AN IDEAL BASED ZERO DIVISOR GRAPH OF GAMMA NEAR-RINGS

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ABSTRACT

In this paper, we study the notion of ideal based zero divisor graph structure of Gamma Near-ring M with respect to reflexive ideal I of M denoted by ΓI(M) whose vertices are the set \( \{x \in M - I | \text{there exists } y \in M - I \text{ such that } x \Gamma y \subseteq I \} \) with distinct vertices x and y are adjacent if and only if \( x \Gamma y \subseteq I \).

Keywords: ideal, graph, zero-divisor, diameter, cycle, Girth, clique.

INTRODUCTION

The concept of a Gamma near-rings [9] was introduced by Satyanarayana and the ideal theory in Gamma near-rings was studied by Bh. Satyanarayana and G.L.Booth.

Let (M, +) be a group (not necessarily abelian) and \( \Gamma \) be a nonempty set. Then M is said to be a \( \Gamma \)-near ring if there exists a mapping \( \Gamma \times \Gamma \times M \rightarrow M \) (denote the image of \( (m, \alpha,m_2) \) by \( m\alpha m_2 \) for \( m, m_2 \in M \) and \( \alpha \in \Gamma \)) satisfying the following conditions.

1. \( (m_1 + m_2)\alpha m_3 = m_1\alpha m_3 + m_2\alpha m_3 \) and
2. \( (m_1\alpha m_2)\alpha m_3 = m_1\alpha (m_2\alpha m_3) \) for all \( m_1, m_2, m_3 \in M \) and \( \alpha, \alpha_2 \in \Gamma \).

Furthermore, M is said to be a zero symmetric \( \Gamma \)-near ring if \( xa0 = 0 \) for all \( x \in M \) and \( \alpha \in \Gamma \) (where 0 is an additive identity in M.)

A normal subgroup L of M is called a left (resp right) ideal of M if u \( a(x + v) - u \alpha v \in L \) (resp \( x \alpha u \in L \)) for all \( x \in L, \alpha \in \Gamma \) and \( u, v \in M \). A normal subgroup I of M is called an ideal if I is both left and right ideal of M. An ideal I of M is said to be reflexive if \( ab \in I \iff b \alpha a \in I \) for \( a, b \in M \) and \( \alpha \in \Gamma \). A proper ideal P of M is said to be prime if for any ideals A, B of M such that \( A \Gamma B \subseteq P \), we have \( A \subseteq P \) or \( B \subseteq P \). An ideal P is called completely prime if \( A \Gamma B \subseteq P \) implies \( a \in P \) or \( b \in P \). It is clear that if I is a reflexive ideal of M then I is prime iff I is completely prime.

In this paper, we study the undirected graph \( \Gamma_{I}(M) \) of Gamma near rings for any completely reflexive ideal I of M. Throughout this paper M stands for a non zero Gamma near -ring with zero element and I is a completely reflexive ideal of M. For distinct vertices x and y of a Graph G, let d(x, y) be the length of the shortest path from x to y. The diameter of a connected graph is the supremum of the distances between vertices. For any graph G, the girth of G is the length of a shortest cycle in G and is denoted by \( gr(G) \). If G has no cycle, we define the girth of G to be infinite. A clique of a graph is a maximal subgraph and the number of graph vertices in the largest clique or graph G, denoted by \( \omega(G) \) is called the clique number of G. A graph G is bipartite with vertex classes \( V_1, V_2 \) if the set of all vertices of G is \( V_1 \cup V_2 \), \( V_1 \cap V_2 = \emptyset \), and edge of G joins a vertex from \( V_1 \) to a vertex of \( V_2 \).

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Theorem: 1
a) If \( I = (0) \) then \( \Gamma(I)(M) = \Gamma(M) \)
b) Let \( I \) be a nonzero completely reflexive ideal of \( M \). Then \( \Gamma(I)(M) = \Phi \) if and only if \( I \) is a completely prime ideal of \( M \).

Proof:
a) Proof is Obvious.
b) Suppose that \( I \) is a completely prime ideal of \( M \). For any \( x, y \in \Gamma(I)(M) \), either \( m - x \) or \( x - y \) is an edge in \( \Gamma(I)(M) \). Hence \( \Gamma(I)(M) \) is empty.

Conversely suppose that \( \Gamma(I)(M) = \Phi \). Therefore if \( x \in M - I \) and \( x - y \in I \), \( \alpha \in \Gamma \) for some \( y \in M \). We must have \( y \in I \). (otherwise \( x \) is a vertex of \( \Gamma(I)(M) \)). Hence \( I \) is a completely prime ideal of \( M \).

Theorem: 2 Let \( I \) be a completely reflexive ideal of a gamma near-ring \( M \). Then \( \Gamma(I)(M) \) is connected and \( \operatorname{diam}(\Gamma(I)(M)) \leq 3 \). Furthermore if \( \Gamma(I)(M) \) contains a cycle, then \( \operatorname{gr}(\Gamma(I)(M)) \leq 7 \).

Proof: Let \( x, y \in M - I \), with \( x - y \) be an edge in \( \Gamma(I)(M) \) and suppose that \( m - y \) is not an edge in \( \Gamma(I)(M) \) for some \( m \in M - I \). Then \( x_{1} \Gamma y_{1} \in I \) for some \( x_{1} \in x > - I \), \( y_{1} \in y > - I \) and \( m \Gamma y_{1} \notin I \). But for each \( x_{1} \Gamma y_{1} \in I \), \( m \Gamma y_{1} \notin I \). Then \( x - y' \) is an edge in \( \Gamma(I)(M) \) for some \( y' \in y > - I \).

Therefore \( \operatorname{gr}(\Gamma(I)(M)) \leq 7 \).

Theorem: 3 Let \( x, y \in M - I \), with \( x - y \) be an edge in \( \Gamma(I)(M) \) and suppose that \( m - y \) is not an edge in \( \Gamma(I)(M) \) for some \( m \in M - I \). Then \( x_{1} \Gamma y_{1} \in I \) for some \( x_{1} \in x > - I \), \( y_{1} \in y > - I \) and \( m \Gamma y_{1} \notin I \). Then \( x_{1} \in x > - I \) and \( m \Gamma y_{1} \notin I \). Then \( x - y' \) is an edge in \( \Gamma(I)(M) \) for some \( y' \in y > - I \).

Theorem: 4 Let \( x_{1} \in x > - I \) and \( a-x-b \) be a path in \( \Gamma(I)(M) \), then either \( I \cup \{ x_{1} \} \) is an ideal of \( M \) for some \( x_{1} \in x > - I \) or \( a-x-b \) is contained in a cycle of length \( \leq 4 \).

Proof: Let \( a-x-b \) be a path in \( \Gamma(I)(M) \). Then there exists \( x_{1}, x_{2} \in x > - I \), \( a_{1} \in a > - I \), \( b_{1} \in b > - I \) such that \( a_{1} \Gamma x_{1} \in I \) and \( b_{1} \Gamma x_{2} \in I \). If \( a_{1} \Gamma b_{1} \in I \) for some \( a' \in a > - I \) and \( b' \in b > - I \). Then \( a-x-b-a \) is contained in a cycle of length \( \leq 4 \). So let us assume that \( a_{1} \Gamma b_{1} \notin I \) for all \( a_{1} \in a > - I \) and \( b_{1} \in b > - I \).

Case: (i) \( x_{1} \neq x_{2} \), then either \( I \cap I_{a_{1}} \cap I_{b_{1}} \in I \cup \{ x_{1} \} \) or there exists \( c \in I \cap I_{a_{1}} \cap I_{b_{1}} \) such that \( c \in I \cup \{ x_{1} \} \). Then \( f_{a_{1}} \Gamma c, f_{b_{1}} \Gamma b_{1} \in I \). In the first case, \( I \cup \{ x_{1} \} \) is an ideal. In the second case \( a-x-b-a \) is contained in a cycle of length \( \leq 4 \).

Case: (ii) \( x_{1} = x_{2} \), then clearly \( a_{1} > \cap b_{1} > I \). Then for each \( z \in a_{1} > \cap b_{1} > - I \). We have, \( z \Gamma x_{1} \in x_{1} > \cap x_{2} \in I \). Clearly either \( x_{1} \neq x \) or \( x_{2} \neq x \). Say \( x_{1} \neq x \). Then we have a path \( a-x_{1}-b \) and hence \( a-x_{1}-b \) is contained in a cycle of length \( \leq 4 \).

Theorem: 5 Let \( I \) be a completely reflexive ideal of \( M \). Then \( \Gamma(I)(M) \) can be neither a pentagon nor a hexagon.

Proof: Suppose that \( \Gamma(I)(M) \) is a b-c-d-e-a-a pentagon. Then by theorem 4, For one of the vertices say \( b_{1} \), \( I \cup \{ b_{1} \} \) is an ideal of \( M \) for some \( b_{1} \in b > - I \). Then in the pentagon, there exists \( d_{1} \in d > - I \) and \( e_{1} \in e > - I \) such that \( d_{1} \Gamma e_{1} \in I \). Since \( I \cup \{ b_{1} \} \) is an ideal, \( b_{1} \gamma d_{1} = b_{1} = b_{1} \gamma e_{1} \) for some \( \gamma, e_{1} \in \Gamma \). But \( b_{1} \gamma (d_{1} \gamma e_{1}) \in I \cup \{ b_{1} \} \). Then \( b_{1} = b_{1} \gamma e_{1} = (b_{1} \gamma d_{1}) \gamma e_{1} = b_{1} \gamma (d_{1} \gamma e_{1}) \in I \). (i.e., \( b_{1} \in I \) which is a contradiction. The proof for the hexagon is the same.

Theorem: 6 Let \( I \) be a reflexive ideal of a Gamma near ring \( M \) and let \( x, y \in M - I \) Then
1. If \( x+I \) is adjacent to \( y+I \) in \( \Gamma(M) \) then \( x \) is adjacent to \( y \) in \( \Gamma(M) \)
2. If \( x \) is adjacent to \( y \) in \( \Gamma(M) \) and \( x+I \neq y+I \) then \( x+I \) is adjacent to \( y+I \) in \( \Gamma(M) \)
3. If \( x \) is adjacent to \( y \) in \( \Gamma(M) \) and \( x+I = y+I \) then \( x^{2}, y^{2} \in I \).
Clearly there is a strong relationship between $\Gamma_i(M)$ and $\Gamma \left( \frac{M}{I} \right)$.

Let I be an ideal of a gamma near-ring M. One can verify that the following method can be used to construct a graph $\Gamma_i(M)$. Let $\{a_\lambda \}_\lambda \subseteq R$ be a set of coset representatives of the vertices of $\Gamma \left( \frac{M}{I} \right)$. For each $\lambda \in I$, define a graph $G_i$ with vertices $\{a_\lambda + i/\lambda \in \Lambda \}$ where edges are defined by the relationship $a_\lambda + i$ is adjacent to $a_{\beta} + i$ in $G_i$ if $a_\lambda + i$ is adjacent to $a_{\beta} + i$ in $\Gamma \left( \frac{M}{I} \right)$.

Theorem: 7 Let I be a completely reflexive ideal of M. Then the following are hold

(i) If M has identity, then $\Gamma_i(M)$ has no cut vertices.

(ii) If M has no identity and if I is a nonzero completely reflexive ideal of M then $\Gamma_i(M)$ has no cut vertices.

Proof: Suppose that the vertex x of $\Gamma_i(M)$ is cut vertex. Let u-x-w be a path in $\Gamma_i(M)$. Since x is a cut vertex, x lies in every path from u to w.

i) Assume that M is Gamma near ring with identity. For any $u, v \in \Gamma_i(M)$, there exists a path $u\rightarrow w$ which shows x(± 1) in $\Gamma_i(M)$ is not a cut vertex. Suppose x = 1. Then there exists $u_1 \in I < u > I, w_1 < w > I \gamma \in \Gamma$ and $t_1, t_2 \in M - I$ such that $u_1 y_1 t_1, w_1 y_1 t_2 \in I$ which implies $u_1, w_1 \in \Gamma_i(M)$. Since $\Gamma_i(M)$ is connected, there exists $m, m_2 \in M - I \cup \{x\}$ such that $u_1 - m - w_1 \in I$ is a path in $\Gamma_i(M)$ which implies $u - m - w - 1 = u (or) u - m - m_1 - w_1 - 1 = u$ is a cycle in $\Gamma_i(M)$ contradicting x=1 is a cut vertex.

ii) Let M be a $\Gamma$-near-ring without identity and I be a non-zero completely reflexive ideal of M. Since $u - x - w$ is a path from u to w, there exists $u_1 \in I < u > I, w_1 < w > I$ and $x_1, x_2 \in I \gamma \in \Gamma$ such that $u_1 y_1 x_1 \in I$ and $w_1 y_2 x_2 \in I$.

Case: (i) $x_1 = x_2$

If $u_1 + I = x_1 + I$ then $u_1 y_1 w_1 \in I \Rightarrow u$ is adjacent to w. Similarly, if $x_2 + I = w_1 + I, u$ is adjacent to w. So assume that $u_1 + I \neq x_1 + I$ and $x_2 + I \neq w_1 + I$. Let $0 \neq I \in I$. Then $u_1 y_1 w_1 \in I$ and $w_1 y_2 x_2 \in I$ which implies $u_1 y(x_1 + I, w_1 y + I) \in I$. If $x = x_1 + I$ then $x \neq x_1 \Rightarrow u- x_1- w$ is a path in $\Gamma_i(M)$, otherwise, $u-x_1+w$ is a path in $\Gamma_i(M)$. Thus there exists a path from u to w not passing through x which is a contradiction.

Case: (ii) Either $x_1$ or $x_2$ equal to x.

Without loss of generality, let us assume that $x_1 = x$ and $x_2 \neq x$. Then $u_1 y x \in I$ and $x_1 y w_1 \in I \Rightarrow u_1 y x_1 \in I$ and $x_2 y w_1 \in I$. Also we have a path u-x_1-w which is a contradiction.

Case: (iii) Neither $x_1$ nor $x_2$ equal to x.

If $x_1 y x_2 \in I$ then we have a path u-x_1-x_2-w which is a contradiction. So assume that $x_1 y x_2 \neq x$, then we have a path u-x_1-yx_2-w which is a contradiction.

Thus x cannot be a cut vertex.

Definition: 8 Using the notation as in the above construction, we call the subset $a_\lambda + I$ a column of $\Gamma_i(M)$. If $a_\lambda + I \in I$ then we call $a_\lambda + I$ a connected column of $\Gamma_i(M)$.

Lemma: 9 Let I be a reflexive ideal of a Gamma near-ring M. Then $gr(\Gamma_i(M)) \leq gr(\Gamma \left( \frac{M}{I} \right))$. In particular if $\Gamma \left( \frac{M}{I} \right)$ contains a cycle then so does $\Gamma_i(M)$ and therefore $gr(\Gamma_i(M)) \leq gr(\Gamma \left( \frac{M}{I} \right)) \leq 4$.

Proof: If $gr(\Gamma \left( \frac{M}{I} \right)) = \infty$ we are done. So suppose $gr(\Gamma \left( \frac{M}{I} \right)) = n < \infty$.

Let $x_1 + I - x_2 + I - \cdots - x_n + I - x_1 + I$ be a cycle in $\Gamma \left( \frac{M}{I} \right)$ through n distinct vertices

Then $x_1 - x_2 - \cdots - x_n - x_1$ is a cycle in $\Gamma_i(M)$ of length n. Hence $gr(\Gamma_i(M)) \leq n$.

Lemma: 10 Let I be a reflexive ideal of a Gamma near ring M. If $|I| \geq 3$ and $\Gamma_i(M)$ contains a connected column, then $gr(\Gamma_i(M)) = 3$
Proof: Let \( x + I \) be a connected column of \( \Gamma_i(M) \). Then \( x^2 \in I \). Let \( i, j \in I - \{0\} \). Then \( x-(x+i)-(x+j)-x \) is a cycle of length 3 in \( \Gamma_i(M) \).

Lemma: 11 Let \( I \) be a reflexive ideal of a gamma near ring \( M \). If \( I \neq 0 \) and \( \Gamma_i(M) \) has only one vertex, then 
\[
\text{gr} \Gamma_i(M) = \begin{cases} 
3 \text{ if } |I| \geq 3 \\
\infty \text{ if } |I| = 2 
\end{cases}
\]

Proof: If \( \Gamma_i(M) \) has only one vertex then \( \Gamma_i(M) \) consist of a single connected column. Thus \( \Gamma_i(M) \) is a complete graph, and therefore has a cycle of length 3 unless \( \Gamma_i(M) \) has only two vertices.

Lemma: 12 Let \( I \) be a reflexive ideal of a gamma near ring \( M \). If \( I \) has two elements, \( \Gamma_i(M) \) has at least two vertices and \( \Gamma_i(M) \) has at least two vertices, and \( \Gamma_i(M) \) has at least one connected column, then \( \text{gr} \Gamma_i(M) = 3 \)

Proof: Let \( x+I \) be a connected column of \( \Gamma_i(M) \). Then \( x^2 \in I \). Let \( y+I \) be a vertex adjacent to \( x+I \) in \( \Gamma_i(M) \). Write \( I = \{0,i\} \). Then \( y-x-x+1-y \) is a cycle of length 3 in \( \Gamma_i(M) \).

Theorem: 13 Let \( I \) be a nonzero reflexive ideal of a gamma near ring \( M \) that is not a completely prime ideal. Then 
\[
\text{gr} \Gamma_i(M) = \infty \text{ if } \Gamma(M) \text{ has only one cut vertex } & |I| = 2 \\
4 \text{ if } \text{gr} \Gamma_i(M) > 3 \text{ and } \Gamma_i(M) \text{ has no connected columns} \\
3 \text{ otherwise }
\]

Proof: The only remaining case is \( I \neq \{0\} \), \( \Gamma_i(M) \) has no connected columns, and \( \text{gr} \Gamma_i(M) > 3 \).

Since \( \Gamma_i(M) \) has no connected columns, \( \Gamma(M) \) must have at least two vertices. By lemma 9, \( \text{gr} \Gamma_i(M) \leq 4 \). Assume \( x-y-z-x \) is a cycle in \( \Gamma_i(M) \) of length 3 and we provide a contradiction. Since \( \text{gr} \Gamma_i(M) > 3 \), \( x+I-y+I-z+I-x+I \) cannot be a cycle in \( \Gamma(M) \). Therefore we have either \( x+I=y+I \), \( y+I=z+I \) (or) \( z+I=x+I \). If \( x+I=y+I \), then \((x+1)(y+1) = (x+1)(y+I) = 0+I \) and so \( x+I \) is a connected column of \( \Gamma(M) \). But this is a contradiction. We get a similar contradiction if \( y+I=z+I \) (or) \( z+I=x+I \). Hence \( \text{gr} \Gamma_i(M) = 4 \).

Theorem: 14 Let \( I \) be a nonzero reflexive ideal of a gamma near ring \( M \). Then \( \Gamma_i(M) \) is bipartite if and only if either
\[\begin{align*}
a) & \quad \text{gr} \Gamma_i(M) = \infty \quad \text{(or)} \\
b) & \quad \text{gr} \Gamma_i(M) = 4 \quad \text{and } \Gamma(M) \text{ is bipartite.}
\end{align*}\]

Proof: Suppose that \( \Gamma_i(M) \) is bipartite. Since \( \Gamma(M) \) is isomorphic to a subgraph (or) \( \Gamma_i(M) \), \( \Gamma(M) \) is bipartite (or a single vertex). By theorem 13, \( \text{gr} \Gamma_i(M) \) is 3,4,\( \infty \). By theorem 1 of sec 1.2 of Bollobas (1979), a graph is bipartite if and only if it does not contain an odd cycle. Hence \( \text{gr} \Gamma_i(M) \neq 3 \).

If \( \text{gr} \Gamma_i(M) = \infty \), then by theorem 13, \( \Gamma_i(M) \) is a graph on two vertices and therefore bipartite. Suppose \( \text{gr} \Gamma_i(M) = 4 \) and \( \Gamma(M) \) is bipartite. Let \( W_1, W_2 \) be the two vertex classes of \( \Gamma(M) \). Define \( V_j = \{x+i/i \in I, x+i \in W_j \} \) for \( j = 1, 2 \). Then \( V_1 \cap V_2 = \emptyset \) and the vertex set of \( \Gamma_i(M) \) is \( V_1 \cup V_2 \).

Let \( x \) and \( y \) be adjacent vertices of \( \Gamma_i(M) \). Without loss of generality, say \( x \in V_1 \). By theorem 13, \( \Gamma_i(M) \) has no connected columns. Thus \( x+I \neq y+I \). Hence \( x+I-y+I \) is an edge in \( \Gamma(M) \). By theorem 6, since \( x+I \in W_1, y+I \in W_2 \). Therefore \( y \in V_2 \). Hence all edges of \( \Gamma_i(M) \) join vertices from \( V_1 \) to those of \( V_2 \). Thus \( \Gamma_i(M) \) is bipartite.

Theorem: 15 Let \( I \) be a reflexive ideal of \( M \) and let \( S \) be a clique in \( \Gamma_i(M) \) such that \( x^2 \neq 0 \) for all \( x \in S \). Then \( S \cup I \) is a reflexive ideal of \( M \).

Proof: Suppose that \( x, y \in S \cup I \). Consider the following three cases

Case: (i) If \( x, y \in I \) then \( xy \in S \cup I, x \in I \)
Case: (ii) If \( x, y \in S \) with \( xy \notin I \) then for all \( c \in S \) \( c \Gamma(xy) \in I \) and hence \( S \cup \{xy\} \) is a clique. Now since \( S \) is a clique, \( xy \in S \).

Case: (iii) If \( x \in I \) and \( y \in S \) then \( xy \notin I \) and hence for any \( c \in S \) \( c \Gamma(xy) \in I \). Therefore \( xy \in S \). Now let \( x \in S \cup I \) and \( r \in M \). Suppose that \( r, x \notin I, a \in I \). If \( rfx \notin I \) then \( rfx \subseteq S \cup I \). If \( rfx \notin I \). Since for any \( c \in S \), \( (rfx) \Gamma \subseteq I \). We have \( rf \in S \).

**Theorem: 16** Let \( I \) be a nonzero reflexive ideal of \( M \) and \( a \in \Gamma(I(M)) \) adjacent to every vertex of \( \Gamma(I(M)) \). Then \( I \colon a \) is a maximal element of the set \( \{(I \colon x) / x \in M\} \). Moreover \( I \colon a \) is a completely prime ideal.

**Proof:** Let \( V = V(\Gamma(I(M))) \). Choose \( 0 \neq x \in I \). It is easy to see that \( a \neq a + x \in \Gamma(I(M)) \). Thus \( a \Gamma(a + x) \in I \) and hence \( a^2 \in I \).

Therefore \( I \cup I = (I : a) \) and so for any \( x \in M,(I : x) \) is contained in \( I \cup I = (I : a) \). Thus the first assertion holds.

Now we prove that \( (I : a) \) is a completely prime ideal. Let \( xy \in (I : a) \) and \( x, y \notin I(a) \). Therefore \( x \Gamma y a \in I \). If \( y \Gamma a \notin I \) then \( x \in (I : y \Gamma a) \). We know that \( (I : a) \subseteq (I : y \Gamma a) \). And therefore \( (I : a) = (I : y \Gamma a) \). Hence \( x \in (I : a) \) which is a contradiction.

**Theorem: 17** Let \( I \) be a non-zero reflexive ideal of \( M \). Then the followings are hold.

a) If \( P_1 \) and \( P_2 \) are completely prime ideals of \( M \) and \( I = P_1 \cap P_2 \neq \emptyset \).

   Then \( \Gamma(I) \) is a complete bipartite graph

b) If \( I \neq 0 \) is a reflexive ideal of \( M \) for which \( I = \sqrt{I} \) then \( \Gamma(I(M)) \) is a complete bipartite graph if and only if there exists prime ideals \( P_1 \) and \( P_2 \) of \( M \) such that \( I = P_1 \cap P_2 \).

**Proof:**

a) Let \( a, b \in M \) – \( I \) with \( ab \in I \). Then \( ab \in P_1 \) and \( ab \in P_2 \). Since \( P_1 \) and \( P_2 \) are completely prime, we have \( a \in P_1 \) or \( b \in P_2 \) and \( a \in P_2 \) (or) \( b \in P_2 \). Therefore suppose \( a \in \frac{P_1}{P_2} \) and \( b \in \frac{P_2}{P_1} \). Thus \( \Gamma(I(M)) \) is a complete bipartite graph with parts \( \frac{P_1}{P_2} \) and \( \frac{P_2}{P_1} \).

b) Suppose that the parts of \( \Gamma(I(M)) \) are \( V_1 \) and \( V_2 \). Set \( P_1 = V_1 \cup I \) and \( P_2 = V_2 \cup I \). It is clear that \( I = P_1 \cap P_2 \). We now prove that \( P_1 \) is a reflexive ideal of \( M \).

To show this let \( a, b \in P_1 \).

**Case:** (i) If \( a, b \in I \) then \( a \gamma b \in I \) and so \( a \gamma b \in P_1 \).

**Case:** (ii) If \( a, b \in V_1 \), \( \gamma \in \Gamma \) then there is \( c \in V_2 \) such that \( c \gamma a \in I \) and \( c \gamma b \in I \). So \( c \in \Gamma(a \gamma b) \in I \). If \( a \gamma b \in I \) then \( a \gamma b \in P_1 \). Otherwise \( a \gamma b \in V_1 \Rightarrow a \gamma b \in P_1 \).

**Case:** (iii) If \( a \in V_1 \) and \( b \in I \) then \( a \gamma b \in I \). So there is \( c \in V_2 \) such that \( c \in \Gamma(a \gamma b) \in I \Rightarrow a \gamma b \in V_1 \) and \( a \gamma b \in P_1 \). Now let \( r \in M \) and \( a \in P_1 \).

**Case:** (1) If \( a \in I \) then \( r \gamma a \in I \) and so \( r \gamma a \in P_1 \).

**Case:** (2) If \( a \in V_1 \), then there exists \( c \in V_2 \) such that \( c \gamma a \in I \). So \( c \in \Gamma(r \gamma a) \in I \). If \( r \gamma a \in I \) then \( r \gamma a \in P_1 \). And so \( r \gamma a \notin I \) then \( r \gamma a \in V_2 \Rightarrow r \gamma a \in P_1 \subseteq M \). We now prove \( P_1 \) is prime. For proving this let \( a \gamma b \in P_1 \) and \( a, b \in P_1 \).

Since \( P_1 = V_1 \cup I \) or \( a \gamma b \in V_1 \) and so in any case there exists \( c \in V_2 \) such that \( c \in \Gamma(r \gamma a) \in I \). Thus \( a \Gamma(c \gamma b) \in I \). If \( c \gamma b \in I \) then by the definition of \( \Gamma(I(M)) \) we have \( b \in V_1 \) which is a contradiction. Hence \( c \gamma b \notin I \) and \( c \gamma b \in V_1 \).

Therefore \( c^2 b \in I \). Since \( I = \sqrt{I} \) \( c^2 \notin I \). Hence \( c^2 \in V_2 \) so \( b \in V_1 \) which is a contradiction. Therefore \( P_1 \) is a completely prime ideal of \( M \).

**REFERENCES**


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