Given a configuration of pebbles on the vertices of a connected graph $G$, a pebbling move (or pebbling step) is defined as the removal of two pebbles from a vertex and placing one pebble on an adjacent vertex. The cover pebbling number $\mu(G)$ of a graph, is the least positive integer $m$ such that however the $m$ pebbles are placed on the vertices of $G$, we can eventually put a pebble on every vertex. In this paper we compute the cover pebbling number of the cube of a path.

**Key words:** Graphs, Pebbling, Cover pebbling, Cube of a path.

1. INTRODUCTION

Pebbling, one of the latest evolutions in graph theory proposed by Lagaris and Saks has been the topic of vast investigation with significant observations, Having Chung [1] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Hurlbert published a survey of pebbling results in [4]. Given a connected graph $G = (V, X)$, where $V$ is the set of all vertices and $X$ is the set of all edges, we distribute certain number of pebbles on the vertices in some configuration. Precisely, a configuration on a graph $G$ is a function from $V(G)$ to $\mathbb{N} \cup \{0\}$ representing a placement of pebbles on $G$. The size of the configuration is the total number of pebbles placed on the vertices. A pebbling move is the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. In pebbling a target vertex is selected and the aim is to move a pebble to the target vertex. The minimum number of pebbles, such that regardless of their initial placement and regardless of the target vertex, we can pebble target vertex is called the pebbling number of $G$. In cover pebbling, the aim is to cover all the vertices with pebbles i.e., to move a pebble to every vertex of the graph simultaneously. The minimum number of pebbles required such that, regardless of their initial placement on $G$, there is a sequence of pebbling moves, at the end of which, every vertex has at least one pebble on it, is called the cover pebbling number of $G$. In [2], Crul et al determine the cover pebbling number for path complete graphs, Fuses.

In this paper, we determine the cover pebbling number for cube of the path.

**Theorem: 1.1** [7] For a path on $n$ vertices, $\pi(P_n) = 2^{n-1}$

**Theorem: 1.2** [2] For a path on $n$ vertices $\mu(P_n) = 2^n - 1$

**Theorem: 1.3** [3] The cover pebbling number of the $n$-cube is $\mu(Q_n) = 3^n$.

**Theorem: 1.4** [2] For a complete graph $K_t$, $\mu(K_t) = 2t - 1$.

2 COVER PEBBLING NUMBER FOR CUBE OF THE PATH

2.1 Definition [7]

Let $G = (V(G), E(G))$ be a connected graph. The $p^{th}$ power of $G$ denoted by $G^p$ is the graph obtained from $G$ by adding the edge $uv$ to $G$ whenever $2 \leq d(u, v) \leq p$ in $G$, that is, $G^p = (V(G), E(G) \cup uv: 2 \leq d(u, v) \leq p \text{ in } G)$. If $p = 1$, we defined $G^1 = G$. We know that if $p$ is large enough, that is $p \geq n - 1$ then $G^p = K_n$.

We start with the cover pebbling number for cube of the path $P_t$.
Theorem: 1 The cover pebbling number for $P_4^3$ is $\mu(P_4^3) = 7$.

Proof: If we place six pebbles on $v_1$, then we use 6 pebbles to pebble $v_2, v_3, v_4$. So $v_1$ remains uncovered. Thus $\mu(P_4^3) \geq 7$.

Now consider the distribution of seven pebbles on the vertices of $P_4^3$. If either $P(v_i) \geq 3, P(v_j) \geq 3, P(v_k) = 0$ for $i \neq j \neq k$ and $i,j,k \in \{1,2,3,4\}$ (or) $P(v_i) \geq 1, P(v_j) \geq 5, P(v_k) = 0$ for $i \neq j \neq k$ and $i,j,k \in \{1,2,3,4\}$ then we are done. Otherwise $P(v_i) = 7, P(v_j) = 0$ for $i \neq j$ and $i,j \in \{1,2,3,4\}$ and so we are done. Thus $\mu(P_4^3) \leq 7$.

Theorem: 2 The cover pebbling number for $P_5^3$ is $\mu(P_5^3) = 11$.

Proof: Consider the configuration such that $p(v_1) = 10$ and $p(v_i) = 0$ for all $v_i \in V(P_5^3) - \{v_1\}$ then we use 10 pebbles to pebble $v_2, v_3, v_4, v_5$. So $v_1$ remains uncovered. Thus $\mu(P_5^3) \geq 11$.

Now consider the distribution of eleven pebbles on the vertices of $P_5^3$.

If either $p(v_i) \geq 3, p(v_j) \geq 5, p(v_k) = 0$ for $i \neq j \neq k$ and $i,j,k \in \{1,2,3,4,5\}$ (or) $p(v_i) \geq 3, p(v_j) \geq 3, p(v_k) \geq 1, p(v_5) = 0$ for $i \neq j \neq k \neq s$ and $i,j,k,s \in \{1,2,3,4,5\}$ then we are done. Otherwise $P(v_i) = 11, P(v_j) = 0$ for $i \neq j$ and $i,j \in \{1,2,3,4,5\}$. So we can move one pebble to all vertices of $P_5^3$ and we are done. Thus $\mu(P_5^3) \leq 11$.

Theorem: 3 The cover pebbling number for $P_6^3$ is 15.

Proof: Consider the configuration such that $p(v_1) = 14$ and $p(v_i) = 0$ for all $v_i \in V(P_6^3) - \{v_1\}$ then we use 14 pebbles to pebble $v_2, v_3, v_4, v_5, v_6$. So $v_1$ remains uncovered. Thus $\mu(P_6^3) \geq 15$.

Now consider the distribution of fifteen pebbles on the vertices of $P_6^3$.

If either $p(v_i) \geq 3, p(v_j) \geq 3, p(v_k) \geq 1, p(v_5) = 0$ for $i \neq j \neq k \neq s$ and $i,j,k,s \in \{1,2,3,4,5,6\}$ (or) $p(v_i) \geq 1, p(v_j) \geq 7$ for $i \neq j$ and $i,j \in \{1,2,3,4,5,6\}$ then we are done. Otherwise $p(v_i) = 15, p(v_j) = 0$ for $i \neq j$ and $i,j \in \{1,2,3,4,5,6\}$.

So we can move one pebble to all vertices of $P_6^3$ and we are done. Thus $\mu(P_6^3) \geq 15/\mu(P_6^3) \geq 15$.

Theorem: 4 $\mu(P_6^3) = \frac{3\mu(P_{s+1}) - 2}{2\mu(P_{s+1}) + \mu(P_s) - 2}$ when $n = 3s + 1, s = 1, 2, 3, \ldots$

Proof: In $P_6^3$, clearly $v_{1,3} < i < n-3$, is adjacent to six vertices $v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v_{i+3}$ and $v_i$, $n-3 \leq i \leq n$, is adjacent to three vertices $v_{i-3}, v_{i-2}, v_{i-1}$. Therefore to pebble the vertex $v_i$ we use the path $v_{i-3}, v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_{i+3}$, \ldots So to pebble a vertex of $P_6^3$, we use any one of the following paths $v_{n-6}, v_{n-5}, v_n$ (or) $v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}$, $v_{n-1}$ (or) $v_{n-8}, v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3}$.

Case: (i) when $n = 3s + 1$,

$3\mu(P_{s+1}) - 3$ pebbles are placed on $v_1$, we use $\mu(P_{s+1}) - 1$ pebbles to pebble the vertices of the path $v_{n-3}, v_{n-4}, v_{n-5}, v_{n-6}$, \ldots, $v_3$, we use $\mu(P_{s+1}) - 1$ pebbles to pebble the vertices of the path $v_{n-1}, v_{n-4}, v_{n-7}$, \ldots, $v_3$ and we use $\mu(P_{s+1}) - 1$ pebbles to pebble the vertices of the path $v_{n-2}, v_{n-5}, v_{n-8}$, \ldots, $v_2$. Then no pebbles will remain to cover $v_1$. Hence $\mu(P_{s+1}) \geq 3\mu(P_{s+1}) - 2$ when $n = 3s + 1, s = 1, 2, 3, \ldots$

We now use induction to show that $\mu(P_{3s+1}^3) \leq 3\mu(P_{s+1}) - 2$ when $n = 3s + 1, s = 1, 2, 3, \ldots$

The assertion is clear for $s = 1$ by theorem 1. Therefore we assume it is true for all $P_{3s-1}^3$ when $1 \leq s' < s$.

Consider an arbitrary configuration of $P_{3s+1}^3$ having $3\mu(P_{s+1}) - 2$ pebbles. Clearly we can cover $v_{3s-1}, v_{3s}, v_{3s+1}$ in a finite number of moves with $3^2$ pebbles or less, since $p(v_{3s-1}) = p(v_{3s}) = p(v_{3s+1}) = 2'$. Hence $\mu(P_{s+1}) - 1 \leq \mu(P_{3s-2}^3) \leq 3\mu(P_{s+1}) - 2$ pebbles. Thus the number of pebbles is enough to cover $P_{3s-1}^3$ by hypothesis. Thus $\mu(P_{s+1}) - 2 \leq \mu(P_{3s-2}^3) \leq 3\mu(P_{s+1}) - 2$ when $n = 3s + 1, s = 1, 2, 3, \ldots$

A similar inductive proof works also for $\mu(P_{n}^m)$ when $n = 3s-1,3s$ and yields the following results.
Case: (ii) \( \mu(P_n^3) = 2\mu(P_{s+1}) + \mu(P_s) - 2 \), when \( n = 3s, s = 2,3,4 \ldots \ldots \) 

Case: (iii) \( \mu(P_n^3) = \mu(P_{s+1}) + 2\mu(P_s) - 2 \), when \( n = 3s - 1, s = 2,3,4 \ldots \ldots \) 

Hence, \( \mu(P_n^3) = \begin{cases} 2\mu(P_{s+1}) + \mu(P_s) - 2, & \text{when } n = 3s, s = 2,3,4 \ldots \ldots \ldots \\ \mu(P_{s+1}) + 2\mu(P_s) - 2, & \text{when } n = 3s - 1, s = 2,3, \ldots \ldots \ldots \end{cases} \)

REFERENCES


Source of support: Nil, Conflict of interest: None Declared