t) dt = \int_0^1 d^2(x_{2n-1}, x_{2n}) \phi(t) dt \\ \leq \int_0^1 c \max\{d(Sy_{2n-1}, Ty_{2n}) \nu(Ax_{2n}, Bx_{2n-1}), d(x_{2n-1}, Sy_{2n-1}) \nu(y_{2n}, Ax_{2n}), \\ d(x_{2n}, x_{2n-1}) d(Sy_{2n-1}, Ty_{2n}), d(Sy_{2n-1}, SAx_{2n}) d(Ty_{2n}, TBx_{2n-1})\} \phi(t) dt$$

$$= \int_0^c \max \{ d(x_{2n-1}, x_{2n}) \nu(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n-1}) \nu(y_{2n}, y_{2n-1}), d(x_{2n}, x_{2n-1}) d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n-1}) d(x_{2n}, x_{2n}) \} \phi(t) dt$$

From which it follows that

$$\int_0^c d(x_{2n-1}, x_{2n}) \phi(t) dt \leq \int_0^c \max \{ \nu(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}) \} \phi(t) dt \quad (3)$$

Applying inequality (2), we get

$$\begin{aligned} \int_0^c \nu(Ax_{2n}, Bx_{2n-1}) \nu(BSy_{2n-1}, ATy_{2n}) \phi(t) dt &= \int_0^c \nu^2(y_{2n-1}, y_{2n}) \phi(t) dt \\ &\leq \int_0^c \max \{ d(Sy_{2n-1}, Ty_{2n}) \nu(Ax_{2n}, Bx_{2n-1}), d(x_{2n-1}, Sy_{2n-1}) \nu(y_{2n}, Ax_{2n}), \\ &\quad \nu(y_{2n-1}, y_{2n}) \nu(Ax_{2n}, Bx_{2n-1}), \nu(Ax_{2n}, BSy_{2n-1}) \nu(Bx_{2n-1}, ATy_{2n}) \} \phi(t) dt \\ &= \int_0^c \max \{ d(x_{2n-1}, x_{2n}) \nu(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n-1}) \nu(y_{2n}, y_{2n-1}), \\ &\quad \nu(y_{2n-1}, y_{2n}) \nu(y_{2n-1}, y_{2n}), \nu(y_{2n-1}, y_{2n}) \nu(y_{2n}, y_{2n-1}) \} \phi(t) dt \end{aligned}$$

From which it follows that

$$\int_0^c \nu(y_{2n-1}, y_{2n}) \phi(t) dt \leq \int_0^c \max \{ \nu(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n}) \} \phi(t) dt \quad (4)$$

(3) and (4) can be written as

$$\begin{aligned} \int_0^c d(x_{n-1}, x_n) \phi(t) dt &\leq \int_0^c \max \{ \nu(y_{n-1}, y_n), d(x_{n-1}, x_n) \} \phi(t) dt \\ \int_0^c \nu(y_{n-1}, y_n) \phi(t) dt &\leq \int_0^c \max \{ \nu(y_{n-1}, y_n), d(x_{n-1}, x_n) \} \phi(t) dt \end{aligned}$$

which can be again written as

$$\int_0^c d(x_{n+1}, x_n) \phi(t) dt \leq \int_0^c \max \{ \nu(y_{n+1}, y_n), d(x_{n+1}, x_n) \} \phi(t) dt \quad (5)$$

$$\int_0^c \nu(y_{n+1}, y_n) \phi(t) dt \leq \int_0^c \max \{ \nu(y_{n+1}, y_n), d(x_{n+1}, x_n) \} \phi(t) dt \quad (6)$$

From (5) and (6), by induction, we get

$$\begin{aligned} \max \left\{ \int_0^c d(x_{n+1}, x_n) \phi(t) dt, \int_0^c \nu(y_{n+1}, y_n) \phi(t) dt \right\} &\leq \int_0^c \max \{ \nu(y_1, y_2), d(x_1, x_2) \} \phi(t) dt \\ &= \int_0^c M_{d, \nu} \phi(t) dt, \end{aligned}$$

$$\text{where } M_{d, \nu} = \max \{ \nu(y_1, y_2), d(x_1, x_2) \}$$

Therefore, $\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = \lim_{n \rightarrow \infty} \nu(y_{n+1}, y_n) = 0$.

Suppose that X is (Σ) d -complete. We have,

$$\sum_{k=1}^n d(x_k, x_{k+1}) \leq M_{d, \nu} \sum_{k=1}^n c^k, \quad n \geq 1.$$

which implies that $\sum_{k=1}^{+\infty} d(x_k, x_{k+1}) < \infty$. Therefore, $x_n \rightarrow z$ for some $z \in X$. Let A be continuous and $w = Az$. Then,

$$\lim_{n \rightarrow \infty} \nu(y_{2n-1}, w) = \lim_{n \rightarrow \infty} \nu(Ax_{2n}, Az) = 0 \text{ and therefore, } \lim_{n \rightarrow \infty} \nu(y_{2n}, w) = 0 \text{ since}$$

$$\lim_{n \rightarrow \infty} \nu(y_{2n-1}, y_{2n}) = 0 \text{ and } Y \text{ satisfies } (W_4). \text{ Hence, } \lim_{n \rightarrow \infty} \nu(y_n, w) = 0.$$

Using (3), we have

$$\int_0^{d(Sw, x_{2n})} \varphi(t) dt \leq \int_0^{c \max\{\nu(y_{2n-1}, y_{2n}), d(Sw, z)\}} \varphi(t) dt$$

On letting $n \rightarrow \infty$ and using 1 – continuity of d , we have

$$\int_0^{d(Sw, z)} \varphi(t) dt \leq \int_0^{c d(Sw, z)} \varphi(t) dt$$

$$\Rightarrow d(Sw, z) \leq c d(Sw, z)$$

which implies that

$$Sw = z = SAz$$

Using (4), we have

$$\int_0^{\nu(y_{2n-1}, Bz)} \varphi(t) dt \leq \int_0^{c \max\{d(x_{2n-1}, x_{2n}), \nu(w, Bz)\}} \varphi(t) dt$$

On letting $n \rightarrow \infty$ and using 1 – continuity of ν , we have

$$\int_0^{\nu(w, Bz)} \varphi(t) dt \leq \int_0^{c \nu(w, Bz)} \varphi(t) dt$$

$$\Rightarrow \nu(w, Bz) \leq c \nu(w, Bz)$$

which implies that

$$Bz = w = BSw$$

Again, using (3), we have

$$\int_0^{d(z, Tw)} \varphi(t) dt \leq \int_0^{c \max\{\nu(w, w), d(Tw, z)\}} \varphi(t) dt$$

$$\Rightarrow d(z, Tw) \leq c d(Tw, z)$$

which implies that

$$Tw = z = TBz$$

As, $Az = w$, we have

$$ATw = w.$$

The same results hold if one of the mappings B , S and T is continuous instead of A .

To prove uniqueness, let SA and TB have a second fixed point z' in X .

On using (3), we have

$$\int_0^{d(z, z')} \varphi(t) dt \leq \int_0^{c \max\{v(w, w'), d(z, z')\}} \varphi(t) dt$$

which implies that

$$\int_0^{d(z, z')} \varphi(t) dt \leq \int_0^c v(w, w') \varphi(t) dt \quad (7)$$

On using (4), we have

$$\int_0^{v(w, w')} \varphi(t) dt \leq \int_0^{c \max\{d(z, z'), v(w, w')\}} \varphi(t) dt$$

which implies that

$$\int_0^{v(w, w')} \varphi(t) dt \leq \int_0^c d(z, z') \varphi(t) dt \quad (8)$$

From (7) and (8), we have

$$\int_0^{d(z, z')} \varphi(t) dt \leq \int_0^c v(w, w') \varphi(t) dt \leq \int_0^{c^2 d(z, z')} \varphi(t) dt$$

$$\Rightarrow \int_0^{d(z, z')} \varphi(t) dt \leq \int_0^{c^2 d(z, z')} \varphi(t) dt$$

$$\Rightarrow \int_0^{d(z, z')} \varphi(t) dt \leq \int_0^{c^2 d(z, z')} \varphi(t) dt \leq \dots \leq \int_0^{c^n d(z, z')} \varphi(t) dt$$

$$\Rightarrow 0 \leq \int_0^{d(z, z')} \varphi(t) dt \leq \lim_{n \rightarrow \infty} \int_0^{c^n d(z, z')} \varphi(t) dt$$

$$\Rightarrow d(z, z') = 0.$$

which implies that

$$z = z'$$

This proves the uniqueness of z . Similarly, the uniqueness of w can be proved. This completes the proof.

Corollary: 3.2 Let (X, d) and (Y, v) be two 1-complete semi-metric spaces. Let S be a mapping of X into Y and T be a mapping of Y into X satisfying the inequalities

$$\int_0^{d(Ty, Ty') d(TSx, TSx')} \varphi(t) dt \leq \int_0^{c \max\{d(Ty, Ty') v(Sx, Sx'), d(x', Ty) v(y', Sx) d(x, x') d(Ty, Ty'), d(Ty, TSx) d(Ty', TSx')\}} \varphi(t) dt$$

$$\int_0^{v(Sx, Sx') v(STy, STy')} \varphi(t) dt \leq \int_0^{c \max\{d(Ty, Ty') v(Sx, Sx'), d(x', Ty) v(y', Sx) v(y, y') v(Sx, Sx'), v(Sx, STy) v(Sx', STy')\}} \varphi(t) dt$$

for all x, x' in X and y, y' in Y , where $0 \leq c < 1$. If either X is (Σ) d - complete and Y satisfies (W_4) or Y is (Σ) v - complete and X satisfies (W_4) and one of the mappings S and T is continuous, then TS has a unique fixed point z in X and ST has a unique fixed point w in Y . Further, $Sz = w$ and $Tw = z$.

Proof: Putting $A = B = S$ and $S = T = T$ in theorem 3.1, we can obtain the result.

REFERENCES

- [1] Aamri M., Bennani S. and Moutawakil D. El, *Related fixed point theorems for two pairs of mappings on two symmetric spaces*, Bulletin of Mathematical Analysis and Applications, Vol. 4(3), (2012), 43- 47.
- [2] Aliouche A. and Fisher B., *A related fixed point theorem for two pairs of mappings on two complete metric spaces*, Hacettepe Journal of Mathematics and Statistics, Vol. 34, (2005), 39 - 45.
- [3] Cho Y.J., Kang S.M. and Kim S.S., *Fixed points in two metric spaces*, Novi Sad Journal Math., 29 (1), (1999), 47-53.
- [4] Fisher B. and Rao K.P.R, *A related fixed point theorem on three metric space*, Hacettepe Journal of Mathematics and Statistics, Vol. 36 (2), (2007), 143 - 146.
- [5] Jain S. and Fisher B., *A related fixed point theorem for three metric spaces*, Hacettepe Journal of Mathematics and Statistics, Vol. 31, (2002), 19 - 24.
- [6] R.K. Namdeo, N.K. Tiwari, B. Fisher and K. Tas, *Related fixed point theorems on two complete and compact metric spaces*, Internat. J. Math. and Math. Sci., Vol. 21, No. 3, (1998), 559-564.
- [7] S.H. Cho, G.Y. Lee and J.S. Bae, *On coincidence and fixed point theorems in symmetric spaces*, Fixed Point Theory Appl. 2008, Art. ID 562130, (2008), 9 pages.
- [8] W.A. Wilson, *On semi-metric spaces*, Amer. J. Math., 53, (1931), 361 – 373.

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