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BEST ONE-SIDED APPROXIMATION OF UNBOUNDEDFUNCTION

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ABSTRACT

In this work, we shall study characterize of one-sided approximation of unbounded functions in weighted space and we shall compare between different norms such as $\|f\|_{p,\alpha}$ *,* $\|f\|_{\delta,p\alpha}$ *and* $\|f\|^{\Phi}_{\delta,p\alpha}$ *where* f *is any unbounded function by using the function* $\Phi_v = \Phi_v(x)$ *such that:*

$$\boldsymbol{\Phi_v(x) = sin^4\left(\frac{\pi}{4m}\right)\{(sin^4 m(x-x_v))/(sin^4(x-x_v)/2) + (sin^4 m(x+x_v))/(sin^4(x+x_v)/2)\}}$$

where m be natural number, $v=\{0,1,2,\dots\dots,m-1\}$*,* $x \in X = [0,1]$ *or* $x \in X = [-1,1]^d$ *(x is mult-variable) and* $x_v = \pi - (2v+1)\pi/2m$*. Alsowe shall deduce degree of best one sided approximation in weighted by using some result of degree of best approximation in* L_p*-space, also we discuss the relation between degree of best one-sided approximation of unbounded function and degree of best one sided approximation of it's derivative.*

1. INTRODUCTION

Throughout this paper, we use the weight function $\omega(x) = e^{-\alpha x}$ which is non-negative measurable function on $(0,\infty)$. For $1 \le p \le \infty$ the weighted space:

$L_{p,\alpha} = \{f \mid f: X \longrightarrow R \text{ , such that } |f(x)\omega_\alpha(x)| \le M, \alpha \ge 1\}$ such that for the function f,

$$\|f\|_{p,\alpha} = \left[\int_a^b |f(x)|^p e^{-\alpha p x}\, dx\right]^{\frac{1}{p}} < \infty$$

Also, $\|f\|_{\infty,\alpha} = \left\{sup\{|f(x)e^{-\alpha x} : x \in \mathrm{X}|\}\right\} < \infty$. The local norm of the function f is define by:

$$\|f\|_{\delta,p,\alpha} = \left[\int_{\mathrm{X}} [sup\{|f(x)e^{-\alpha x}| : x \in \mathrm{N}(\delta,\mathrm{t})\}]^p dx\right]^{\frac{1}{p}}$$

where $\delta > 0$ and $N(\delta,t) = \{y \in \mathrm{X} : |t-y| \le \delta\}$. By using the function:

$$\psi(u,\delta) = \delta\Phi(u) + \delta^2$$

where $\Phi(u) = \begin{cases} (1-u^2)^{\frac{1}{2}} & if \ \mathrm{X} = [-1,1] \\ u(1-u)^{\frac{1}{2}} & if \ \mathrm{X} = [0,1] \end{cases}$

We can define another local norm as: $\|f\|^{\Phi}_{\delta,p,\alpha} = \left[\int_{\mathrm{X}} [sup\{|f(x)e^{-\alpha x}| : x \in \mathrm{N}((\psi,\delta),\mathrm{x})\}]^p dx\right]^{\frac{1}{p}}$ also we mention $\omega_r^{\varphi}(f,x,\delta)_{p,\alpha} = \underbrace{sup}_{0 \le h \le \delta} \left\|\Delta^r_{h\varphi} f(x)\right\|_{p,\alpha}$ (the ordinary Ditzian-Totik modulus of smoothness for f)

where: $\Delta^r_{h\varphi} f(x) = \sum_{i=0}^{r} (-1)^{i+r} C_i^r f(x+i\varphi h), x + i\varphi h \in [0,1]$

And we shall define the algebraic polynomial F_m such that:

$F_m(x) = \Phi_m(x) = \Phi_v(\arccos x)$ *on* $x \in [0,1]$ *and* $\Phi_{j,m}(x) = \prod_{s=1}^{d} F_{j_s,m}(x_s)$

where $x \in [-1,1]^d$. Suppose P_n is the polynomial of best approximation of the function f in the weighted space which is a Banach Space. And suppose that:

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$Q_n^{\mp}(f,x) = P_n(f,x) \mp \sum_{j\in Z} \Phi_{j,m}(x)\|f - P_n(f,x)\|_{\infty,\alpha(X_j)}$

where $j = (j_1, j_2, \dots \dots, j_d)$, j $\in Z$ $such\ that\ Z = \{0,1,2, \dots \dots, m-1\}^d$.And $X_j = [z_{j_1}, z_{j_1+1}] \times [z_{j_1+1}, z_{j_1+2}] \times \dots \dots \dots \dots \times [z_{j_d}, z_{j_d+1}]$.

Also we define $\grave{Z} = \{0,1,2, \dots, m\}^d$ and $\grave{X}_j = [z_{j_1-1}, z_{j_1+1}] \times [z_{j_1+1}, z_{j_1+2}] \times \dots \dots \dots \dots \times [z_{j_d-1}, z_{j_d+1}]$.Clear (as we shall prove in theorem (3.4)) that:

$Q_n^-(f,x) \le f(x) \le Q_n^+(f,x)$. We define the degree of best one-sided approximation of function f as:

$E_n^{\sim} = inf\|P_n^+ + P_n^-\|$ where $P_n^{\mp} \in \boldsymbol{P}_n$(where $\boldsymbol{P}_n$ is the space of all polynomials of degree n).

2. AUXILIARY RESULTS

Here we brief some results which we needed in our work:

Lemmas 2.1: [1] Let $j \in Z$ then:
(i) $\Phi_{j,m}(x) \in \boldsymbol{P}_{4m-2}^d$.
(ii)$\Phi_{j,m}(x) \ge 1 \quad \forall x \in X_j$.

Lemma2.2: Let$a_j \ge 0, j \in Z$ then:

$$\left\|\sum\nolimits_{j\in Z} a_j\ \Phi_{j,m}(x)\right\|_{p,\alpha(X)} \le c \sum\nolimits_{j\in Z} a_j^p (z_j - z_{j-1})^{\frac{1}{p}}$$

Proof: By using lemma 2.1(ii) we get:

$$\sum\nolimits_{j\in Z} a_j\ \Phi_{j,m}(x)\, e^{-\alpha p x} \le \sum\nolimits_{j\in Z} a_j\ \Phi_{j,m}(x)$$

Also, $\left\|\sum_{j\in Z} a_j\ \Phi_{j,m}(x)\right\|_{p,\alpha(X)} x = \left\|\sum_{j\in Z} a_j\ \Phi_{j,m}(x) e^{-\alpha p x}\right\|_{p(X)}$

$$\le \left\|\sum\nolimits_{j\in Z} a_j\ \Phi_{j,m}(x)\right\|_{p(X)}$$

$$\le c\left[\sum\nolimits_{j\in Z} a_j^p (z_j - z_{j-1})\right]^{\frac{1}{p}}.$$

Lemma 2.3: [1] Suppose that u, v$\in [-1,1]$ and $0 < t \le \frac{1}{2}$ then:

$|u - v| \le \psi(t,u)\ where\ \psi(t,u) \le 6\psi(t,u)$ and $\psi(t,v) \le 4\psi(t,u)$.

Lemma2.4: Let$f \in L_{p,\propto}(X)$; $(1 \le p \le \infty)$ and $0 < t \le \frac{1}{2}$ then:
$\left\|\psi(t,.)^{-\frac{1}{p}}\|f\|_{p,\alpha(N(t,.))}\right\|_{p,\alpha} \le c\|f\|_{p,\alpha(X)}$

Proof:
$\left\|\psi(t,.)^{-\frac{1}{p}}\|f\|_{p,\alpha(N(t,.))}\right\|_{p,\alpha} = \left\|\psi(t,.)^{-\frac{1}{p}}\right\|_{p,\alpha(X)} \left\|\|f\|_{p,\alpha(N(t,.))}\right\|_{p,\alpha(X)}$

$$\le \|\psi(t,.)^p\|_{p,\alpha(X)}\|f\|_{p,\alpha(X)} \le c_p\|f\|_{p,\alpha(X)}.$$

Lemma2.5: [2] If f is bounded measurable function on$[a,b], a,b \in R$, then:

$$\int_a^b f(x)dx \approx (b-a)n^{-1}\sum_{i=1}^n f(x_i)\ where\ x_i = a + \frac{1}{2n}(b-a)(2i-1).$$

Lemma2.6: [3] Let $m\ and\ \delta$ be any number such that $m\delta \le \frac{1}{4}$ then for $f \in L_{p,\alpha}$ we get:
$\|f\|_{m\delta,p,\alpha}^{\Phi} \le c_p m^{\frac{2d}{p}}\|f\|_{\delta,p,\alpha}^{\Phi}$.

Lemma 2.7: [1] Suppose that $x \in X = [-1,1]^d, 0 \le t \le \frac{1}{2}$ and $N = [2\pi/t]$ then:

$\grave{X}_j \subset N(t,x), x \in X_j$

We can write the local norm $\|f\|^{\Phi}_{\delta,p,\alpha}$ of the unbounded functions f with form globally norm $\|f_\delta\|_{p,\alpha(\mathrm{X})}$ as the following:

$$\|f\|^{\Phi}_{\delta,p,\alpha} = \|f\|^{\Phi}_{\delta,p,\alpha(\mathrm{X})} = \big\|\|f\|_{\infty,N(\delta,.)}\big\|_{p,\alpha(\mathrm{X})} = \|f_\delta\|_{p,\alpha(\mathrm{X})}$$

where $f_\delta(\mathrm{X}) = sup\{|f(t)|: t \in N(\delta,x\}$.

Also, we can define $N(\delta,x)^\sigma$ as the following:

$$N(\delta,x)^\sigma = N(\psi(\delta,x),x)^\sigma = \prod_{s:\sigma_s=1} N(\psi(\delta,x_s),x_s)$$
$$= N(\psi(\delta,x_1),x_1) \dots \dots N(\psi(\delta,x_d),x_d)$$

Also, $\psi(\delta,x)^\sigma = \prod_s \psi(\delta,x_s) = \prod_{s=1}^d \psi(\delta,x_s)$.

3. MAIN RESULTS

Now, we prove the following theorem where x is a single variable that is $x \in [0,1]$

Theorem 3.1: Suppose that $P_n \in \boldsymbol{P}_n$ then:
$\|P_n\|^{\Phi}_{\delta,p,\alpha} \le c_p[1 + \max(n\delta, n^2\delta^2)^{\frac{1}{p}}\|P_n\|_{p,\alpha}$ Where $1 \le p \le \infty$

Proof: We shall the use the following equality:

$$P_n(t) - P_n(x) = \int_x^t \grave{P}_n(u)du\,, \quad x,t \in \mathrm{X}$$

So, $\big|\|P_n\|^{\Phi}_{\delta,p,\alpha} - \|P_n\|_{p,\alpha}\big| \le [\int_{\mathrm{X}}(\int_{\mathrm{N(x,\delta)}}(\grave{P}_n(u)du)^p\omega^p_\alpha(x)dx]^{\frac{1}{p}}$

$$\le [c_1\int_{\mathrm{X}}\frac{1}{[N(x,\delta)]^p}\int_{[\mathrm{N}(x,\delta)]^p}(\delta\Phi(u)+\delta^2)^p(\grave{P}_n(u)du)^p\omega^p_\alpha(x)dx]^{\frac{1}{p}}$$

$$\le c_1^{\frac{1}{p}}[\left(\frac{\delta\Phi(u)+\delta^2}{\mathrm{N(x,\delta)}}\right)^p]^{\frac{1}{p}}[\int_{\mathrm{X}}[\grave{P}_n(x)]^p\omega^p_\alpha(x)dx]^{\frac{1}{p}}$$

$$\le c_p\left\|(\delta\Phi+\delta^2)\grave{P}_n(x)\right\|_{p,\alpha}$$

$$\le c_p[\delta\left\|\Phi\grave{P}_n\right\|_{p,\alpha} + \delta^2\left\|\grave{P}_n\right\|_{p,\alpha}]$$

$$\le [\max(n\delta, n^2\delta^2)]\|P_n\|_{p,\alpha}$$

$$= [1 + max(n\delta, n^2\delta^2)]\|P_n\|_{p,\alpha}$$

Then: $\|P_n\|^{\Phi}_{\delta,p,\alpha} \le c_p[1 + \max(n\delta, n^2\delta^2)^{\frac{1}{p}}\|P_n\|_{p,\alpha}$ where $1 \le p \le \infty$.

Now, we shall prove same theorem where x is multivariable that is $x \in [-1,1]^d$

Theorem 3.2: Suppose that $P_n \in \boldsymbol{P}^d_n$ then:
$\|P_n\|^{\Phi}_{\delta,p,\alpha} \le c_d[1 + \max(n\delta, n^2\delta^2)^{\frac{1}{p}}\|P_n\|_{p,\alpha}$ Where $1 \le p \le \infty$

Proof: We shall us the following equality:

$$P(t) - P(x) = \sum_{\substack{\alpha,|\alpha|\ge 1\\ \alpha_s=0,1}} \int_{x^{(\alpha)}}^{t^{(\alpha)}} D^\alpha P_n(x^{(1-\alpha)} + u^{(\alpha)})du^{(\alpha)}$$

By using lemmas (2.3) and (2.4) we get:

$$\left|\|P_n\|^{\Phi}_{\delta,p,\alpha} - \|P_n\|_{p,\alpha}\right| \le [\int_{X} |P_n(\xi_x) - P_n(x)|^p \omega^p_\alpha(x)dx]^{\frac{1}{p}}$$

$$\le [\int_{X} \left[\sum_{|\alpha|\ge 1} \int_{N(x,\delta)^\alpha} D^\alpha P_n(x^{(1-\alpha)} + u^{(\alpha)})du^{(\alpha)}\right]^p \omega^p_\alpha(x)dx\,]^{\frac{1}{p}}$$

$$\le \left[c_1 \sum_{|\alpha|\ge 1} \int_{X} \frac{1}{[N(x,\delta)^\alpha]^p} \int_{N(x,\delta)^\alpha} ([\delta\Phi(u)^\alpha + \delta^2]^p \left[D^\alpha P_n\left(x^{(1-\alpha)} + u^{(\alpha)}\right)du^{(\alpha)}\right]^p \omega^p_\alpha(x)dx\right]^{\frac{1}{p}}$$

$$\le c_1^{\frac{1}{p}} [(\frac{\delta\Phi(u)^{(\alpha)}+\delta^2}{N(x,\delta)})^{p\alpha}]^{\frac{1}{p}} [\int_{X} (D^\alpha P_n(x^{(1-\alpha)} + u^{(\alpha)})du^{(\alpha)})^p \omega^p_\alpha(x)dx]^{\frac{1}{p}}$$

$$\le c_p \sum_{|\alpha|\ge 1} [\int_{X} |D^\alpha P_n(x)|^p [\delta\Phi(x) + \delta^2]^\alpha \omega^p_\alpha(x)dx]^{\frac{1}{p}}$$

$$\le c_p \sum_{|\alpha|\ge 1} \|[\delta\Phi(x) + \delta^2]^\alpha D^\alpha p_n\|_{p,\alpha}$$

$$\le c_p \sum_{|\alpha|\ge 1} [\delta^{|\alpha|} \|\Phi^\alpha D^\alpha p_n\|_{p,\alpha} + \delta^{2^{|\alpha|}} \|D^\alpha P_n\|_{p,\alpha}]$$

$$\le c_p \sum_{|\alpha|\ge 1} [\max(n\delta, n^2\delta^2)]^{|\alpha|} \|P_n\|_{p,\alpha}$$

$$\le c_p [(1 + \max(n\delta, n^2\delta^2))^d - 1] \|P_n\|_{p,\alpha}$$

So, $\|P_n\|^{\Phi}_{\delta,p,\alpha} \le c_p [1 + \max(n\delta, n^2\delta^2)^{\frac{1}{p}} \|P_n\|_{p,\alpha}$ where $1 \le p \le \infty$.

By using theorem 3.2 we get the following corollary:

Corollary 3.3: Suppose that $P_n \in \boldsymbol{P}^d_n$ then:

$$\|P_n\|_{p,\alpha(X)} \le \|P_n\|^{\Phi}_{p,\alpha(X)} \le c_d \|P_n\|_{p,\alpha(X)}$$

Theorem 3.4: For any function f on $X = [-1,1]^d$

$$Q^-_n(f,x) \le f(x) \le Q^+_n(f,x)$$

Proof: Let $x \in X = [-1,1]^d$ and by using lemma (2.1) we get:

$$Q^+_n(f,x) = P_n(x) + \sum_{j\in Z} \Phi_{j,m}(x) \|f(x) - P_n(x)\|_{\infty(X_j)}$$

$$\ge P_n(x) + \|f(x) - P_n(x)\|_{\infty(X_j)}$$

$$\ge P_n(x) + |f(x) - P_n(x)| = f(x).$$

$$Q^-_n(f,x) = P_n(x) - \sum_{j\in Z} \Phi_{j,m}(x) \|f(x) - P_n(x)\|_{\infty(X_j)}$$

$$\le P_n(x) - \|f - P_n(x)\|_{\infty(X_j)}$$

$$\le P_n(x) - |f(x) - P_n(x)| = f(x)$$

So, $Q^-_n(f,x) \le f(x) \le Q^+_n(f,x)$.

Theorem 3.5: Suppose that $f \in L_{\infty,\alpha(X)}$ then:

$$E^{\sim}_n(f)_{p,\alpha} \le c_d E_n(f)^{\Phi}_{1/n,\alpha,p} \le c_d E^{\sim}_n(f)_{p,\alpha}$$

Proof: Suppose that $P^+_n, P^-_n \in \boldsymbol{P}^d_n$ such that $p^-_n \le f(x) \le p^+_n$, $x \in X$

And $E^{\sim}_n(f)_{p,\alpha} = \|P^+_n - P^-_n\|_{p,\alpha}$

So by using corollary (3.3)

$E_n(f)^{\Phi}_{1/n,\alpha,p} \leq E_n^{\sim}(f)^{\Phi}_{1/n,\alpha,p} \leq \|P_n^+ - P_n^-\|^{\Phi}_{1/n,p,\alpha} \leq c_p\|P_n^+ - P_n^-\|_{p,\alpha}$

$=c_p E_n^{\sim}(f)_{p,\alpha}$

Then: $E_n(f)^{\Phi}_{1/n,\alpha,p} \leq c_d E_n^{\sim}(f)_{p,\alpha}$

Also by using lemmas (2.2), (2.6), (2.7) and theorem (3.5), we get:

$$E_n^{\sim}(f)_{p,\alpha} \leq \|Q_n^+ - Q_n^-\|_{p,\alpha} = 2\left\|\sum_{j\in Z}\Phi_{j,m}(x)\|f-P_n\|_{\infty(X_j)}\right\|_{p,\alpha}$$

$$\leq c\left[\sum_{j\in Z}(Z_j - Z_{j-1})\|f-P_n\|^p_{\infty(X_j)}\right]^{\frac{1}{p}}$$

$$\leq c\left[\sum_{j\in Z}\int_{X_j}\|f-P_n\|^p_{\infty(X_j)}\,dx\right]^{\frac{1}{p}}$$

$$\leq c\left[\sum_{j\in Z}\int_{X_j}\|f-p_n\|^p_{\infty,(N(2\pi/m-1,x)}\,dx\right]^{\frac{1}{p}}$$

$$\leq c\left[\sum_{j\in Z}\int_{X}\|f-p_n\|^p_{\infty,(N(2\pi/m-1,x)}\,dx\right]^{\frac{1}{p}}$$

$$\leq c_p\|f-p_n\|^{\Phi}_{1/n,p,\alpha}$$

$$= c_p E_n(f)^{\Phi}_{1/n,p,\alpha}$$

So, $E_n^{\sim}(f)_{p,\alpha} \leq c_p E_n(f)^{\Phi}_{1/n,\alpha,p} \leq c_p E_n^{\sim}(f)_{p,\alpha}$.

Theorem 3.6: Suppose that $f \in L_{p,\alpha(X)}$ such that $1 \leq p \leq \infty$and let

$\sum_{v=1}^{\infty} V^{2d/p-1}\, E_v(f)_{p,\alpha} < \infty$, and let $f = F$ almost

everywhere then:$E_n^{\sim}(f)_{p,\alpha} \leq c_p n^{-2d/p}\sum_{v=1}^{\infty} V^{2d/p-1}\, E_v(f)_{p,\alpha}$.

Proof: Since $E_v(f)_{p,\alpha} = \|f - Q_v\|_{p,\alpha(X)}$, $v = 0,1,2 \ldots \ldots$

Since for $n \in N, \sum_{v=1}^{m}(Q_{n.2} - Q_{n.2^{v-1}}) = Q_{n.2^N} - Q_n$

And since $F = f$almost everywhere

So, $\|F - Q_{n.2^{v-1}}\|_{\infty,\alpha(X)\to 0(\,as\ N\to\infty)}$

And, $F - Q_n(x) = \sum_{v=1}^{\infty}(Q_{n.2} - Q_{n.2^{v-1}})$

Then by using theorems (3.2) and (3.5) we get:

$$E_v^{\sim}(f)_{p,\alpha} \leq c_p E_n(f)^{\Phi}_{1/n,\alpha,p} = c_p E_n(F - Q_n)^{\Phi}_{1/n,\alpha,p}$$

$$\leq c_p\|F - Q_n\|^{\Phi}_{1/n,p,\alpha}$$

$$\leq c_p\sum_{v=1}^{\infty}\|Q_{n.2^v} - Q_{n.2^{v-1}}\|^{\Phi}_{1/n,p,\alpha}$$

$$\leq c_p\sum_{v=1}^{\infty} c_p(2^v)^{\frac{2d}{p}}\|Q_{n.2^v} - Q_{n.2^{v-1}}\|_{p,\alpha}$$

$$\leq c_p\sum_{v=1}^{\infty} 2^{2vd/p}\,[E_{n.2^v}(f)_{p,\alpha} + E_{n.2^{v-1}}(f)_{p,\alpha}]$$

$\leq c_p n^{-2d/p} \sum_{v=1}^{\infty} (2^v.n)^{2d/p} \, [E_{n.2^v}(f)_{p,\alpha} + E_{n.2^{v-1}}(f)_{p,\alpha}]$

$\leq c_p n^{-2d/p} \sum_{v=n}^{\infty} v^{2d/p-1} E_n(f)_{p,\alpha}$.

Before we prove direct theorem of best one-sided approximation in weight space we shall refer to same theorem of best approximation in weight space by using $B_n(f,x)$-operator, where x is single variable.

Theorem 3.7: [4] Let $f \in L_{p,\alpha}(X)$(single case)then $\|f(x) - B_n(f,x)\|_{p,\alpha} \leq c\omega_2^{\varphi}(f,x,\delta)_{p,\alpha} + \tau_2^{\varphi}(f,\delta)_{p,\alpha}$ where $B_n(f,x)$is Bernstein Polynomial of .

Theorem 3.8: (Direct Theorem of onesided approximation in single case) Let f be any function on [0, 1] and let: $Q_n^{\mp}(f,x) = P_n + \Phi_m(x)\|f - P_n\|_{\infty(X)}$ be an operator (where p_n is a best approximation of f) then

$E_n^{\sim}(f)_{p,\alpha} \leq c\omega_2^{\varphi}(f,x,\delta)_{p,\alpha} + \tau_2^{\varphi}(f,\delta)_{p,\alpha}$.

Proof: By using lemma (2.5) and (3.5) and (3.7) we get:

$E_n^{\sim}(f)_{p,\alpha} \leq c_p E_n(f)_{1/n,p,\alpha}^{\Phi} \leq c_p E_n^{\sim}(f)_{p,\alpha}$

$Then: \; E_n^{\sim}(f)_{p,\alpha} \leq c_p E_n(f)_{1/n,p,\alpha}^{\Phi}$

Since $\int_a^b f(x)dx \approx \frac{(b-a)}{n} \sum_{i=1}^{n} f(x_i) \; where \; x_i = a + \frac{1}{2n}(b-a)(2i-1).$

So, we have $(\frac{b-a}{n} \sum_{i=1}^{n} (f^p(x_i)\omega_\alpha{}^p(x_i))^{\frac{1}{p}} \approx (\int_a^b (f^p(x)\omega_\alpha{}^p(x)dx)^{\frac{1}{p}}$

That is $(\frac{b-a}{n} \sum_{i=1}^{n} \left(f^p(x_i)\omega_\alpha{}^p(x_i)\right)^{\frac{1}{p}} \leq c_p \left(\int_a^b (f^p(x)\omega_\alpha{}^p(x)dx\right)^{\frac{1}{p}}$Then, $\|f\|_{1/n,p,\alpha} \leq c_p \|f\|_{p,\alpha}$

Thus, $E_n^{\sim}(f)_{p,\alpha} \leq c_p E_n(f)_{1/n,p,\alpha}^{\Phi} \leq c_p E_n(f)_{1/n,p,\alpha} \leq c_p c_p E_n(f)_{p,\alpha}$

So, $E_n^{\sim}(f)_{p,\alpha} \leq c_p c_p c\omega_2^{\varphi}(f,x,\delta)_{p,\alpha} + \tau_2^{\varphi}(f,\delta)_{p,\alpha} = c_p c\omega_2^{\varphi}(f,x,\delta)_{p,\alpha} + \tau_2^{\varphi}(f,\delta)_{p,\alpha}$.

Before, we prove inverse theorem of best one-sided approximation in weight space we shall refer to same theorem of best approximation in weight space by using $B_n(f,x)$-operator, where x is single variable.

Theorem 3.9: [4] Let$f \in L_{p,\alpha(X)}$(single case) such that ($1\leq p < \infty$) then:

$\tau_2^{\varphi}(f,\Delta,n^{-1})_{p,\alpha} \leq \frac{1}{n} \sum_{k=0}^{n} \|f - B_k(f)\|_{p,\alpha}$.

Theorem 3.10: (Inverse Theorem of one-sided approximation **in single case)** Let f be any function on [0,1] and let: $Q_n^{\mp}(f,x) = P_n + \Phi_m(x)\|f - P_n\|_{\infty(X)}$ be an operator where P_n is a best approximation of f

then: $\tau(f,\frac{1}{n})_{p,\alpha} \leq \frac{c}{n} \sum_{s=0}^{n} \begin{cases} \|f - L_n\|_{p,\alpha} + \|f^{\sim} - L_n\|_{p,\alpha} \; if \; p = 1,\infty \\ \|f - L_n\|_{p,\alpha} \qquad if \; 1 < p < \infty \end{cases}$

Proof: By using theorems (3.5) and (3.9) we get:

$E_n^{\sim}(f)_{p,\alpha} \leq c_p E_n(f)_{1/n,\alpha,p}^{\Phi} \leq c_p E_n^{\sim}(f)_{p,\alpha}$

That is $E_n^{\sim}(f)_{p,\alpha} \leq c_p E_n(f)_{1/n,\alpha,p}^{\Phi}$

But we have $E_n(f)_{p,\alpha} \leq E_n(f)_{1/n,p,\alpha}^{\Phi} \leq c_p E_n^{\sim}(f)_{p,\alpha}$

Then: $E_n(f)_{p,\alpha} \leq c_p E_n^{\sim}(f)_{p,\alpha}$

So, $\sum_{i=1}^{N} E_n(f)_{p,\alpha} \leq c_p \sum_{i=1}^{N} E_n^{\sim}(f)_{p,\alpha}$

Thus: $\tau_2^{\varphi}(f,\Delta,n^{-1})_{p,\alpha} \leq \frac{1}{n} \sum_{k=0}^{n} \|f - B_k(f)\|_{p,\alpha}$.

Now we discuss the relation between degree of best one-sided approximation of unbounded function$f$ anddegree of best one-sidedapproximation of its derivative, where x is single variable.

Theorem 3.12: Let$f \in L^{(1)}{}_{p,\alpha}$ be any function on [0, 1] (i.e. f and $\acute{f}$ in$L_{p,\alpha}$) and let:

$Q_n^{\mp}(f,x) = P_n + \Phi_m(x)\|f - P_n\|_{\infty(X)}$ be an operator where p_n is a best approximation of f then

$E_n^{\sim}(\acute{f})_{p,\alpha} \leq c_p(4m-2)\tau\left(f, \frac{1}{n}\right)_{p,\alpha}$.

Proof: Since $Q_n^-(f,x) \leq f(x) \leq Q_n^+(f,x), x \in [0,1]$ Then one of the cases is true:

$\grave{Q}_n^-(f,x) \leq \grave{f}(x) \leq \grave{Q}_n^+(f,x)$, $\grave{Q}_n^+(f,x) \leq \grave{f}(x) \leq \grave{Q}_n^-(f,x)$, $\grave{Q}_n^-(f,x) \leq \grave{Q}_n^+(f,x) \leq \grave{f}(x)$, $\grave{f}(x) \leq \grave{Q}_n^-(f,x) \leq \grave{Q}_n^+(f,x)$,$\grave{Q}_n^+(f,x) \leq \grave{Q}_n^-(f,x) \leq \grave{f}(x)$ or$\grave{f}(x) \leq \grave{Q}_n^+(f,x) \leq \grave{Q}_n^-(f,x)$.

Where $\grave{Q}_n^{\mp}(f,x) = \grave{P}_n \mp \grave{\Phi}_m(x)\|f(x) - P_n\|_{\infty(X)}$

If $\grave{Q}_n^-(f,x) \leq \grave{f}(x) \leq \grave{Q}_n^+(f,x)$, $\grave{Q}_n^+(f,x) \leq \grave{f}(x) \leq \grave{Q}_n^-(f,x)$

Then, by similar way of proof of theorem 3.5 and using lemma 2.1(i) and Bernstein inequality we get:

$$
\begin{aligned}
E_n^{\sim}(\grave{f})_{p,\alpha} &\leq \left\|\grave{Q}_n^+(f,x) - \grave{Q}_n^-(f,x)\right\|_{p,\alpha} \\
&= \left\|\grave{P}_n - \grave{P}_n + 2(\grave{\Phi}_m(x)\|f(x) - P_n\|_{\infty(X)})\right\|_{p,\alpha} \\
&= 2\left\|\grave{\Phi}_m(x)\|f(x) - P_n\|_{\infty(X)}\right\|_{p,\alpha} \\
&\leq 2(4m-2)\left\|\Phi_m(x)\|f(x) - P_n\|_{\infty(X)}\right\|_{p,\alpha} \\
&\leq c_1(4m-2)\left[(1)\|f(x) - P_n(x)\|^p{}_{\infty(X)}\right]^{\frac{1}{p}} \\
&\leq c_1(4m-2)[\textstyle\int_X (\|f(x) - P_n(x)\|^p{}_{\infty(X)}dx)]^{\frac{1}{p}} \\
&\leq c_1(4m-2)\left[\textstyle\int_X (\|f(x) - P_n(x)\|^p{}_{\infty,(N(2\pi/m-1,x))}dx\right]^{\frac{1}{p}} \\
&\leq c_1(4m-2)\|f - P_n(x)\|^{\Phi}_{1/n,p,\alpha} \\
&= c_1(4m-2)E_n(f)_{1/n,p,\alpha} \\
&\leq c_1c_p(4m-2)E_n(f)_{p,\alpha} \\
&\leq cc_1c_p(4m-2)\tau\left(f, \tfrac{1}{n}\right)_{p,\alpha} = c_p(4m-2)\tau\left(f, \tfrac{1}{n}\right)_{p,\alpha}.
\end{aligned}
$$

If $\grave{Q}_n^-(f,x) \leq \grave{Q}_n^+(f,x) \leq \grave{f}(x)$,$\grave{f}(x) \leq \grave{Q}_n^-(f,x) \leq \grave{Q}_n^+(f,x)$,$\grave{Q}_n^+(f,x) \leq \grave{Q}_n^-(f,x) \leq \grave{f}(x)$
or
$\grave{f}(x) \leq \grave{Q}_n^+(f,x) \leq \grave{Q}_n^-(f,x)$.

Then:
$E_n^{\sim}(\grave{f})_{p,\alpha} = E_n(\grave{f})_{p,\alpha} \leq \left\|\grave{f} - \grave{Q}_n^{\mp}(f,x)\right\|_{p,\alpha} \leq c_p\left\|f - Q_n^{\mp}(f,x)\right\|_{p,\alpha} \leq c_p\omega\left(f, \Delta_n(x)\right)$.
where $\Delta_n(x) = \max\left(\frac{\sqrt{1-x^2}}{n}, \frac{1}{n^2}\right)$ $n = 1,2, \ldots$ and $\Delta_0(x) = 1$.

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