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GENERALIZATON OF FIXED POINT THEOREMS IN PARTIAL CONE METRIC SPACES

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ABSTRACT

The object of this paper is to generalize the Fixed Point theorems in the partial cone metric spaces given by Sönmez [1] using the concept of normality of cone. Results proved in this paper are generalization of Fixed Point results established for metric spaces.

Keywords: *Partial Cone Metric Space, Fixed Point, Contractive Mapping.*

AMS (1991) Subject classification: *54H25, 47H10.*

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1. INTRODUCTION:

Huang and Zhang [3] initiate the concept of cone metric space and generalized the concept of metric spaces. In this space they replace the set of real numbers by the ordered Banach space. Recently many papers on cone metric spaces have been appeared e.g. see [4], [6], and main topological properties of such spaces have been obtained. The usual metric which is defined on cone metric spaces suggest that the self-distance for any point is zero but in partial cone metric space this distance need not zero. Specially, from the point of sequence, a convergent sequence need not have unique limit. The concept of partial cone metric play a very important role not only in topology but also in other branches of science and applied science involving mathematics especially in computer science, information science, and biological science.

In this paper we extend the results proved by Sönmez [1] and established fixed point results which have been proved in cone metric space e.g. see [2].

2. PRELIMINARIES:

Definition 2.1 [1]: Let E be a real Banach space and P be a subset of E . The set P is called a cone if

(C1) P is closed, nonempty and $P \neq \{0\}$, here 0 is the zero vector of E;

(C2) $a, b \in \Re, a, b \geq 0, \quad x, y \in P \Rightarrow ax + by \in P$;

(C3) $x \in P \ and \ -x \in P \ \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$. While $x << y$ if and only if $y - x \in P^0$, where P^0 denotes the interior of P.

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Proposition 2.2: Let P be a cone in a real Banach space E. If for $a \in P$ and $a \leq ka$, for some $k \in [0,1)$ then $a = 0$.

Definition 2.3 [1]: Let P be a cone in a real Banach space E then P is called normal, if there exists a constant $K > 0$ such that for all $x, y \in E$

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 2.4 [1]: Let X be a nonempty set, E be a real Banach space. Suppose that the mapping $d : X \times X \to E$ satisfies:

(CM1) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(CM 2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(CM 3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.
We note that any cone metric space is a topological Hausdorff space without the assumption of normality on E.

In this paper, we always suppose that E is a Banach space, P is a cone in E with $P^0 \neq \phi$ and "$\leq$" is partial ordering with respect to P.

Definition 2.5 [1]: A partial cone metric on a nonempty set X is a function $p : X \times X \to E$ such that for all $x, y, z \in X$;

(p1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;

(p2) $0 \leq p(x, x) \leq p(x, y)$;

(p3) $p(x, y) = p(y, x)$;

(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial cone metric space is a pair (X, p) such that X is a nonempty set and p is a partial cone metric on X.

Remark: 2.6 It is clear that, if $p(x, y) = 0$, then from (p1) and (p2) $x = y$ but converse may not be true.

A cone metric space is a partial cone metric space. But there are partial cone metric spaces which are not cone metric spaces. The following examples will illustrate it better

Example: 2.7 Let $E = \Re^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \Re^+$ and $p : X \times X \to E$ defined as $p(x, y) = (\max\{x, y\}, \alpha \max\{x, y\})$ where $\alpha \geq 1$ is a constant. Then (X, p) is a partial cone metric space which is not a cone metric space.

Example 2.8: Let $E = \ell_1$, $P = \{ \{x_n\} \in \ell_1 : x_n \geq 0\}$, $X = \{ \{x_n\} : \{x_n\} \in (\Re^+)^\omega, \sum x_n < \infty :\}$ where $(\Re^+)^\omega$ is the set of all infinite sequence over $\Re^+$, and $p : X \times X \to E$ defined by,

$p(x, y) = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, ..., \max\{x_n, y_n\}, ...)$.
Then (X, p) is a partial cone metric space, but not cone metric space.

Let (X, d) be a partial cone metric space, $x \in X$ and A be a non-empty subset of X. The distance between the set A and the singleton set $\{x\}$, and that between two subsets A and B of X given as

$$p(x, A) = \inf\{ p(x, a):\ a \in A\}\ ,\quad p(A, B) = \inf\{ p(a, b):\ a \in A,\ b \in B\}$$

Throughout this paper (X, p) will denote a partial cone metric space with respect to cone P.

Theorem: 2.9 [1] Any partial cone metric space (X, p) is a topological space.

Proof: For $c \in P^0$ let $B_p(x, c) = \{ y \in X : p(x, y) << c + p(x, x)\}$ and $\beta = \{B_p(x, c) : x \in X, c \in P^0\}$. Then $\tau_p = \{U \subset X : \forall x \in U, \exists B \in \beta, x \in B \subset U\}$ is a topology on X.

Definition: 2.10 [1] Let (X, p) be a partial cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in P^0$ there is $n_0 \in$ **N** such that for all $n > n_0$ $p(x_n, x) << c + p(x, x)$ then $\{x_n\}$ is said to be convergent, and $\{x_n\}$ converges to x. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Theorem 2.11 [1]: Let (X, p) be a partial cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $p(x_n, x) \to p(x, x),\ (n \to \infty)$.

We note that if (X, p) be a partial cone metric space, P be a normal cone with normal constant K, if $p(x_n, x) \to p(x, x)\ (n \to \infty)$ then $p(x_n, x_n) \to p(x, x)\ (n \to \infty)$.

Lemma: 2.12 [1] Let $\{x_n\}$ be a sequence in a partial cone metric space (X, p). If a point x is the limit of $\{x_n\}$ and $p(y, y) = p(x, y)$, then y is the limit of $\{x_n\}$.

Definition 2.13 [1]: Let (X, p) be a partial cone metric space. $\{x_n\}$ be a sequence in X. $\{x_n\}$ is Cauchy sequence if there is $a \in P$ such that for every $\varepsilon > 0$ there is $n_0 \in$ **N** such that for all $n, m > n_0$

$$\| p(x_n, x_m) - a \| < \varepsilon .$$

We call a partial cone metric space complete if every Cauchy sequence in space is convergent in space.

For more basic results we refer [1].

3. MAIN RESULTS:

Theorem: 3.1 Let (X, p) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfying the contractive condition, $p(Tx, Ty) \le ap(x, y) + bp(Tx, x) + cp(Ty, y) + dp(Tx, y) + ep(Ty, x)$ …(3.1.1) Where a, b, c, d, e all are nonnegative constant such that $a + b + c + d + e < 1$. Then T has unique Fixed Point in X.

Proof:
Existence of Fixed point: Let $x_0 \in X$ be arbitrary point of X. We define iterative sequence $\{x_n\}$ by $Tx_n = x_{n+1}$.

First we shall show that this sequence is a Cauchy sequence.
Taking $x = x_n$, $y = x_{n-1}$ in (3.1.1) we get

$$p(Tx_n, Tx_{n-1}) \le ap(x_n, x_{n-1}) + bp(Tx_n, x_n) + cp(Tx_{n-1}, x_{n-1}) + dp(Tx_n, x_{n-1}) + ep(Tx_{n-1}, x_n)$$

$$\Rightarrow p(x_{n+1}, x_n) \le ap(x_n, x_{n-1}) + bp(x_{n+1}, x_n) + cp(x_n, x_{n-1}) + dp(x_{n+1}, x_{n-1}) + ep(x_n, x_n)$$

$$\Rightarrow p(x_{n+1}, x_n) \le ap(x_n, x_{n-1}) + bp(x_{n+1}, x_n) + cp(x_n, x_{n-1}) + d[p(x_{n+1}, x_n) + p(x_n, x_{n-1}) - p(x_n, x_n)] + ep(x_n, x_n)$$

Writing $p(x_n, x_{n+1}) = p_n$ we get

$$p_n \le ap_{n-1} + bp_n + cp_{n-1} + d[p_n + p_{n-1}] + (e-d)p(x_n, x_n)$$

$$\Rightarrow \quad p_n \le (a+c+d)P_{n-1} + (b+d)p_n + (e-d)p(x_n, x_n) \tag{3.1.2}$$

Again using (3.1.1) with $x = x_{n-1}$, $y = x_n$ and using symmetry of inequality we get

$$p_n \le (a+b+e)p_{n-1} + (c+e)p_n + (d-e)p(x_n, x_n) \tag{3.1.3}$$

adding (3.1.2) and (3.1.3) we get

$$2p_n \le (2a+b+c+d+e)p_{n-1} + (b+c+d+e)p_n$$

$$\Rightarrow \quad p_n \le \frac{(2a+b+c+d+e)}{2-(b+c+d+e)} p_{n-1}$$

$$\Rightarrow \quad p_n \le \lambda p_{n-1}, \quad \text{where} \quad \lambda = \frac{(2a+b+c+d+e)}{2-(b+c+d+e)} < \frac{a+1}{1+a} = 1, \text{ hence } \lambda < 1.$$

Thus $p_n \le \lambda p_{n-1}$ implies that

$$p_n \le \lambda^n p_0 \tag{3.1.4}$$

Now let $m, n \in \mathbf{N}$ and $m > n$, then

$$p(x_n, x_m) \le p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{m-1}, x_m) - [p(x_{n+1}, x_{n+1}) + p(x_{n+2}, x_{n+2}) + p(x_{m-1}, x_{m-1})]$$

$$\Rightarrow p(x_n, x_m) \le p_n + p_{n+1} + \ldots + p_{m-1}$$

$$\Rightarrow p(x_n, x_m) \le \lambda^n p_0 + \lambda^{n+1} p_0 + \lambda^{n+2} p_0 + \ldots$$

$$\Rightarrow p(x_n, x_m) \le \lambda^n [+\lambda + \lambda^2 + \ldots] p_0$$

$$\Rightarrow p(x_n, x_m) \le \frac{\lambda^n p_0}{1-\lambda} \tag{3.1.5}$$

Hence by normality we have

$$\|p(x_n, x_m)\| \le \frac{\lambda^n K}{1-\lambda}\|p_0\|,$$ here K is the normal constant of cone.

Now since $\lambda < 1$ hence $\|p(x_n, x_m)\| \le \frac{\lambda^n K}{1-\lambda}\|p_0\| \to 0,$ as $n \to \infty$.

Therefore $\{x_n\}$ is a Cauchy sequence, and X is complete hence it must be convergent in X, let $\lim_{n\to\infty} x_n = u \in X$, and therefore

$$\lim_{n\to\infty} p(x_n, u) = p(u,u) = \lim_{n\to\infty} p(x_n, x_n) = 0 \tag{3.1.6}$$

Now consider

$p(Tu,u) \le p(Tu,Tx_n) + p(Tx_n,u) - p(Tx_n,Tx_n)$, and using (3.1.1) we get

$$p(Tu,u) \le ap(u,x_n) + bp(Tu,u) + cp(Tx_n,x_n) + dp(Tu,x_n) + ep(Tx_n,u) + p(Tx_n,u) - p(Tx_n,Tx_n)$$

$$\le ap(u,x_n) + bp(Tu,u) + cp(x_{n+1},x_n) + dp(Tu,x_n) + ep(x_{n+1},u) + p(x_{n+1},u)$$

$$\le ap(u,x_n) + bp(Tu,u) + cp(x_{n+1},x_n) + dp(Tu,x_n) + (e+1)p(x_{n+1},u)$$

$$\le ap(u,x_n) + bp(Tu,u) + cp(x_{n+1},x_n) + d[p(Tu,u) + p(u,x_n) - p(u,u)] + (e+1)p(x_{n+1},u)$$

So using (3.1.4) we have

$$p(Tu,u) \le \frac{a+d}{1-b-d}p(u,x_n) + \frac{c\lambda^n p_0}{(1-b-d)(1-\lambda)} + \frac{(e+1)}{1-b-d}p(x_{n+1},u)$$

Hence

$$\|p(Tu,u)\| \le \frac{(a+d)K}{1-b-d}\|p(u,x_n)\| + \frac{c\lambda^n K\|p_0\|}{(1-b-d)(1-\lambda)} + \frac{(e+1)K}{1-b-d}\|p(x_{n+1},u)\| \to 0 \text{ as } n \to \infty.$$

So we have $\|p(Tu,u)\| = 0$ therefore $p(Tu,u) = 0$ or $Tu = u$.

Thus u is a Fixed Point of T.

UNIQUENESS:

If v is another Fixed Point of T then $Tv = v$.

Using (3.1.1) with $x = u,\ y = v$ we get

$$\Rightarrow \quad p(Tu,Tv) \le ap(u,v) + bp(Tu,u) + cp(Tv,v) + dp(Tu,v) + ep(Tv,u)$$

$$\Rightarrow \quad p(u,v) \le ap(u,v) + bp(u,u) + cp(v,v) + dp(u,v) + ep(v,u)$$

$$\Rightarrow \qquad p(u,v) \le (a+d+e)\,p(u,v) + cp(v,v) \tag{3.1.7}$$

Again Using (3.1.1) with $x = v,\ y = u$ we get

$$p(u,v) \le (a+d+e)\,p(u,v) + bp(v,v) \tag{3.1.8}$$

Adding (3.1.7) and (3.1.8) we get

$$2p(u,v) \le 2(a+d+e)\,p(u,v) + (b+c)\,p(v,v)$$

$$\Rightarrow \qquad p(u,v) \le \frac{(b+c)}{2(1-a-d-e)}\,p(v,v) = \mu\ \ p(v,v)$$

Where $\mu = \dfrac{(b+c)}{2(1-a-d-e)} < \dfrac{(b+c)}{2(b+c)} = \dfrac{1}{2}$

Thus $\quad \mu < \dfrac{1}{2}$

Therefore $\quad p(u,v) \le \mu\ p(v,v) \le \mu[\,p(v,u) + p(u,v) - p(u,u)]$

Or $\quad p(u,v) \le 2\mu\ p(u,v)$, but $\mu < \dfrac{1}{2}$ so $2\mu < 1$, hence by Proposition **2.2** we have $\quad p(u,v) = 0$ hence $u = v$, Thus Fixed Point is unique.

Now we prove some consequences of above theorem,

If we take $d = e$ in theorem **3.1** we get the following theorem

Theorem: 3.2 Let (X, p) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfying the contractive condition, $p(Tx,Ty) \le ap(x, y) + bp(Tx, x) + cp(Ty, y) + d[\,p(Tx, y) + p(Ty, x)]$ Where a,b,c,d all are nonnegative constant such that $a+b+c+2d < 1$. Then T has unique Fixed Point in X.

If we take $b = c,\ d = e$ in theorem **3.1** we get the following theorem

Theorem 3.3: Let (X, p) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfying the contractive condition, $p(Tx,Ty) \le ap(x, y) + b[\,p(Tx, x) + p(Ty, y)] + d[\,p(Tx, y) + p(Ty, x)]$ Where a,b,d all are nonnegative constant such that $a+2b+2d < 1$. Then T has unique Fixed Point in X.

All above theorems are the extension of theorems proved in [1].

If we take $b = c = d = e = 0,\ 0 < a < 1$ in theorem **3.1** we get the following theorem

Theorem: 3.4 [1, Theorem 6] Let (X, p) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfying the contractive condition, $p(Tx,Ty) \le ap(x, y)$ for all $x, y \in X$, Where $a \in (0,1)$. Then T has unique Fixed Point in X.

Remark: 3.5 If we take $b = c = d = e = 0,\ 0 < a < 1$ then there is no harm in the argument as we use in the proof of uniqueness in theorem 3.1.

If we take $b = c,\ d = e = a = 0, 0 < b < \frac{1}{2}$ in theorem **3.1** we get the following theorem.

Theorem: 3.6 [1, Theorem 7] Let (X, p) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose that the mapping $T : X \to X$ satisfying the contractive condition, $p(Tx, Ty) \le b[\,p(Tx, x) + p(Ty, y)]$ where $b \in (0, \frac{1}{2})$. Then T has unique Fixed Point in X.

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