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ON ϕ- SYMMETRIC KENMOTSU MANIFOLDS ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

R. N. Singh[1] & Giteshwari Pandey[2]*

[1]Department of Mathematical Sciences, A.P.S.University, Rewa (M.P.)-486003, India.
[2]Department of Mathematical Sciences, A.P.S. University, Rewa (M.P.), India.

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ABSTRACT

The object of the present paper is to study ϕ-symmetric Kenmotsu manifolds with respect to semi-symmetric metric connection. We study locally ϕ-symmetric, ϕ-symmetric, ϕ-recurrent and locally pseudo projective ϕ-symmetric Kenmotsu manifolds with respect to semi-symmetric metric connection.

Key-Words: *semi-symmetric metric connection, Kenmotsu manifold, locally ϕ-symmetric, ϕ-recurrent, locally pseudo-projective ϕ-symmetric, η- Einstein manifold.*

AMS Subject Classification(2000):*53C15, 53C25.*

1. INTRODUCTION

In 1977, T. Takahashi [18] introduced the notion of local ϕ-symmetry on a Sasakian manifold. Generalizing the notion of ϕ-symmetry, one of the authors, U. C. De [4] introduced the notion of ϕ-recurrent Sasakian manifold. In the context of contact geometry the notion of ϕ-symmetry is introduced and studied by E. Boeckx, P. Buecken and L. Vanhecke [1] with several examples. In 2008, ϕ-recurrent $N(\kappa)$-contact metric manifolds and ϕ-recurrent (κ,μ) - contact metric manifolds were studied by authors [6] and [11] respectively.

On the other hand, the idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [8]. A linear connection ∇^* in an n-dimensional differentiable manifold M^n is said to be semi-symmetric connection if its torsion tensor T^* is of the form
$T^*(X,Y) = u(X)Y - u(Y)X,$

where u is 1-form. In addition, the connection ∇^* is said to be semi-symmetric metric connection if it satisfies the condition
$$(\nabla^*_X g)(Y,Z) = 0, \tag{1}$$

for all X, Y, Z ∈ TM, where TM is the Lie algebra of vector fields of the manifold M^n. In 1932, H.A. Hayden [10] defined a semi-symmetric metric connection on a Riemannian manifold and this was further studied by K. Yano [20], U.C. De and J. Sengupta [3], G.Pathak and U.C.De [14], R.N.Singh and K.P.Pandey [16], R. N. Singh and M. K. Pandey [17] and many others.

In the present paper, we study ϕ-symmetric Kenmotsu manifolds with respect to semi-symmetric metric connection. In section 2, some preliminary results are recalled. Section 3 contains the expression for curvature tensor (resp. Ricci tensor) with respect to semi-symmetric metric connection and relationship between curvature tensors (resp. Ricci tensor) with respect to semi-symmetric metric connection and Levi-Civita connection. Section 4 deals with locally ϕ −symmetric Kenmotsu manifolds admitting semi-symmetric metric connection. Section 5 is devoted to the study of ϕ −symmetric Kenmotsu manifolds admitting semi-symmetric metric connection. ϕ-recurrent Kenmotsu manifolds admitting semi-symmetric metric connection are studied in section 6 and it is obtained that if a Kenmotsu manifold is ϕ-recurrent with respect to semi-symmetric metric connection then (M^n, g) is an η-Einstein manifold with respect to Levi-Civita connection. The last section admits locally pseudo projective ϕ-symmetric Kenmotsu manifolds with respect to semi-symmetric metric connection.

*Corresponding author: Giteshwari Pandey**
Department of Mathematical Sciences, A.P.S. University, Rewa (M.P.), India.

2. PRELIMINARIES

If on an odd dimensional differentiable manifold M^n of differentiability class C^{r+1}, there exists a vector valued real linear function ϕ, a 1-form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \tag{2}$$

$$\eta(\phi X) = 0, \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{4}$$

for arbitrary vector fields X and Y, then (M^n, g) is said to be an almost contact metric manifold and the structure (ϕ, ξ, η, g) is called an almost contact metric structure to M^n. In view of equations (2), (3) and (4), we have

$$\eta(\xi) = 1, \tag{5}$$

$$g(X, \xi) = \eta(X), \tag{6}$$

$$\phi\xi = 0. \tag{7}$$

An almost contact metric manifold is called Kenmotsu manifold [12] if

$$(\nabla_X \phi)Y = \eta(Y)\phi X - g(X, \phi Y)\xi, \tag{8}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{9}$$

where ∇ is the Levi-Civita connection of g. Also the following relations hold in Kenmotsu manifold

$$(\nabla_X \eta)(Y) = g(X, Y) + \eta(X)\eta(Y), \tag{10}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{11}$$

$$R(\xi, X)Y = -R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{12}$$

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{13}$$

$$Ric(X, \xi) = -(n-1)\eta(X), \tag{14}$$

$$Q\xi = -(n-1)\xi, \tag{15}$$

where Q is the Ricci operator, i.e.

$$g(QX, Y) = Ric(X, Y), \tag{16}$$

$$Ric(\phi X, \phi Y) = Ric(X, Y) + (n-1)\eta(X)\eta(Y), \tag{17}$$

for arbitrary vector fields X, Y, Z on M^n. Let M^n be an n-dimensional Kenmotsu manifold and ∇ be the Levi-Civita connection on M^n. The relations between the semi-symmetric metric connection ∇^* and the Levi-Civita connection ∇ of a Kenmotsu manifold (M^n, g) is given by [20]

$$\nabla^*_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \tag{18}$$

3. CURVATURE TENSOR OF A KENMOTSU MANIFOLD WITH RESPECT TO SEMI-SYMMETRIC METRIC CONNECTION

Let R and R^* be the curvature tensors of the Levi-Civita connection ∇ and the semi-symmetric metric connection ∇^* respectively given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and

$$R^*(X, Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z. \tag{19}$$

Using equations (8), (9) and (18) in equation (19), we have

$$R^*(X,Y)Z = R(X,Y)Z + 3\{g(X,Z)Y - g(Y,Z)X\} + 2\eta(Z)\{\eta(Y)X - \eta(X)Y\} + 2\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi. \quad (20)$$

From above equation,we have

$$\begin{aligned} 'R^*(X,Y,Z,U) &= 'R(X,Y,Z,U) + 3\{g(X,Z)g(Y,U) - g(Y,Z)g(X,Y)\} \\ &\quad +2\eta(Z)\{\eta(Y)g(X,U) - \eta(X)g(Y,U)\} + 2\eta(U)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \end{aligned} \quad (21)$$

where $'R^*(X,U,Z,U) = g(R^*(X,Y)Z,U)$. Putting $X = W = e_i$ in above equation and summing over i, $1 \leq i \leq n$, we get

$$Ric^*(Y,Z) = Ric(Y,Z) - (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z), \quad (22)$$

where Ric^* and Ric are the Ricci tensor of the connection ∇^* and ∇ respectively. Contracting above equation, we get

$$r^* = r - (3n^2 - 7n + 4), \quad (23)$$

where r^* and r are the scalar curvatures of the connection ∇^* and ∇ respectively.

4. LOCALLY ϕ-SYMMETRIC KENMOTSU MANIFOLDS WITH RESPECT TO SEMI-SYMMETRIC METRIC CONNECTION

Definition: 4.1 *A Kenmotsu manifold M^n is said to be locally ϕ-symmetric [18] if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \quad (24)$$

for all vector fields X, Y, Z, W orthogonal to vector field ξ.

Analogous to the definition of locally ϕ-symmetric Kenmotsu manifolds, we define

Definition: 4.2 *A Kenmotsu manifold M^n is said to be locally ϕ-symmetric with respect to semi-symmetric metric connection if*

$$\phi^2((\nabla^*_W R^*)(X,Y)Z) = 0, \quad (25)$$

for all vector fields X, Y, Z, W orthogonal to vector field ξ.

Theorem: 4.1 *A Kenmotsu manifold is locally ϕ-symmetric with respect to semi-symmetric metric connection ∇^* if and only if it is so with respect to Levi-Civita connection ∇.*

Proof: From equation (18), we have

$$(\nabla^*_W R^*)(X,Y)Z = (\nabla_W R^*)(X,Y)Z + \eta(R^*(X,Y)Z)W - g(W, R^*(X,Y)Z)\xi. \quad (26)$$

Now, differentiating equation (20) covariantly with respect to W, we get

$$\begin{aligned} (\nabla_W R^*)(X,Y)Z &= (\nabla_W R)(X,Y)Z - 2g(W,X)\eta(Z)Y - 2g(W,Z)\eta(X)Y \\ &\quad +4\eta(X)\eta(Z)\eta(W)Y + 2g(W,Y)\eta(Z)X + 2g(W,Z)\eta(Y)X \\ &\quad -4\eta(Y)\eta(Z)\eta(W)X - 2g(X,Z)g(W,Y)\xi + 2g(X,Z)\eta(Y)\eta(W)\xi \\ &\quad +2g(Y,Z)g(W,X)\xi - 2g(Y,Z)\eta(X)\eta(W)\xi. \end{aligned} \quad (27)$$

Using equations (13),(22)and (27) in equation (26),we get

$$\begin{aligned} (\nabla^*_W R^*)(X,Y)Z &= (\nabla_W R)(X,Y)Z - 2g(W,X)\eta(Z)Y - 2g(W,Z)\eta(X)Y \\ &\quad +4\eta(X)\eta(Z)\eta(W)Y + 2g(W,Y)\eta(Z)X + 2g(W,Z)\eta(Y)X \\ &\quad -4\eta(Y)\eta(Z)\eta(W)X - 5g(X,Z)g(W,Y)\xi + 4g(X,Z)\eta(Y)\eta(W)\xi \\ &\quad +5g(Y,Z)g(W,X)\xi - 4g(Y,Z)\eta(X)\eta(W)\xi + 2g(X,Z)\eta(Y)W \\ &\quad -2g(Y,Z)\eta(X)W - g(W,R(X,Y)Z)\xi - 2g(W,Y)\eta(X)\eta(Z)\xi \\ &\quad -2g(W,X)\eta(Y)\eta(Z)\xi. \end{aligned} \quad (28)$$

Operating ϕ^2 on both sides of equation (28) and using equations (2) and (7), we get

$$\begin{aligned} \phi^2((\nabla^*_W R^*)(X,Y)Z) &= \phi^2((\nabla_W R)(X,Y)Z) + 2g(W,X)\eta(Z)Y - 4\eta(X)\eta(Z)\eta(W)Y \\ &\quad -2g(W,Y)\eta(Z)X - 2g(W,Z)\eta(Y)X + 2g(W,Y)\eta(X)\eta(Z)\xi + 4\eta(Y)\eta(Z)\eta(W)X \\ &\quad -2g(X,Z)\eta(Y)W + 2g(X,Z)\eta(Y)\eta(W)\xi + 2g(Y,Z)\eta(X)W - 2g(Y,Z)\eta(X)\eta(W)\xi \end{aligned} \quad (29)$$

If we consider X, Y, Z and W are orthogonal to ξ, then equation (29) yields

$$\phi^2((\nabla^*_W R^*)(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{30}$$

This completes the proof.

5. ϕ-SYMMETRIC KENMOTSU MANIFOLDS WITH RESPECT TO SEMI-SYMMETRIC METRIC CONNECTION

Definition: 5.1 *A Kenmotsu manifold M^n is said to be ϕ-symmetric([18] if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{31}$$
for arbitrary vector fields X, Y, Z, W.

Definition: 5.2 *A Kenmotsu manifold M^n is said to be ϕ-symmetric with respect to semi-symmetric metric connection if*

$$\phi^2((\nabla^*_W R^*)(X,Y)Z) = 0, \tag{32}$$
for arbitrary vector fields X, Y, Z, W.

Theorem: 5.1 *If M^n be a ϕ-symmetric Kenmotsu manifold with respect to semi-symmetric metric connection ∇^* then the manifold is an η-Einstein manifold .*

Proof: Let us consider a ϕ-symmetric Kenmotsu manifold with respect to semi-symmetric metric connection. Then by virtue of equations (2) and (31), we have

$$-(\nabla^*_W R^*)(X,Y)Z + \eta((\nabla^*_W R^*)(X,Y)Z)\xi = 0, \tag{33}$$

from which it follows that

$$-g((\nabla^*_W R^*)(X,Y)Z,U) + \eta((\nabla^*_W R^*)(X,Y)Z)\eta(U) = 0. \tag{34}$$

Using equation (28) in above equation, we get
$$\begin{aligned}
&-g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) + 2(\nabla_W \eta)(X)\eta(Z)g(Y,U)\\
&\quad +2(\nabla_W \eta)(Z)\eta(X)g(Y,U) - 2(\nabla_W \eta)(Y)\eta(Z)g(X,U) - 2(\nabla_W \eta)(Z)\eta(Y)g(X,U)\\
&\quad\quad -\eta(R(X,Y,Z))g(W,U) - g(X,Z)g(W,U)\eta(Y) + g(Y,Z)g(W,U)\eta(X)\\
&\quad\quad\quad -2(\nabla_W \eta)(X)\eta(Y)\eta(W)\eta(U) + 2(\nabla_W \eta)(Y)\eta(X)\eta(Z)\eta(U)\\
&\quad\quad\quad\quad +\eta\big(R(X,Y,Z)\big)\eta(W)\eta(U) + g(X,Z)\eta(Y)\eta(W)\eta(U) - g(Y,Z)\eta(X)\eta(W)\eta(U) = 0.
\end{aligned} \tag{35}$$

Let $\{e_i\}$,i=1,2,3.....n be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in equation (35) and taking summation over $i,\ 1 \le i \le n$, we get
$$\begin{aligned}
&-(\nabla_W Ric)(Y,Z) + \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) + (4-2n)(\nabla_W \eta)(Y)\eta(Z) + (2-2n)(\nabla_W \eta)(Z)\eta(Y)\\
&-\eta(R(W,Y)Z) - g(W,Z)\eta(Y) - 2(\nabla_W \eta)(\xi)\eta(Y)\eta(Z) + \eta(R(\xi,Y)Z)\eta(W) + \eta(Y)\eta(Z)\eta)(W) = 0.
\end{aligned} \tag{36}$$

The second term of equation (36) by putting $Z = \xi$ takes the form

$$\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi), \tag{37}$$

which is denoted by E. In this case E vanishes. Namely, we have

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W(R(e_i,Y)\xi),\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) - g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi). \tag{38}$$

at p $\in M^n$. In local co-ordinates $\nabla_W e_i = W^j \Gamma^h_{ji} e_h$, where Γ^h_{ji} are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$,where δ_{ij} is the Kronecker delta and hence the Christoffel symbols are zero. Therefore $\nabla_W e_i$ = 0. Also we have

$$g(R(e_i,\nabla_W Y)\xi,\xi) = 0, \tag{39}$$

since R is skew-symmetric . Using equation (39) and $\nabla_W e_i = 0$ in equation (38), we get

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W(R(e_i,Y)\xi),\xi) - g(R(e_i,Y)\nabla_W \xi,\xi). \tag{40}$$

In view of $g(R(e_i,Y)\xi,\xi) = -g(R(\xi,\xi)Y,e_i) = 0$ and $(\nabla_W g) = 0$, we have

$$g(\nabla_W(R(e_i,Y)\xi,)\xi) + g(R(e_i,Y)\xi,\nabla_W\xi) = 0, \tag{41}$$

which implies

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,\nabla_W\xi) - g(R(e_i,Y)\nabla_W\xi,\xi).$$

Since R is skew-symmetric, we have

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = 0. \tag{42}$$

Using equation (42) in equation (36), we get

$$(\nabla_W Ric)(Y,\xi) = (4-2n)Ric(Y,W) - (4-2n)\eta(Y)\eta(W). \tag{43}$$

Now, we know that

$$(\nabla_W Ric)(Y,\xi) = \nabla_W(Ric(Y,\xi)) - Ric(\nabla_W Y,\xi) - Ric(Y,\nabla_W\xi), \tag{44}$$

which on using equations (9), (14) takes the form

$$(\nabla_W Ric)(Y,\xi) = -(n-1)g(Y,W) - Ric(Y,W). \tag{45}$$

Form equations (43) and (45), we have

$$Ric(Y,W) = (n-3)g(Y,W) + (4-2n)\eta(Y)\eta(W), \tag{46}$$

which shows that M^n is an η −Einstein manifold.

6. ϕ -RECURRENT KENMOTSU MANIFOLDS WITH RESPECT TO SEMI-SYMMETRIC METRIC CONNECTION

Definition: 6.1 *A Kenmotsu manifold M^n is said to be ϕ-recurrent ([4]) if there exists a non-zero 1-form A such that*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z, \tag{47}$$
for arbitrary vector fields X, Y, Z, W.

If X,Y,Z and W are orthonormal to vector field ξ, then the manifold is called locally ϕ-recurrent manifold.

Definition: 6.2 *A Kenmotsu manifold M^n is said to be ϕ-recurrent with respect to semi-symmetric metric connection if*

$$\phi^2((\nabla^*_W R^*)(X,Y)Z) = A(W)R^*(X,Y)Z, \tag{48}$$
for arbitrary vector fields X, Y, Z, W.

Theorem: 6.1 *A ϕ-recurrent Kenmotsu manifold with respect to semi-symmetric metric connection is an η-Einstein manifold .*

Proof: From equations (2) and (48) ,we get

$$-((\nabla^*_W R^*)(X,Y)Z) + \eta((\nabla^*_W R^*)(X,Y)Z)\xi = A(W)g(R^*(X,Y)Z,U), \tag{49}$$

from which, we have

$$-g((\nabla^*_W R^*)(X,Y)Z,U) + \eta((\nabla^*_W R^*)(X,Y)Z)\eta(U) = A(W)R^*(X,Y)Z. \tag{50}$$

Using equations (20) and (28) in above equation ,we get

$$\begin{aligned}
&-g((\nabla_W R)(X,Y)Z,U) + 2(\nabla_W\eta)(X)\eta(Z)g(Y,U) + 2(\nabla_W\eta)(Z)\eta(X)g(Y,U) - 2(\nabla_W\eta)(Y)\eta(Z)g(X,U)\\
&\quad -2(\nabla_W\eta)(Z)\eta(Y)g(X,U) - \eta(R(X,Y)Z)g(W,U) - g(X,Z)g(W,U)\eta(Y)\\
&\quad +g(Y,Z)g(W,U)\eta(X) + \eta((\nabla_W R)(X,Y)Z)\eta(U) - 2(\nabla_W\eta)(X)\eta(Y)\eta(Z)\eta(U)\\
&\quad +2(\nabla_W\eta)(Y)\eta(X)\eta(Z)\eta(U) + \eta(R(X,Y)Z)\eta(W)\eta(U) + g(X,Z)\eta(Y)\eta(W)\eta(U) - g(Y,Z)\eta(X)\eta(Z)\eta(U)\\
&\quad = A(W)g(R(X,Y)Z,U) + 3A(W)g(X,Z)g(Y,U)\\
&\quad -3A(W)g(Y,Z)g(X,U) - 2A(W)\eta(X)\eta(Z)g(Y,U) + 2A(W)\eta(Y)\eta(Z)g(X,U)\\
&\quad -2A(W)\eta(Y)\eta(U)g(X,Z) + 2A(W)\eta(X)\eta(U)g(Y,Z).
\end{aligned} \tag{51}$$

Let $\{e_i\}$, i=1,2,3.....n be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in equation (51) and taking summation over i, $1 \le i \le n$, we get

$$-(\nabla_W Ric)(Y,Z) + \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) + (4-2n)(\nabla_W\eta)(Y)\eta(Z) + (2-2n)(\nabla_W\eta)(Z)\eta(Y)$$
$$-\eta(R(W,Y)Z) - g(W,Z)\eta(Y) - 2(\nabla_W\eta)(e_i)\eta(Y)\eta(Z)\eta(e_i) + \eta(R(\xi,Y)Z)\eta(W) + \eta(Y)\eta(Z)\eta)(W)$$
$$= A(W)Ric(Y,Z) + (5-3n)A(W)g(Y,Z) - (4-2n)A(W)\eta(Y)\eta(Z), \quad (52)$$

which by putting $Z = \xi$, gives

$$(\nabla_W Ric)(Y,\xi) - \eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) - (4-2n)(\nabla_W\eta)(Y) = -A(W)Ric(Y,\xi) + (n-1)A(W)\eta(Y). \quad (53)$$

Using equation (42) in equation (53), we get

$$(\nabla_W Ric)(Y,\xi) = (4-2n)(\nabla_W\eta)(Y) + 2(n-1)A(W)\eta(Y). \quad (54)$$

Now, we know that

$$(\nabla_W Ric)(Y,\xi) = \nabla_W(Ric(Y,\xi)) - Ric(\nabla_W Y,\xi) - Ric(Y,\nabla_W\xi), \quad (55)$$

which on using equations (8) and (14) takes the form

$$(\nabla_W Ric)(Y,\xi) = -(n-1)g(Y,W) - Ric(Y,W). \quad (56)$$

Form equations (54) and (56), we have

$$Ric(Y,W) = (n-3)g(Y,W) - (4-2n)\eta(Y)\eta(W) - 2(n-1)A(W)\eta(Y). \quad (57)$$

Replacing Y and W by ϕY and ϕW respectively in above equation and using equations (4) and (17), we get

$$Ric(Y,W) = (n-3)g(Y,W) + (4-2n)\eta(Y)\eta(W), \quad (58)$$

which shows that M^n is an η −Einstein manifold.

Theorem: 6.2 *In a ϕ-recurrent Kenmotsu manifold admitting semi-symmetric metric connection, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by equation (65).*

Proof: By virtue of equations (2) and (48), we have

$$(\nabla^*_W R^*)(X,Y)Z = \eta((\nabla^*_W R^*)(X,Y)Z)\xi - A(W)R^*(X,Y)Z. \quad (59)$$

Using equations (20) and (28) in above equation ,we get

$$(\nabla_W R)(X,Y)Z - 2(\nabla_W\eta)(X)\eta(Z)Y - 2(\nabla_W\eta)(Z)\eta(X)Y + 2(\nabla_W\eta)(Y)\eta(Z)X$$
$$+2(\nabla_W\eta)(Z)\eta(Y)X + \eta(R(X,Y)Z)W + g(X,Z)\eta(Y)W - g(Y,Z)\eta(X)W$$
$$= \eta((\nabla_W R)(X,Y)Z)\xi - 2(\nabla_W\eta)(X)\eta(Y)\eta(Z)\xi + 2(\nabla_W\eta)(Y)\eta(X)\eta(Z)\xi + \eta(R(X,Y)Z)\eta(W)\xi$$
$$+g(X,Z)\eta(Y)\eta(W)\xi - g(Y,Z)\eta(X)\eta(W)\xi - A(W)R(X,Y)Z - 3A(W)g(X,Z)Y$$
$$+3A(W)g(Y,Z)X + 2A(W)\eta(X)\eta(Z)Y - 2A(W)\eta(Y)\eta(Z)X + 2A(W)\eta(Y)g(X,Z)\xi$$
$$-2A(W)\eta(X)g(Y,Z)\xi. \quad (60)$$

Taking inner product of above equation with respect to ξ, we get

$$A(W)\eta(R(X,Y)Z) = A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \quad (61)$$

Writing two more equations by the cyclic permutations of X, Y and Z from equation (61) and adding them to equation (61),we get

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z)$$
$$= A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + A(X)[g(W,Z)\eta(Y)$$
$$-g(Y,Z)\eta(W)] + A(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)]. \quad (62)$$

Using equation (13) in (62) ,we get

$$A(W)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] + A(X)[g(Y,Z)\eta(W) - g(W,Z)\eta(Y)] + A(Y)[g(W,Z)\eta(X)$$
$$-g(X,Z)\eta(W)] = 0. \quad (63)$$

Now putting $Y = Z = e_i$ in equation (63) and taking summation over i, $1 \leq i \leq n$, we get

$$A(W)\eta(X) = A(X)\eta(W), \tag{64}$$

for all vector fields X, W. Replacing X by ξ in equation (64),we get

$$A(W) = \eta(\rho)\eta(W), \tag{65}$$

for all vector field W, where $A(\xi) = g(\xi,\rho) = \eta(\rho)$, ρ being the vector field associated to the 1-form A i.e.

$$g(X,\rho) = A(X). \tag{66}$$

7.LOCALLY PSEUDO-PROJECTIVE ϕ-SYMMETRIC KENMOTSU MANIFOLDS WITH RESPECT TO SEMI-SYMMETRIC METRIC CONNECTION

Pseudo-projective curvature tensor of M^n with respect to Levi-Civita connection is given by [15]

$$\tilde{P}(X,Y)Z = aR(X,Y)Z + b[Ric(Y,Z)X - Ric(X,Z)Y] - \frac{r}{n}[\frac{a}{(n-1)} + b][g(Y,Z)X - g(X,Z)Y], \tag{67}$$

where a and b are the constants such that $a, b \neq 0$, R, Ric and r are the Riemannian curvature tensor, Ricci tensor and scalar curvature respectively.

Pseudo-projective curvature tensor of M^n with respect to semi-symmetric metric connection is given by

$$\tilde{P}^*(X,Y)Z = aR^*(X,Y)Z + b[Ric^*(Y,Z)X - Ric^*(X,Z)Y] - \frac{r^*}{n}[\frac{a}{(n-1)} + b][g(Y,Z)X - g(X,Z)Y]. \tag{68}$$

Using equations (20), (22) and (23) in above equation, we get

$$\begin{aligned}\tilde{P}^*(X,Y)Z = {} & aR(X,Y)Z + b[Ric(Y,Z)X - Ric(X,Z)Y] + \alpha\{g(Y,Z)X - g(X,Z)Y\} + \beta\{\eta(Y)X - \eta(X)Y\}\eta(Z) \\ & -2a\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi,\end{aligned} \tag{69}$$

where

$$\alpha = -3a + (5-3n)b - \frac{r - 3n^2 + 7n - 4}{n}(\frac{a}{n-1} + b)$$

and

$$\beta = 2a - (4-2n)b.$$

Definition: 7.1 *An n-dimensional Kenmotsu manifold M^n is said to be locally pseudo-projective ϕ-symmetric ([19]), if*

$$\phi^2((\nabla_W \tilde{P})(X,Y)Z) = 0, \tag{70}$$

for all vector fields X, Y, Z and W orthogonal to ξ.

Definition: 7.2 *An n-dimensional Kenmotsu manifold M^n is said to be locally pseudo-projective ϕ-symmetric with respect to semi-symmetric metric connection if*

$$\phi^2((\nabla^*_W \tilde{P}^*)(X,Y)Z) = 0, \tag{71}$$

for all vector fields X, Y, Z and W orthogonal to ξ.

Theorem: 7.1 *A Kenmotsu manifold is locally pseudo-projective ϕ-symmetric with respect to ∇^* if and only if it is so with respect to Levi-Civita connection ∇.*

Proof: From equation (18), we have

$$(\nabla^*_W \tilde{P}^*)(X,Y)Z = (\nabla_W \tilde{P}^*)(X,Y)Z + \eta(\tilde{P}^*(X,Y)Z)W - g(W,\tilde{P}^*(X,Y)Z)\xi. \tag{72}$$

Now differentiating equation (68) with respect to W, we get

$$\begin{aligned}(\nabla_W \tilde{P}^*)(X,Y)Z = {} & a(\nabla_W R^*)(X,Y)Z + b[(\nabla_W Ric^*)(Y,Z)X - (\nabla_W Ric^*)(X,Z)Y] \\ & - \frac{(\nabla_W r^*)}{n}(\frac{a}{(n-1)} + b)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{73}$$

By virtue of equations (27),(22) and (23) above equation reduces to

$$(\nabla_W \tilde{P}^*)(X,Y)Z = a(\nabla_W R)(X,Y)Z - 2a\{g(X,W)\eta(Z)Y - 2ag(Z,W)\eta(X)Y + 4a\eta(X)\eta(Z)\eta(W)\}Y$$
$$+2ag(Y,W)\eta(Z)X + 2ag(Z,W)\eta(Y)X - 4a\eta(W)\eta(Y)\eta(Z)\}X - 2ag(X,Z)g(Y,W)\xi$$
$$+2ag(X,Z)\eta(Y)\eta(W)\xi + 2ag(Y,Z)g(X,\mathrm{W})\xi - 2ag(Y,Z)\eta(X)\eta(W)\xi - (4-2n)b[(\nabla_W\eta)(Y)\eta(Z)X$$
$$+(\nabla_W\eta)(Z)\eta(Y)X] + (4-2n)b[(\nabla_W\eta)(X)\eta(Z)Y + (\nabla_W\eta)(Z)\eta(X)Y] + b[(\nabla_W Ric)(Y,Z)X$$
$$-(\nabla_W Ric)(X,Z)Y] - \frac{(\nabla_W r)}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y] \quad (74)$$

which on using equation (67) reduces to

$$(\nabla_W \tilde{P}^*)(X,Y)Z = (\nabla_W \tilde{P})(X,Y)Z - 2ag(X,W)\eta(Z)Y - 2ag(Z,W)\eta(X)Y + 4a\eta(X)\eta(Z)\eta(W)Y$$
$$+2ag(Y,W)\eta(Z)X + 2ag(Z,W)\eta(Y)X - 4a\eta(W)\eta(Y)\eta(Z)X - 2ag(X,Z)g(Y,W)\xi$$
$$+2ag(X,Z)\eta(Y)\eta(W)\xi + 2ag(Y,Z)g(X,W)\xi - 2ag(Y,Z)\eta(X)\eta(W)\xi - (4-2n)b[(\nabla_W\eta)(Y)\eta(Z)X$$
$$+(\nabla_W\eta)(Z)\eta(Y)X] + (4-2n)b[(\nabla_W\eta)(X)\eta(Z)Y + (\nabla_W\eta)(Z)\eta(X)Y]. \quad (75)$$

Now, taking the inner product of equation (68) with ξ and using equations (20), (22) and (23) , we get

$$\eta(\tilde{P}^*(X,Y)Z) = [-2a + (5-3n)b - \frac{(r-3n^2+7n-4)}{n}\left(\frac{a}{(n-1)} + b\right)][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]W$$
$$+b[Ric(Y,Z)\eta(X) - Ric(X,Z)\eta(Y)]W. \quad (76)$$

Also from equations (20), (22), (23) and (69), we have

$$g(W,\tilde{P}^*(X,Y)Z)\xi = ag(W,R(X,Y)Z)\xi + 3a\{g(X,Z)g(Y,W) - g(Y,Z)g(W,X)\}\xi$$
$$+2a\{g(W,X)\eta(Y) - g(Y,W)\eta(X)\}\eta(Z)\xi + 2a\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\eta(W)\xi$$
$$+b[Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W) + (5-3n)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}$$
$$+(4-2n)\{\eta(X)g(Y,W) - \eta(Y)g(X,W)\}\eta(Z)]\xi - \frac{(r-3n^2+7n-4)}{n}\left(\frac{a}{(n-1)} + b\right)][g(Y,Z)g(X,W)$$
$$-g(X,Z)g(Y,W)]\xi. \quad (77)$$

Now using equations (75), (76) and (77) in equation (72), we get

$$(\nabla^*_W \tilde{P}^*)(X,Y)Z = (\nabla_W \tilde{P})(X,Y)Z + 2a\{g(Y,W)X - g(X,W)Y\}\eta(Z) + 2a\{\eta(Y)X$$
$$-\eta(X)Y\}g(W,Z) - 4a\{\eta(Y)X - \eta(X)Y\}\eta(Z)\eta(W) - 4a\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\eta(W)\xi$$
$$-(4-2n)b\{(\nabla_W\eta)(Y)X - (\nabla_W\eta)(X)Y\}\eta(Z) - (4-2n)b\{\eta(Y)X - \eta(X)Y\}(\nabla_W\eta)(Z)$$
$$+(\alpha - a)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}W + b\{Ric(Y,Z)\eta(X) - Ric(X,Z)\eta(Y)\}W - ag(W,R(X,Y)Z)\xi$$
$$+(\alpha + 2a)\{g(Y,Z)g(W,X) - g(X,Z)g(Y,W)\}\xi + (4-2n+2a)\{g(Y,W)\eta(X) - g(X,W)\eta(Y)\}\eta(Z)\xi$$
$$-b\{Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W)\}\xi \quad (78)$$

Applying ϕ^2 on both sides of above equation and using equations (2) and (7), we get

$$\phi^2((\nabla^*_W \tilde{P}^*)(X,Y)Z) = \phi^2((\nabla_W \tilde{P})(X,Y)Z) + 2a\{g(X,W)\eta(Z) + g(Z,W)\eta(X)\}Y$$
$$-2a\{g(X,W)\eta(Z) + g(Z,W)\eta(X)\}\eta(Y)\xi + 4a\{\eta(Y)X - \eta(X)Y\}\eta(Z)\eta(W) - 2a\{g(Y,W)\eta(Z)$$
$$+g(Z,W)\eta(Y)\}X + 2a\{g(Y,W)\eta(Z) + g(Z,W)\eta(Y)\}\eta(X)\xi - (4-2n)b[-(\nabla_W\eta)(Y)\eta(Z)X$$
$$+(\nabla_W\eta)(Y)\eta(Z)\eta(X)\xi - \eta(Y)(\nabla_W\eta)(Z)X + \eta(Y)(\nabla_W\eta)(Z)\eta(X)\xi] + (4-2n)b[-(\nabla_W\eta)(X)\eta(Z)Y$$
$$+(\nabla_W\eta)(X)\eta(Z)\eta(Y)\xi - \eta(X)(\nabla_W\eta)(Z)Y + \eta(X)(\nabla_W\eta)(Z)\eta(Y)\xi] + [-2a + (5-3n)b$$
$$-\frac{(r-3n^2+7n-4)}{n}\left(\frac{a}{(n-1)} + b\right)][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)](-W + \eta(W)\xi) + b[Ric(Y,Z)\eta(X)$$
$$-Ric(X,Z)\eta(Y)](-W + \eta(W)\xi). \quad (79)$$

If we consider X, Y, Z and W orthogonal to ξ, above equation reduces to

$\phi^2((\nabla^*_W \tilde{P}^*)(X,Y)Z) = \phi^2((\nabla_W \tilde{P})(X,Y)Z)$. (80)

This completes the proof.

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