CHARACTERIZATION OF A PARTIAL ORDER RELATION ON PRE A*-ALGEBRA

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(Received on: 08-03-13; Revised & Accepted on: 22-10-13)

ABSTRACT

This manuscript is a classification on Pre-A*-algebra \(A\) in sight of it is like a partially ordered set. Using a binary operation in Pre-A*-algebra, an observation is made on Pre A*-Algebra as a partially ordered set with respect to binary operation \(\land\) and obtained consequent results. It is also grant access to an equivalent condition for a Pre A*-algebra become a Boolean algebra.

Key words: A*-algebra, Pre-A*-algebra, Boolean algebra, partially ordered set, Ada, homomorphism.


INTRODUCTION

In a draft manuscript entitled “The Equational theory of Disjoint Alternatives”, E. G. Manes (1989) introduced the concept of Ada (Algebra of disjoint alternatives) \((A, \land, \lor, \neg, 0, 1, 2)\) which is however differs from the definition of the Ada of E. G. Manes (1993) later paper entitled “Adas and the equational theory of if-then-else”. While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras and the later concept is based on C-algebras \((A, \land, \lor, \neg, \sim)\) introduced by Fernando Guzman and Craig C. Squir (1990). P. Koteswara Rao (1994) first introduced the concept of A*-algebra \((A, \land, \lor, *, \neg, 0, 1, 2)\) not only studied the equivalence with Ada, C-algebra, Ada’s connection with 3-Ring, Stone type representation but also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra.


Boolean algebra depends on two element logic. C-algebra, Ada, A*-algebra and our Pre A*-algebra are regular extensions of Boolean logic to 3 truth values, where the third truth value stands for an undefined truth value. The Pre A*-algebra structure is denoted by \((A, \land, \lor, \neg, \sim)\) where \(A\) is non-empty set, \(\land, \lor\) are binary operations and \(\neg, \sim\) is a unary operation.

In this paper we define a relation \(\leq\) on Pre A*-algebra with respect to the binary operation \(\lor\) and we discuss the properties of a Pre A*-algebra like a poset. We find the necessary conditions for a poset to become a lattice. We also present a equivalent condition for a Pre A*-algebra become a Boolean algebra. For any \(a \in A\) define \(A_a = \{x \in A / a \lor x = x\}\) and \(x^a = a \lor x^\sim\) then \((A_a, \land, \lor, \sim)\) is a Pre A*-algebra. We also define a mapping \(\alpha_{a,b}\) from \(A_a\) to \(A_b\) by \(\alpha_{a,b}(x) = b \lor x\) for all \(x \in A_a\) is a homomorphism of Pre A*-algebras.

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1. PRELIMINARIES

1.1. Definition: The relation R on a set A is called a partial order on A when R(≤) is reflexive, anti-symmetric, and transitive. Under these conditions, the set A is called a partially ordered set or a poset. Frequently we write (A, R) or (A, ≤) to denote that A is partially ordered by the relation R(≤). Since the relation ≤ on the set of real numbers is the prototype of a partial order it is common to write ≤ to represent an arbitrary partial order can be described as follows:

1. For all a ∈ A, a ≤ a (symmetry)
2. For all a, b ∈ A, a ≤ b, b ≤ a, then a = b (anti symmetry)
3. For all a, b, c ∈ A, a ≤ b and b ≤ c, then a ≤ c (transitivity)

Two elements a and b in A are said to be comparable under ≤ if either a ≤ b or b ≤ a; otherwise they are incomparable. If every pair of elements of A are comparable, then we say that the poset is totally ordered.

1.2. Definition: An algebra (A, ∧, ∨, (−) ~) where A is a non-empty set with 1, ∧, ∨ are binary operations and (−) ~ is a unary operation satisfying

(a) x~x = x ∀x ∈ A
(b) x ∧ x = x, ∀x ∈ A
(c) x ∧ y = y ∧ x, ∀x, y ∈ A
(d) (x ∧ y) ~ = x ~ ∨ y ~ ∀x, y ∈ A
(e) x ∧ (y ∧ z) = (x ∧ y) ∧ z, ∀x, y, z ∈ A
(f) x ∧ (y ∨ z) = (x ∧ y) ∨ (x ∧ z), ∀x, y, z ∈ A
(g) x ∧ y = x ∧ (x ~ ∨ y), ∀x, y ∈ A is called a Pre A*-algebra.

1.1. Example: 3 = {0, 1, 2} with operations ∧, ∨, (−) ~ defined below is a Pre A*-algebra.

<table>
<thead>
<tr>
<th>∧</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>∨</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>x</th>
<th>x~</th>
</tr>
</thead>
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<tr>
<td>0</td>
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<td>0</td>
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<td>0</td>
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</table>

1.1. Note: The elements 0, 1, 2 in the above example satisfy the following laws:

(a) 2~ = 2
(b) 1 ∧ x = x for all x ∈ 3
(c) 0 ∨ x = x for all x ∈ 3
(d) 2 ∧ x = 2 ∨ x = 2 for all x ∈ 3.

1.2. Example: 2 = {0, 1} with operations ∧, ∨, (-)~ defined below is a Pre A*-algebra.

<table>
<thead>
<tr>
<th>∧</th>
<th>0</th>
<th>1</th>
<th>∨</th>
<th>0</th>
<th>1</th>
<th>x</th>
<th>x~</th>
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<tbody>
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</tbody>
</table>

1.2. Note:

(i) (2, ∨, ∧, (−)) is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.

(ii) The identities 1.2(a) and 1.2(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.2(b) to 1.2(g).

1.3. Definition: Let A be a Pre A*-algebra. An element x ∈ A is called a central element of A if x ∨ x~ = 1 and the set {x ∈ A| x ∨ x~ = 1} of all central elements of A is called the centre of A and it is denoted by B (A).

1.1. Theorem: Let A be a Pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations ∧, ∨, (−)~
1.1. Lemma: [Satyanarayana.A, (2012).] Every Pre A*-algebra with 1 satisfies the following laws

(a) \( x \lor 1 = x \lor x^\sim \)  
(b) \( x \land 0 = x \land x^\sim \)

1.2. Lemma: [7] Every Pre A*-algebra with 1 satisfies the following laws.

(a) \( x \land (x^\sim \lor x) = x \lor (x^\sim \land x) = x \)  
(b) \( (x \lor x^\sim) \land y = (x \land y) \lor (x^\sim \land y) \)  
(c) \( (x \lor y) \land z = (x \land z) \lor (x^\sim \land y \land z) \)

1.4. Definition: Let \((A_1, \lor, \land, (-)^\sim)\) and \((A_2, \lor, \land, (-)^\sim)\) be a two Pre A*- algebras. A mapping \( f : A_1 \to A_2 \) is called a Pre A*-homomorphism if

(i) \( f(a \land b) = f(a) \land f(b) \)  
(ii) \( f(a \lor b) = f(a) \lor f(b) \)  
(iii) \( f(a^\sim) = (f(a))^\sim \)

The homomorphism \( f : A_1 \to A_2 \) is onto, then \( f \) is called epimorphism.

The homomorphism \( f : A_1 \to A_2 \) is one-one then \( f \) is called monomorphism

The homomorphism \( f : A_1 \to A_2 \) is one-one and onto then \( f \) is called an isomorphism, and \( A_1, A_2 \) are isomorphic, denoted in symbol \( A_1 \cong A_2 \).

2. Pre A*-algebra as a poset with respect to Binary Operation \( \lor \)

2.1 Definition: Let \( A \) be a Pre A*-algebra. Define \( \leq \) on \( A \) by \( x \leq y \) if and only if \( y \lor x = x \lor y = y \).

2.1 Lemma: If \( A \) is a Pre A*-algebra, then \((A, \leq)\) is a poset.

Proof: Since \( x \lor x = x \), \( x \leq x \) for all \( x \in A \).

Therefore \( \leq \) is reflexive.

Suppose that \( x, y, z \in A \), \( x \leq y \) and \( y \leq z \).

Then we have \( y \lor x = x \lor y = y \) and \( z \lor y = y \lor z = y \).

Now \( z = y \lor z = x \lor y \lor z = x \lor z \).

Therefore \( x \lor z = z \lor x = z \), i.e., \( x \leq z \).

This shows that \( \leq \) is transitive.

Suppose that \( x, y \in A \), \( x \leq y \) and \( y \leq x \).

Then we have \( y \lor x = x \lor y = y \) and \( x \lor y = y \lor x = y \).

This shows that \( x = y \).

Therefore \( \leq \) is antisymmetric.

Therefore \((A, \leq)\) is poset.

2.1. Note: If \( A \) is a Pre A*-algebra with 1, 0, 2 then \( 0 \leq x (0 \lor x = x \lor 0 = x) \), for all \( x \in A \) and \( x \leq 2 (2 \lor x = x \lor 2 = 2) \). This shows that 2 is the greatest element and 0 is the least element of the poset.
The Hasse diagram of the poset \((A, \leq)\) is given by

\[
\begin{array}{c}
2 \\
0 \\
1
\end{array}
\]

Diagram: 2.1

We know that \(A \times A\) is a Pre \(A^*\)-algebra under point wise operation. The Hasse diagram is given below

\[
A \times A = \{a_1 = (1, 1), a_2 = (1, 0), a_3 = (1, 2), a_4 = (0, 1), a_5 = (0, 0), a_6 = (0, 2), a_7 = (2, 1), a_8 = (2, 0), a_9 = (2, 2)\}
\]

Diagram: 2.2

Observe that, \(x \leq a_9\), i.e., \((x \vee a_9 = a_9 \vee x = a_9)\) and \(a_5 \leq x\) \((x \vee a_5 = a_5 \vee x = x)\) for all \(x \in A \times A\). This shows that \(a_9\) is the greatest element and \(a_5\) is the least element of \(A \times A\).

We have \(2 \times 3 = \{a_1 = (1,1), a_2 = (0,0), a_3 = (1,0), a_4 = (0,1), a_5 = (0,2), a_6 = (1,2)\}\) is Pre \(A^*\)-algebra under point wise operation having four central elements two non-central elements and no element is satisfying the property that \(a^\sim = a\).

The Hasse diagram for \((2 \times 3, \leq)\) as given below

Diagram: 2.3

Observe that, \(x \leq a_6\), i.e., \((x \vee a_6 = a_6 \vee x = a_6)\) and \(a_2 \leq x\) \((x \vee a_2 = a_2 \vee x = x)\) for all \(x \in 2 \times 3\). This shows that \(a_6\) is the greatest element and \(a_2\) is the least element of \(2 \times 3\).

2.1. Theorem: In the poset \((A, \leq)\), for any \(x \in A\), Supremum \(\{x, x^\sim\} = x \vee x^\sim\) infimum \(\{x, x^\sim\} = x \wedge x^\sim\).

Proof: We have \((x \vee x^\sim) \vee x = x \vee x^\sim\) and \(x^\sim \vee (x \vee x^\sim) = x \vee x^\sim\)

Therefore, \(x \leq x \vee x^\sim\) and \(x^\sim \leq x \vee x^\sim\).

Hence \(x \vee x^\sim\) is an upper bound of \(\{x, x^\sim\}\)
Suppose $n$ is an upper bound of $\{x, x^\sim\}$
That is, $x \leq n, x^\sim \leq n \Rightarrow n \lor x = n$ and $n \lor x^\sim = n$

Now $n \lor (x \lor x^\sim) = (n \lor x) \lor x^\sim = n \lor x^\sim = n$

This shows that $x \lor x^\sim \leq n$

Therefore $x \lor x^\sim$ is a least upper bound of $\{x, x^\sim\}$

This shows that supremum of $\{x, x^\sim\} = x \lor x^\sim$

Again we have $(x \land x^\sim) \lor x = x$ and $(x \land x^\sim) \lor x^\sim = x^\sim$

Therefore $x \land x^\sim \leq x$ and $x \land x^\sim \leq x^\sim$

Hence $x \land x^\sim$ is a lower bound of $\{x, x^\sim\}$

Suppose $m$ is a lower bound of $\{x, x^\sim\}$
That is, $m \leq x, m \leq x^\sim \Rightarrow m \lor x = x$ and $m \lor x^\sim = x^\sim$

Now $m \lor (x \lor x^\sim) = (m \lor x) \lor (m \lor x^\sim) = m \lor x^\sim = m$.

This shows that $m \leq x \land x^\sim$

Therefore $x \land x^\sim$ is greatest lower bound of $\{x, x^\sim\}$.

This shows that infimum of $\{x, x^\sim\} = x \land x^\sim$.

2.2. Theorem: In a poset $(A, \leq)$ with $1$, for any $x, y \in A$, $\sup\{x, y\} = x \lor y$.

Proof: We have $(x \lor y) \lor x = x \lor y$ and $(x \lor y) \lor y = x \lor y$

Therefore, $x \leq x \lor y$ and $y \leq x \lor y$.

Hence $x \lor y$ is an upper bound of $\{x, y\}$

Suppose $m$ is an upper bound of $\{x, y\}$
That is, $x \leq m, y \leq m \Rightarrow m \lor x = m$ and $m \lor y = m$

Now $m \lor (x \lor y) = (m \lor x) \lor (m \lor y) = m \lor y = m$.

This shows that $x \lor y \leq m$

Therefore $x \lor y$ is a least upper bound of $\{x, y\}$

This shows that supremum of $\{x, y\} = x \lor y$.

In general for a Pre A*-algebra with $1$, $x \land y$ need not be the greatest lower bound of $\{x, y\}$ in $(A, \leq)$. For example $2 \lor x = 2 \land x = 2, \forall x \in A$ is not a greatest lower bound. However we have the following.

2.3. Theorem: In a poset $(A, \leq)$ with $1$, for any $x, y \in B(A)$, $\inf\{x, y\} = x \land y$

Proof: If $x, y \in B(A)$, then we have $x \lor (x \land y)$ and $y \lor (x \land y) = y$

This shows that, $x \land y \leq x$ and $x \land y \leq y$.

Hence $x \land y$ is a lower bound of $\{x, y\}$
Suppose m is a lower bound of \{x, y\}, then \( m \lor x = x \), \( m \lor y = y \).

Now \( m \lor (x \land y) = (m \lor x) \land (m \lor y) = x \land y \).

Therefore \( m \leq x \land y \).

Hence \( \inf \{x, y\} = x \land y \).

**2.4. Theorem:** In the poset \((A, \leq)\), if \( x, y \in B(A) \), then \( x \land y \leq x \land x^{\sim} \)

**Proof:**

\[
(x \land x^{\sim}) \lor (x \land y) = [(x \land x^{\sim}) \lor x] \land [(x \land x^{\sim}) \lor y]
\]

\[
= x \land (0 \lor y)
\]

\[
= x \land y
\]

Therefore \( x \land y \leq x \land x^{\sim} \).

**2.5. Theorem:** In the poset \((A, \leq)\), if \( x \leq y \), then for any \( z \in A \),

(a) \( z \land x \leq z \land y \)

(b) \( z \lor x \leq z \lor y \)

**Proof:** If \( x \leq y \), then \( x \lor y = y \).

(a) \( (z \land x) \lor (z \land y) = z \land (x \lor y) = z \land y \).

Therefore \( z \land x \leq z \land y \).

(b) \( (z \lor x) \lor (z \lor y) = z \lor (x \lor y) = z \lor y \).

Therefore \( z \lor x \leq z \lor y \).

Now we are giving the following equivalent conditions for \( x \leq y \).

**2.2. Lemma:** In a Pre \( A^*\)-algebra

(i) \( x \leq y \iff x \lor (x^{\sim} \land y) = (x^{\sim} \land y) \lor x = y \)

(ii) \( x \leq y \iff y \lor (y^{\sim} \land x) = (y^{\sim} \land x) \lor y = y \)

**Proof:**

(i) If \( x \leq y \),

\[
\iff x \lor y = y
\]

\[
\iff x \lor (x^{\sim} \land y) = (x^{\sim} \land y) \lor x = y
\]

(ii) If \( x \leq y \),

\[
\iff y \lor x = y
\]

\[
\iff y \lor (y^{\sim} \land x) = (y^{\sim} \land x) \lor y = y
\]

Now we prove modular type results in the following

**2.3. Lemma:** In the poset \((A, \leq)\), if \( x \leq y \Rightarrow x \lor (y \land z) = y \land (x \lor z) \)

**Proof:** Suppose \( x \leq y \), then \( x \lor y = y \).

Now \( x \lor (y \land z) = (x \lor y) \land (x \lor z) = y \land (x \lor z) \).

If \( x, y \in B(A) \) then by theorem 2.3 \( \inf \{x, y\} = x \land y \). In general \( x \land y \) need not be an upper bound of \( \{x, y\} \) in poset \((A, \leq)\). If \( x \land y \) is an upper bound of \( \{x, y\} \) in poset \((A, \leq)\), then \( A \) becomes Boolean algebra.

Now we have the following theorem.

**2.6. Theorem:** If \( A \) is a Pre \( A^*\)-algebra and \( x \land (x \lor y) = x \) for all \( x, y \in A \) then \((A, \leq)\) is a lattice.

**Proof:** By theorem 2.2, we have every pair of elements have l.u.b and if \( x \lor (x \land y) = x \) for all \( x, y \in A \), then by theorem 2.3 we have every pair of elements have g.l.b. Hence \((A, \leq)\) is a lattice.
Now we present a equivalent condition for a Pre A*-algebra become a Boolean algebra.

2.7. Theorem: The following conditions are equivalent for any Pre A*-algebra \((A, \land, \lor, (-) \sim)\).

1. \(A\) is a Boolean Algebra
2. \(x \land y \leq x\) for all \(x, y \in A\)
3. \(x \land y \leq y\) for all \(x, y \in A\)
4. \(x \land y\) is a lower bound of \(\{x, y\}\) in \((A, \leq)\) for all \(x, y \in A\)
5. \(x \land y\) is an infimum of \(\{x, y\}\) in \((A, \leq)\) for all \(x, y \in A\)
6. \(x \lor x \sim\) is the least element in \((A, \leq)\) for every \(x \in A\)

Proof: (1) \(\Rightarrow\) (2) Suppose \(A\) be a Boolean algebra

Now \(x \lor (x \land y) = x\) (by absorption law)

Therefore \(x \land y \leq x\).

(2) \(\Rightarrow\) (3) Suppose \(x \land y \leq x\) then \(x \lor (x \land y) = x\)

Now \(y \lor (x \land y) = y\).

Hence \(x \land y \leq y\).

(3) \(\Rightarrow\) (4) Suppose that \(x \land y \leq y\) \(\Rightarrow y \lor (x \land y) = y\)

Since \(x \land y \leq y\) then \(x \land y\) is lower bound of \(y\)

Now \(x \lor (x \land y) = x\) (by supposition)

Therefore \(x \land y \leq x\)

\(\Rightarrow x \land y\) is a lower bound of \(x\)

\(x \land y\) is a lower bound of \(\{x, y\}\).

(4) \(\Rightarrow\) (5) Suppose \(x \land y\) is a lower bound of \(\{x, y\}\)

Suppose \(z\) is a lower bound of \(\{x, y\}\) then \(z \leq x, z \leq y\) that is

\(x \lor z = x, y \lor z = y\)

Now \(z \lor (x \land y) = (z \lor x) \land (z \lor y) = x \land y\)

Therefore \(z \leq x \land y\).

\(x \land y\) is the greatest lower bound of \(\{x, y\}\)

Hence \(\text{Inf } \{x, y\} = x \land y\).

(5) \(\Rightarrow\) (6) Suppose \(\text{Inf } \{x, y\} = x \land y\) then \(x, y \in B(A)\)

Now \(\text{Inf } \{x \land x \sim, y\} = x \land x \sim \land y = x \land x \sim\)

\(\Rightarrow x \land x \sim \leq \text{Inf } y\)

Therefore \(x \land x \sim\) is the least element in \((A, \leq)\).

(6) \(\Rightarrow\) (1) Suppose \(x \land x \sim\) is the least element in \(A\) then \(x \land x \sim \leq y\),

for \(y \in A\)
$$\Rightarrow (x \wedge x^\sim) \vee y = y$$

Now $$y \wedge (x \vee y) = [(x \wedge x^\sim) \vee y] \vee (x \vee y)$$

$$= [(x \wedge x^\sim) \vee x] \vee y$$

$$= (x \wedge x^\sim) \vee y = y$$ (by supposition)

Therefore absorption law holds hence $$A$$ is a Boolean algebra.

2.8. Theorem: Let $$A$$ be a pre $$A^*$$-algebra $$x \vee x^\sim$$ is the greatest element in $$(A, \leq)$$ for every $$x \in A$$ then $$A$$ is a Boolean algebra.

Proof: Suppose $$x \vee x^\sim$$ is the greatest element in $$(A, \leq)$$ then

$$y \leq x \vee x^\sim$$ for any $$y \in A$$

$$\Rightarrow (x \vee x^\sim) \vee y = x \vee x^\sim$$

Now $$x \vee (x \wedge y) = [x \wedge (x^\sim \vee x)] \vee (x \wedge y)$$

$$= x \wedge [(x \vee x^\sim) \vee y]$$

$$= x \wedge (x \vee x^\sim)$$ (by supposition)

$$= x$$

Therefore $$x \vee (x \wedge y) = x$$, absorption law holds.

Hence $$A$$ is a Boolean algebra.

2.9. Theorem: Let $$A$$ be a Pre $$A^*$$-algebra and $$a \in A$$. Let $$A_a = \{x \in A / a \vee x = x\}$$. Then $$A_a$$ is closed under the operations $$\wedge$$ and $$\vee$$. Also for any $$x \in A_a$$ define, $$x^a = a \vee x^\sim$$. Then $$(A_a, \wedge, \vee, ^a)$$ is a Pre $$A^*$$-algebra with $$1$$ (here $$a$$ is itself is the identity for $$\vee$$ in $$A_a$$; that is $$1$$ in $$A_a$$).

Proof: Let $$x, y \in A_a$$. Then $$a \vee x = x$$ and $$a \vee y = y$$.

Now $$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) = x \wedge y \Rightarrow x \wedge y \in A_a$$

Also $$a \vee (x \vee y) = (a \vee x) \vee y = x \vee y \Rightarrow x \vee y \in A_a$$

Therefore $$A_a$$ is closed under the operation $$\wedge$$ and $$\vee$$.

$$a \vee x^a = a \vee (a \vee x^\sim) = a \vee x^\sim = x^a \Rightarrow x^a \in A_a$$

Thus $$A_a$$ is closed under $^a$.

Now for any $$x, y, z \in A_a$$

1. $$x^{a^2} = (a \vee x)^2 = a \vee (a \vee x^\sim) = a \vee (a \wedge x) = a \vee x = x$$

2. $$x \wedge x = (a \vee x) \wedge (a \vee x) = x \wedge x = x$$

3. $$x \wedge y = (a \vee x) \wedge (a \vee y) = (a \vee y) \wedge (a \vee x) = y \wedge x$$

4. $$(x \wedge y)^2 = a \vee (x \wedge y)^\sim = a \vee (x \wedge y^\sim)$$

$$= (a \vee x^\sim) \vee (a \vee y^\sim)$$
\( x \lor y = x \lor y \)

(5) \( x \land (y \lor z) = (a \lor x) \land (a \lor y) \land (a \lor z) \)
\[= a \lor \{x \land (y \lor z)\} \lor \{a \lor (x \land (y \lor z))\} \lor \{a \lor (x \land (y \lor z))\}\]
\[= a \lor \{x \land (y \lor z)\} \lor \{a \lor (x \land (y \lor z))\} \lor \{a \lor (x \land (y \lor z))\}\]
\[= (x \land y) \lor z \]

(6) \( x \land (y \lor z) = (a \lor x) \land (a \lor y) \lor (a \lor z) \)
\[= \{a \lor (x \land (y \lor z))\} \lor \{a \lor (x \land (y \lor z))\} \lor \{a \lor (x \land (y \lor z))\} \lor \{a \lor (x \land (y \lor z))\}\]
\[= x \land (y \lor z) \]

(7) \( x \land (x \lor y) = x \land (x \lor y) \)
\[= (x \land (x \lor y)) \lor (x \land y) \lor (x \land (x \lor y)) \lor (x \land y) \]
\[= x \land (x \lor y) \lor (x \land y) \]

Finally \( x \in A \) implies that \( a \lor x = x = x \lor a \). Thus \( (A, \land, \lor, a) \) is a Pre A*-algebra with \( a \) as identity for \( \lor \).

2.10. Theorem: Let \( a, b \) be elements in a Pre A*-algebra \( A \) such that \( a \leq b \). Then the following hold.

(1) \( a \lor b = b \)

(2) The map \( \alpha_{a,b} : A_a \rightarrow A_b \) defined by \( \alpha_{a,b} (x) = b \lor x \) for all \( x \in A_a \) is a homomorphism of Pre A*-algebras.

(3) \( \alpha_{a,b} (B(A_a)) \subseteq B(A_b) \)

(4) If \( a \leq b \leq c \) then \( \alpha_{a,b} \circ \alpha_{b,c} = \alpha_{a,c} \)

(5) \( \alpha_{a,a} \) is the identity map on \( A_a \)

Proof: Suppose that \( a \leq b \)

(1) We have \( a \leq b \Rightarrow a \lor b = b \)

(2) Let \( x, y \in A_a \). Then \( \alpha_{a,b} (x \land y) = b \lor (x \land y) \)
\[= (b \lor x) \land (b \lor y) \]
\[= \alpha_{a,b} (x) \land \alpha_{a,b} (y) \]

and \( \alpha_{a,b} (x \lor y) = b \lor (x \lor y) \)
\[= (b \lor x) \lor (b \lor y) \]
\[= \alpha_{a,b} (x) \lor \alpha_{a,b} (y) \]

Also \( \alpha_{a,b} (x^a) = b \lor x^a \)
\[= b \lor (a \lor x^a) \]
\[= (b \lor a) \lor x \]
\[= b \lor (b \lor x) \]
\[= b \lor (b \land x) \]
\[= b \lor (b \lor x)^b \]
\[= (\alpha_{a,b} (x))^b \]

Therefore \( \alpha_{a,b} \) is a homomorphism of Pre A*-algebras.

(3) Let \( x \in B(A_a) \).

Then \( x \lor x^a = a \) (since \( a \) is identity in \( A_a \) and therefore \( a = x \lor (a \lor x^a) \) (i)

Now \( \alpha_{a,b} (x) \lor [ \alpha_{a,b} (x)]^b = (b \lor x) \lor (b \lor x)^b \)
\[= (b \lor x) \lor [b \lor (b \lor x)^b] \]
\[= (b \lor x) \lor [b \lor (b \land x)^b] \]
\[= (b \lor x) \lor (b \lor x^a) \]
= b ∨ (x ∨ x̄)
= (a ∨ b) ∨ (x ∨ x)
= b ∨ [a ∨ (x ∨ x)]
= b ∨ a (by (i))
= b, which is 1 in A

Therefore \( \alpha_{a,b} (x) \in B(A) \)

Thus \( \alpha_{a,b} (B(A)) \subseteq B(A) \)

(4) Let \( a \leq b \leq c \)

\[
[\alpha_{a,b} \circ \alpha_{b,c}](x) = \alpha_{a,b}[\alpha_{b,c}(x)]
= \alpha_{a,b}[c \lor x]
= b \lor c \lor x
= c \lor x
= \alpha_{a,c}(x)
\]

Therefore \( \alpha_{a,b} \circ \alpha_{b,c} = \alpha_{a,c} \)

(5) \( \alpha_{a,a}(x) = a \lor x = x \) for all \( x \in A \)

Then \( \alpha_{a,a} \) is identity map on \( A \).

CONCLUSION

This manuscript points up the character of the Pre-A*-algebra like a partially ordered set. With respect to binary operation \( \lor \), able to define a relation \( \leq \) on a Pre-A*-algebra and observed that such a Pre-A*-algebra as a partially ordered set with respect to the relation \( \leq \) and derived corresponding results. It has been observed a necessary condition a Pre-A*-algebra to become a lattice. We also present an equivalent condition for a Pre A*-algebra become a Boolean algebra. For any \( a \in A \) defined a set \( A_a = \{x \in A / a \lor x = x\} \), \( x^a = a \lor x \) and observed that \( (A_a, \land, \lor, ^a) \) is a Pre A*-algebra. Also defined a mapping \( \alpha_{a,b} \) from \( A_a \) to \( A_b \) by \( \alpha_{a,b}(x) = b \lor x \) for all \( x \in A_a \) and confirmed a homomorphism of Pre A*-algebras.

REFERENCES


Source of support: Nil, Conflict of interest: None Declared