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FOURIER APPROACH TO FUNCTION APPROXIMATION

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ABSTRACT

Function approximation is fundamental to many real world problems and Fourier method may be considered as one of the powerful technique for numerical approximation of functions .The inverse FT (IFT) is used to reconstruct the data from the coefficients known FT transforms of observed data. However, it is always difficult to predict the accuracy of any such approximation. We have used exponential, periodic and piece wise functions and the simulations assumes the functions as the output of a Gaussian function with infinite frequencies. The Fourier approximation has been strictly evaluated in statistical terms for its accuracy. The RMSE is found to be least (0.1601) for periodic function as compared to exponential (6.007) and piece wise function (2.70) and on evaluation of the approximation from 100 points to 200 points as expected the RMSE values show a marginal decrease However, acceptance of the results of exponential function approximation may depend on the stiffness and the desired accuracy. The results indicate that Fourier approach may viewed as a suitable method for approximating functions exponential, periodic and piece-wise.

Key words: *Fourier transform, Inverse Fourier transform, Gaussian function, approximation-exponential, periodic function, piece-wise function.*

1.1 INTRODUCTION:

The motivation for the Fourier transform comes from the study of Fourier series. In the study of Fourier series, complicated functions are written as the sum of simple waves mathematically represented by sines and cosines. Due to the properties of sine and cosine it is possible to recover the amount of each wave in the sum by an integral. In many cases it is desirable to use Euler's formula, which states that $e^{2\pi i\theta} = \cos 2\pi\theta + i \sin 2\pi\theta$, to write Fourier series in terms of the basic waves $e^{2\pi i\theta}$. This has the advantage of simplifying many of the formulas involved and providing a formulation for Fourier series that more closely resembles the definition followed in this article. This passage from sines and cosines to complex exponentials makes it necessary for the Fourier coefficients to be complex valued. The usual interpretation of this complex number is that it gives both the amplitude (or size) of the wave present in the function and the phase (or the initial angle) of the wave. This passage also introduces the need for negative "frequencies". If θ were measured in seconds then the waves $e^{2\pi i\theta}$ and $e^{-2\pi i\theta}$ would both complete one cycle per second, but they represent different frequencies in the Fourier transform. Hence, frequency no longer measures the number of cycles per unit time, but is closely related.

There is a close connection between the definition of Fourier series and the Fourier transform for functions f which are zero outside of an interval. For such a function we can calculate its Fourier series on any interval that includes the interval .where f is not identically zero. The Fourier transform is also defined for such a function. As we increase the length of the interval on which we calculate the Fourier series, then the Fourier series coefficients begin to look like the Fourier transform and the sum of the Fourier series of f begins to look like the inverse Fourier transform. To explain this more precisely, suppose that T is large enough so that the interval $[-T/2, T/2]$ contains the interval on which f is not identically zero. Then the n-th series coefficient c_n is given by**:**

$$c_n = \int_{-T/2}^{T/2} f(x)\, e^{-2\pi i (n/T) x} dx.$$

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Comparing this to the definition of the Fourier transform it follows that $c_n = \hat{f}(n/T)$ since $f(x)$ is zero outside $[-T/2, T/2]$. Thus the Fourier coefficients are just the values of the Fourier transform sampled on a grid of width $1/T$. As T increases the Fourier coefficients more closely represent the Fourier transform of the function.

1.2 STATISTICAL EVALUATION OF FOURIER APPROACH TO FUNCTION APPROXIMATION:

As is common in statistical literature, the term goodness of fit is used here in the sense our data reasonably come from the Fourier model with the assumption that the model coefficients have been estimated with little uncertainty. It also explains high proportion of the variability in our data and is able to predict new observations with high certainty. The goodness-of-fit statistics and coefficient confidence bounds yield numerical measures that aid to our statistical reasoning.

Goodness-of-Fit Statistics is given by the sum of squares due to error (SSE), R-square, Adjusted R-square and Root mean squared error (RMSE).

1.2.1 SSE (SUM OF SQUARES DUE TO ERROR):

This statistic measures the total deviation of the response values from the fit to the response values. It is also called the summed square of residuals and is usually labeled as *SSE*.

$$SSE = \sum_{i=1}^{n} w_i \left(y_i - \hat{y}_i \right)^2$$

A value closer to 0 indicates that the model has a smaller random error component, and that the fit will be more useful for prediction.

1.2.2 R-SQUARE:

This statistic measures how successful the fit is in explaining the variation of the data. Put another way, R-square is the square of the correlation between the response values and the predicted response values. It is also called the square of the multiple correlation coefficient and the coefficient of multiple determination.

R-square is defined as the ratio of the sum of squares of the regression (*SSR*) and the total sum of squares (*SST*). *SSR* is defined as

$$SSR = \sum_{i=1}^{n} w_i \left(\hat{y}_i - \bar{y} \right)^2$$

SST is also called the sum of squares about the mean, and is defined as

$$SST = \sum_{i=1}^{n} w_i \left(y_i - \bar{y} \right)^2$$

Where $SST = SSR + SSE$. Given these definitions, R-square is expressed as

$$\text{R-square} = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

R-square can take on any value between 0 and 1, with a value closer to 1 indicating that a greater proportion of variance is accounted for by the model. For example, an R-square value of 0.8234 means that the fit explains 82.34% of the total variation in the data about the average.

If we increase the number of fitted coefficients in your model, R-square will increase although the fit may not improve in a practical sense. To avoid this situation, we have used the degrees of freedom adjusted R-square statistic described below.

Note that it is possible to get a negative R-square for equations that do not contain a constant term. Because R-square is defined as the proportion of variance explained by the fit, if the fit is actually worse than just fitting a horizontal line then R-square is negative. In this case, R-square cannot be interpreted as the square of a correlation. Such situations indicate that a constant term should be added to the model.

1.2.3 DEGREES OF FREEDOM ADJUSTED R-SQUARE:

This statistic uses the R-square statistic defined above, and adjusts it based on the residual degrees of freedom. The residual degrees of freedom is defined as the number of response values n minus the number of fitted coefficients m estimated from the response values.

$$v = n - m$$

v indicates the number of independent pieces of information involving the n data points that are required to calculate the sum of squares. Note that if parameters are bounded and one or more of the estimates are at their bounds, then those estimates are regarded as fixed. The degrees of freedom is increased by the number of such parameters.

The adjusted R-square statistic is generally the best indicator of the fit quality when we compare two models that are *nested* — that is, a series of models each of which adds additional coefficients to the previous model.

$$\text{adjusted R-square} = 1 - \frac{SSE(n-1)}{SST(v)}$$

The adjusted R-square statistic may take on any value less than or equal to 1, with a value closer to 1 indicating a better fit. Negative values can occur when the model contains terms that do not help to predict the response.

1.2.4 Root Mean Squared Error:

This statistic is also known as the fit standard error and the standard error of the regression. It is an estimate of the standard deviation of the random component in the data, and is defined as

$$RMSE = s = \sqrt{MSE}$$

WhereMSE is the mean square error or the residual mean square

$$MSE = \frac{SSE}{v}$$

Just as with *SSE*, an *MSE* value closer to 0 indicates a fit that is more useful for prediction.

1. 3 FOURIER APPROXIMATION EXPERIMENTS:

1.3.1 Case1: 1-D Continuous Exponential Function

The original function and its Fourier approximation for 100 data points out of the original function of 2000 data points are plotted in Fig 1. The Fourier coefficients have been reevaluated for 200 points for understanding the accuracy of function approximation. The result is given in Fig 2. The computed the Fourier coefficients are with the 95% confidence bounded limits and the RMSE values have been provided below each figure. The RMSE differences can be seen as the distinguishing characteristic of the accuracy of the approximation.

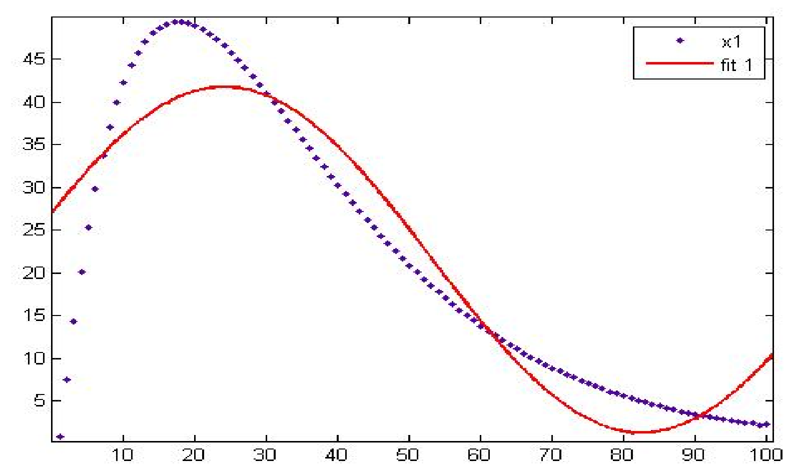

Fig 1 Fourier Simulation result for Case 1-(100 points)

General model Fourier1:

$$f(x) = a_0 + a_1 cos(xw) + b_1 sin$$

(xw)cients (with 95% confidence bounds):

$$a_0 = 21.53\ (20.27, 22.78)$$
$$a_1 = 5.584\ (1.929, 9.239)$$
$$b_1 = 19.46\ (17.36, 21.57)$$
$$w = 0.0538\ (0.05024, 0.05737)$$

Goodness of fit:

SSE: 3464 R-square: 0.8708 Adjusted R-square: 0.8668 RMSE: 6.007

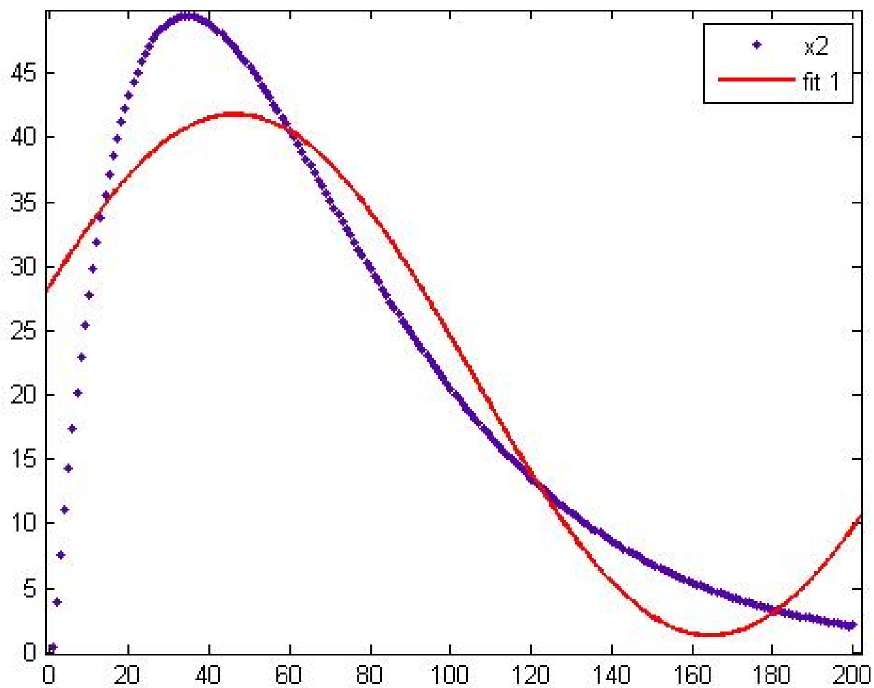

Fig 2 Fourier Simulation result for Case 1-(200 points)

General model Fourier1:

$$f(x) = a_0 + a_1 cos(xw) + b_1 sin(xw)$$

Coefficients (with 95% confidence bounds):

$$a_0 = 21.55\ (20.7, 22.4)$$
$$a_1 = 6.748\ (4.285, 9.21)$$
$$b_1 = 19.07\ (17.55, 20.6)$$
$$w = 0.02659\ (0.02536, 0.02782)$$

Goodness of fit:

SSE: 6567 R-square: 0.8774 Adjusted R-square: 0.8756 RMSE: 5.788

It is to be noted that for both 100 and 200 data point, the analysis at 95% Confidence Interval shows minor changes in the RMSE values from 6.007 to 5.788. There is slightly decrease in the error with the increased in the interpolation from 100 to 200 points. The results are in concordance with the statistical expectation of increased in number of observations.

1.3.2 Case 2: 1-D Continuous Periodic Function

Sampled and reevaluated the original function by 100 points and 200 points are shown in (Fig 3 and 4) respectively. The Fourier coefficients with the 95% confidence bounded limits and RMSE value and distinguishes the both the Interpolation points.

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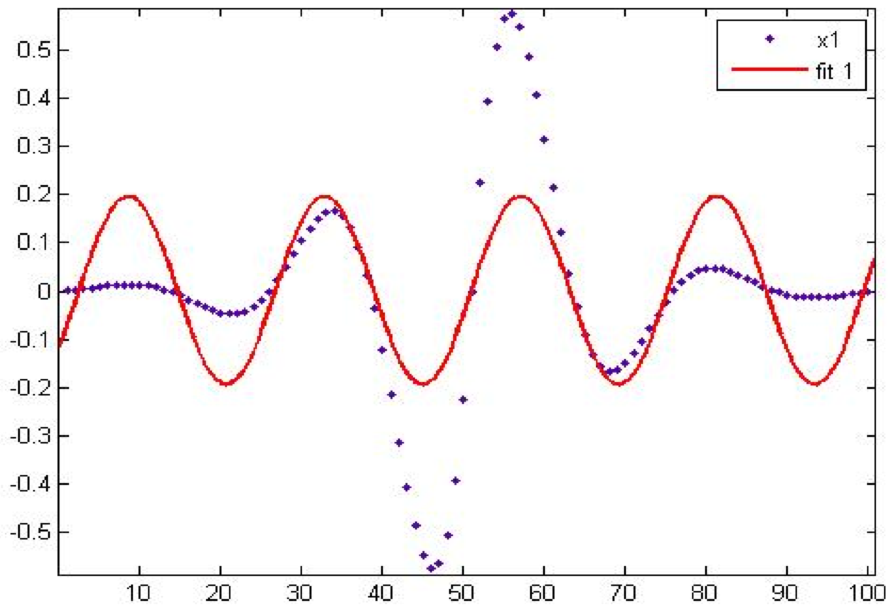

Figure 3 Fourier Simulation result for Case 2 (100 points)

General model Fourier1:

$$f(x) = a_0 + a_1 cos(xw) + b_1 sin(xw)$$

Coefficients (with 95% confidence bounds):

$$\begin{aligned} a_0 &= 0.0006999\ (-0.03111, 0.03251) \\ a_1 &= -0.1164\ (-0.1925, -0.04041) \\ b_1 &= 0.157\ (0.09443, 0.2195) \\ w &= 0.259\ (0.2514, 0.2667) \end{aligned}$$

Goodness of fit:

SSE: 2.461 R-square: 0.4304 Adjusted R-square: 0.4126 RMSE: 0.1601

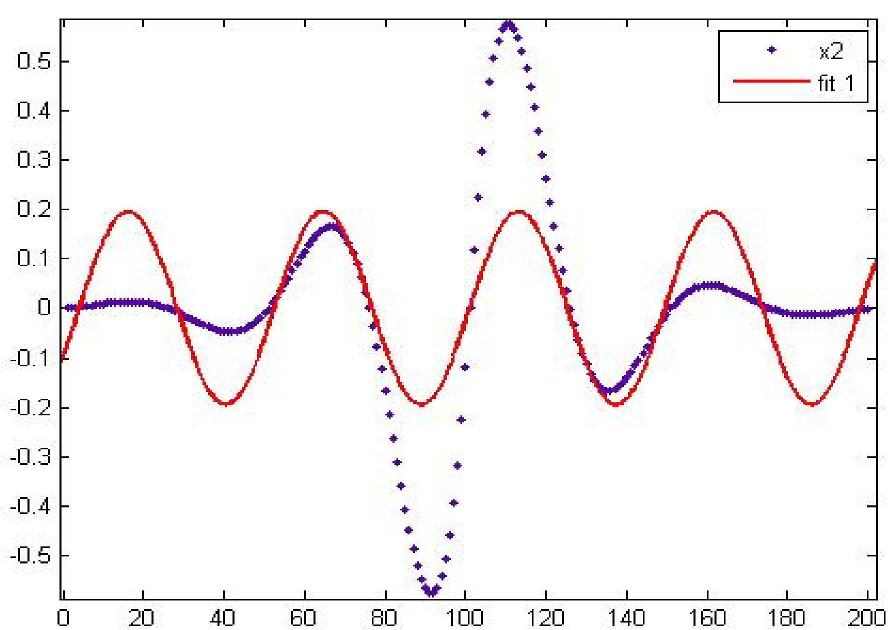

Figure 4 Fourier Simulation result for Case 2(200 points)

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General model Fourier1:

$$f(x) = a_0 + a_1 cos(xw) + b_1 sin(xw)$$

Coefficients (with 95% confidence bounds):

$$\begin{aligned} a_0 &= 0.0003521\ (-0.02175, 0.02246) \\ a_1 &= -0.09631\ (-0.1518, -0.04086) \\ b_1 &= 0.17\ (0.1301, 0.2099) \\ w &= 0.1296\ (0.1269, 0.1322) \end{aligned}$$

Goodness of fit:

SSE: 4.915 R-square: 0.4304 Adjusted R-square: 0.4217 RMSE: 0.1584

The function approximations carried out for 100 points out of the original function and reevaluation for 200 data points show decrease of RMSE from 0.160 to 0.158 (approximately equal) . It implies decreasing level of error in function approximation with more interpolation points

1.3.3 Case 3: Piecewise continuous function

The sampled function and its approximation is plotted in Figure 5. The revaluated function for 200 points is given in figure 6. The computed the Fourier coefficients with the 95% confidence bounded limits and RMSE value are shown below each figure. The values of RMSEs distinguish the both the Interpolation points.

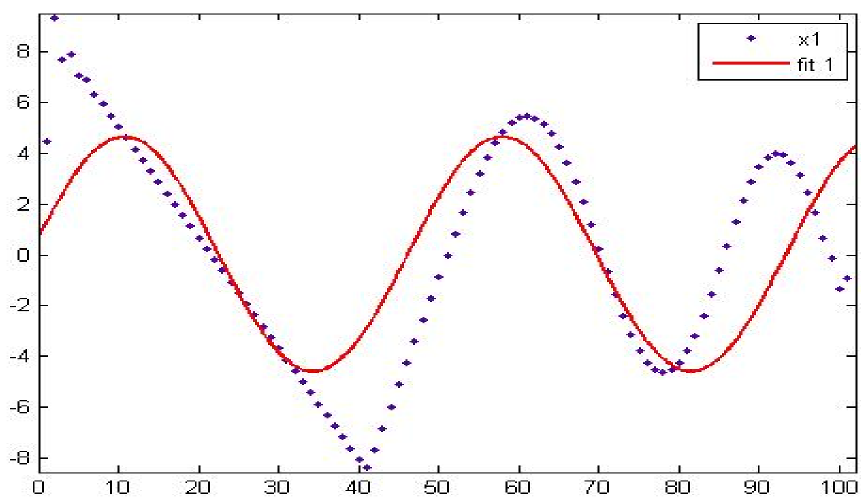

Figure 5 Fourier Simulation result for Case 3 (100 points)

General model Fourier1:

$$f(x) = a_0 + a_1 cos(xw) + b_1 sin(xw)$$

Coefficients (with 95% confidence bounds):

$$\begin{aligned} a_0 &= 0.02967\ (-0.5326, 0.592) \\ a_1 &= 0.6709\ (-0.9133, 2.255) \\ b_1 &= 4.566\ (3.741, 5.392) \\ w &= 0.1332\ (0.1274, 0.1389) \end{aligned}$$

Goodness of fit:

SSE: 755.2 R-square: 0.5831 Adjusted R-square: 0.5702 RMSE: 2.79

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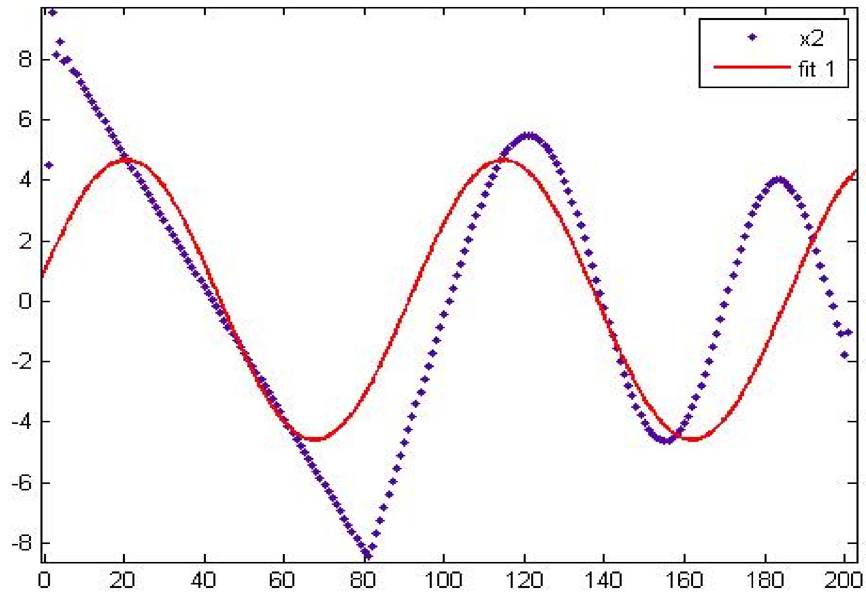

Figure 6 Fourier Simulation result for Case 3 (200 points)

General model Fourier1:

$$f(x) = a_0 + a_1 cos(xw) + b_1 sin(xw)$$

Coefficients (with 95% confidence bounds):

$$\begin{aligned} a_0 &= 0.03641\ (-0.3568, 0.4296) \\ a_1 &= 0.9562\ (-0.1408, 2.053) \\ b_1 &= 4.532\ (3.935, 5.13) \\ w &= 0.06667\ (0.06464, 0.06869) \end{aligned}$$

Goodness of fit:

SSE: 1510 R-square: 0.584 Adjusted R-square: 0.5777 RMSE: 2.769

We have shown the results for 100 sample data points of the original function and the reevaluation of approximation for 200 data points. The analysis of 95% Interval seen in the RMSE decreased from 2.79 to 2.769 with slightly decrease in the error with the increased in the interpolation from 100 to 200 points. It indicates minimal changes error of the function approximation by Fourier method.

The Overall results of three functions are given in Table 1 below for comparison purpose.

Table: 1 Fourier Simulation result for all the three functions

Fourier coefficients	Continuous exponential function	Periodic function	Piecewise continuous function
a_0	21.53	0.0006999	0.02967
a_1	5.584	−0.1164	0.6709
b_1	19.46	0.157	4.566
w	0.0538	0.259	0.1332
SSE	3464	2.461	755.2
$R-square$	0.8708	0.4304	0.5831
Adjusted $R-$square	0.8668	0.4126	0.5702
$RMSE$	6.007	0.1601	2.79

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Table: 2 Statistical Measures of Approximation

	Case(1)	Case (2)	Case(3)
a_0	21.55	0.0003521	0.03641
a_1	6.748	−0.09631	0.9562
b_1	19.07	0.17	4.532
w	0.02659	0.1296	0.06667
SSE	6567	4.915	1510
$R - square$	0.8774	0.4304	0.584
Adjusted R-square	0.8756	0.4217	0.5777
$RMSE$	5.788	0.1584	2.769

1.3 RESULTS AND CONCLUSION:

The results of the simulations and its statistical analyses (Table 1) indicate that continuous periodic functions may be the suitable functions for approximation by Fourier method. However the graphical output shows high deviations from the original data. The nearest approximation is piecewise continuous function and the least approximation functions is the continuous exponential function. The simulation results are keeping with the theoretical expectations of Fourier series expansions of above functions. It is expected that the degree of random error for exponential function are to be higher .Therefore the Fourier method may not be suitable for approximation of continuous exponential function.

RMSE values of the numerical techniques indicate error in approximations. The results show relatively higher error in terms of exactly reconstruction of the function by Fourier method. It appears that there are large variances in the approximating sine and cosine terms in the equations

$$f(x) = a_0 + a_1 cos(xw) + b_1 sin(xw)$$

Contributes to large RMSE's.

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