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I_{gg}-Closed sets

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ABSTRACT

We define $\mathbf{I}_{gg}$ -closed sets in an ideal topological space and discuss their properties.

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1. INTRODUCTION AND PRELIMINARIES

A nonempty collection **I** of subsets of a given set X is said to be an ideal on X if (i). $A \in \mathbf{I}$ and $B \subset A$ implies $B \in \mathbf{I}$ (heredity) and (ii). $A \in \mathbf{I}$ and $B \in \mathbf{I}$ implies $A \cup B \in \mathbf{I}$ (finite additivity). If (X, τ) is a topological space and **I** is an ideal on X, then $(X, \tau, \mathbf{I})$ is called an ideal topological space [4]. For each subset A of X, $A^\star(\mathbf{I}, \tau) = \{x \in X: U_x \cap A \notin \mathbf{I}$, for every open set U_x containing $x\}$, is called the local function of A [4] with respect to **I** and τ. We simply write $A^\star$ instead of $A^\star(\mathbf{I}, \tau)$ in case there is no chance for confusion. We often use the properties of the local function stated in Theorem 2.3 of [3] without mentioning it. Moreover, $cl^\star(A) = A \cup A^\star$ [6] defines a Kuratowski closure operator for a topology $\tau^\star$ on X which is finer than τ. A subset A of an ideal topological space (X, τ) is said to be g-closed [5], if $cl(A) \subset U$ whenever $A \subset U$ and U is open. The complement of a g-closed set is called a g-open set [5]. A subset A of an ideal space $(X, \tau, \mathbf{I})$ is said to be $\mathbf{I}_g$-closed [2], if $cl^\star(A) \subset U$ whenever $A \subset U$ and U is open. The complement of a $\mathbf{I}_g$-closed set is called a $\mathbf{I}_g$-open set [2]. The collection of all g -open sets in a topological space (X, τ) is denoted by τ_g. The g-closure of A denoted by $cl_g(A)$[1] is the intersection of all g-closed sets containing A and the the g-interior of A denoted by $int_g(A)$[1] is the union of all g-open sets contained in A.

For each subset A of X, $A_g^\star(\mathbf{I}, \tau) = \{x \in X: U_x \cap A \notin \mathbf{I}$, for every g-open set U_x containing $x\}$, is called the g-local function of A [1] with respect to **I** and τ_g and is denoted by $A_g^\star$. Also, $cl_g^\star(A) = A \cup A_g^\star$ [1] is a Kurotowski closure operator for a topology $\tau_g^\star = \{X - A: cl_g^\star(A) = A\}$[1] on X which is finer than τ_g.

2. $\mathbf{I}_{gg}$ -CLOSED SETS

Definition: 2.1 A subset A of an ideal topological space $(X, \tau, \mathbf{I})$ is said to be an $\mathbf{I}_{gg}$ -closed set if $A_g^\star \subset U$ whenever $A \subset U$ where U is a g-open set in X, equivalently, $cl_g^\star(A) \subset U$ whenever $A \subset U$ where U is a g-open set in X. A is said to be an $\mathbf{I}_{gg}$-open set if $X - A$ is an $\mathbf{I}_{gg}$-closed set.

Theorem: 2.2 Let $(X, \tau, \mathbf{I})$ be an ideal topological space and $A \subset X$. Then the following are equivalent.
(a) A is $\mathbf{I}_{gg}$ -closed.
(b) For all $x \in cl_g^\star(A)$, $cl_g(\{x\}) \cap A \neq \emptyset$
(c) $cl_g^\star(A) - A$ contains no nonempty g-closed set.
(d) $A_g^\star - A$ contains no nonempty g-closed set.

Proof:
(a) ⟹ (b). Suppose that $x \in cl_g^\star(A)$. If $cl_g(\{x\}) \cap A = \emptyset$, then $A \subset X - cl_g(\{x\})$. Since A is I_{gg}-closed, $cl_g^\star(A) \subset X - cl_g(\{x\})$ which is a contradiction to the fact that $x \in cl_g^\star(A)$.
This proves (b).

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(b) ⟹(c): Suppose $F \subset cl_g^\star(A) - A$, F is a g-closed set and $x \in F$. Since $F \subset X - A$ and F is g-closed, $cl_g(\{x\}) \cap A \subset cl_g(F) \cap A = F \cap A = \emptyset$. Since $x \in cl_g^\star(A)$, by (b), $cl_g(\{x\}) \cap A \neq \emptyset$, a contradiction which proves (c).

(c) ⟹(a): Let U be a g-open set containing A. Since $cl_g^\star(A)$ is closed, X-U is g-closed, $cl_g^*(A) \cap (X\text{-}U)$ is g-closed and $cl_g^\star(A) \cap (X-U) \subset cl_g^\star(A) \cap (X-A) = cl_g^\star(A) - A$. By hypothesis, $cl_g^\star(A) \cap (X-U) = \emptyset$, and so $cl_g^\star(A) \subset U$. Thus, A is $\mathbf{I}_{gg}$-closed.

The equivalence of (c) and (d) follows from the fact that $cl_g^\star(A) - A = A_g^\star - A$.

Theorem: 2.3 Let $(X, \tau, \mathbf{I})$ be a topological space and $A \subset X$. Then the following are equivalent.

(a) A is $\mathbf{I}_{gg}$-closed.

(b) $A \cup (X - cl_g^\star(A))$ is $\mathbf{I}_{gg}$-closed.

(c) $cl_g^\star(A) - A$ is $\mathbf{I}_{gg}$-open.

(d) $A_g^\star - A$ is $\mathbf{I}_{gg}$-open.

Proof:

(a) ⟹(b): Suppose that A is $\mathbf{I}_{gg}$-closed. Let $A \cup (X - cl_g^\star(A)) \subset U$ where U is g-open. Then $X - U \subset (X - A) \cap cl_g^\star(A) = cl_g^\star(A) - A$ where X–U is g-closed. By Theorem 2.2(c), $cl_g^\star(A) - A$ contains no nonempty g-closed set, $X - U = \emptyset$ implies that X=U. Since X is the only g-open set containing A, $A \cup (X - cl_g^\star(A))$ is $\mathbf{I}_{gg}$-closed.

(b) ⟹(a): Suppose that $A \cup (X - cl_g^\star(A))$ is $\mathbf{I}_{gg}$-closed. If F is any g-closed set contained in $cl_g^\star(A) - A$, then $A \cup (X - cl_g^\star(A)) \subset X\text{-}F$ where X–F is g-open. Therefore, $cl_g^\star(A) \cup cl(X - cl_g^\star(A)) \subset X - F$ and so $X \subset X - F$, it follows that $F = \emptyset$. Hence A is $\mathbf{I}_{gg}$-closed.

The equivalence of (b) and (c) follows from the fact that $X - (cl_g^\star(A) - A) = A \cup (X - cl_g^\star(A))$. The equivalence of (c) and (d) follows from the fact that $cl_g^\star(A) - A = A_g^\star - A$.

Theorem: 2.4 For a subset A of an ideal topological space $(X, \tau, \mathbf{I})$, A is $\mathbf{I}_{gg}$-open if and only if $F \subset int_g^\star(A)$ whenever $F \subset A$, where F is a g-closed set in $(X, \tau, \mathbf{I})$.

Proof: Suppose that A is $\mathbf{I}_{gg}$-open. If F is g-closed and $F \subset A$, then $X - A \subset X - F$, and so $cl_g^\star(X - A) \subset X - F$. Therefore, $F \subset int_g^\star(A)$.

Conversely, suppose the condition holds. Let U be a g-open set such that $X - A \subset U$. Then $X - U \subset A$ and so $X - U \subset int_g^\star(A)$ which implies that $cl_g^\star(X - A) \subset U$. Thus X–A is $\mathbf{I}_{gg}$-closed and so A is $\mathbf{I}_{gg}$-open.

Theorem: 2.5 Let $(X, \tau, \mathbf{I})$ be an ideal topological space. Then every g-open subset of $(X, \tau, \mathbf{I})$ is $\tau_g^\star$-closed if and only if every subset of $(X, \tau, \mathbf{I})$ is $\mathbf{I}_{gg}$-closed.

Proof: Suppose that every g-open subset of $(X, \tau, \mathbf{I})$ is $\tau_g^\star$-closed. Let $A \subset U$ where $A \subset X$ and U is g-open. Since U is g-open, $A_g^\star \subset U_g^\star \subset U$ and so A is $\mathbf{I}_{gg}$-closed. Conversely, suppose that every subset of $(X, \tau, \mathbf{I})$ is $\mathbf{I}_{gg}$-closed. Let $A \subset X$ be a g-open set. Since $A \subset A$, $A_g^\star \subset A$, A is a $\tau_g^\star$ -closed set. Hence, every g-open subset of $(X, \tau, \mathbf{I})$ is $\tau_g^\star$-closed if and only if every subset of $(X, \tau, \mathbf{I})$ is $\mathbf{I}_{gg}$-closed.

Theorem: 2.6 Let $(X, \tau, \mathbf{I})$ be an ideal topological space. If A and B are subsets of X such that $A \subset B \subset cl_g^\star(A)$ and A is an $\mathbf{I}_{gg}$-closed subset of X, then B is also an $\mathbf{I}_{gg}$-closed set.

Proof: Since $cl_g^\star(B) - B \subset cl_g^\star(A) - A$ and $cl_g^\star(A) - A$ contains no nonempty g-closed set, $cl_g^\star(B) - B$ contains no nonempty g-closed set. This implies that B is $\mathbf{I}_{gg}$-closed.

Theorem: 2.7 Let $(X, \tau, \mathbf{I})$ be an ideal topological space. If A and B are subsets of X such that $int_g^\star(A) \subset B \subset A$ and A is an $\mathbf{I}_{gg}$-open set, then B is also an $\mathbf{I}_{gg}$-open set.

Theorem: 2.8 Let $(X, \tau, \mathbf{I})$ be an ideal topological space. If A and B are subsets of X such that $A \subset B \subset A_g^\star$ and A is $\mathbf{I}_{gg}$-closed, then A and B are g-closed sets.

Proof: Since $A \subset B \subset A_g^\star$, $A_g^\star = B_g^\star = A^\star = B^\star$ by [1, Theorem 3.10]. Let $A \subset U$ and $U \in \tau$. Since A is $\mathbf{I}_{gg}$-closed, $A^\star = B^\star = B_g^\star = A_g^\star \subset U$ which implies that A and B are g-closed sets.

Theorem: 2.9 Let (X, τ, **I**) be an ideal topological space and $A \subset X$. Then A is $\mathbf{I}_{gg}$-closed if and only if $A = F–N$, where F is $\tau_g{}^\star$-closed and N contains no nonempty g-closed set.

Proof: If A is $\mathbf{I}_{gg}$-closed, then by Theorem 2.2(c), $N = A_g{}^\star – A$ contains no nonempty g-closed set. If $F = cl_g{}^\star(A)$, then F is $\tau_g{}^\star$-closed such that $F–N = (A \cup A_g{}^\star) – (A_g{}^\star – A) = (A \cup A_g{}^\star) \cap ((X– A_g{}^\star) \cup A) = A$. Conversely, suppose that $A = F–N$ where F is $\tau_g{}^\star$-closed and N contains no nonempty g-closed set. Let U be a g-open set such that $A \subset U$. Then $F–N \subset U$ which implies that $F \cap (X–U) \subset N$. Now, $A \subset F$ and $F_g{}^\star \subset F$ implies that $A_g{}^\star \cap (X–U) \subset F_g{}^\star \cap (X–U) \subset F \cap (X–U) \subset N$. By hypothesis, since $A_g{}^\star \cap (X–U)$ is g-closed, $A_g{}^\star \cap (X–U) = \emptyset$, and so $A_g{}^\star \subset U$ which implies that A is $\mathbf{I}_{gg}$-closed.

Theorem: 2.10 Let (X, τ, **I**) be an ideal topological space and $A \subset X$. Then the following hold.
(a) Every g-closed set is $\mathbf{I}_{gg}$-closed.
(b) The union of two $\mathbf{I}_{gg}$-closed sets is $\mathbf{I}_{gg}$-closed.
(c) The intersection of a closed set with an $\mathbf{I}_{gg}$-closed set is $\mathbf{I}_{gg}$-closed.
(d) The union of a g-closed set with an $\mathbf{I}_{gg}$-closed is $\mathbf{I}_{gg}$-closed.

Proof:
(a) If A is g-closed in (X, τ, **I**), then A is $\tau_g{}^\star$-closed. Therefore, $A_g{}^\star \subset A \subset U$ whenever $A \subset U$ and $U \in \tau_g$ implies that A is $\mathbf{I}_{gg}$-closed.

(b) Let A and be B be $\mathbf{I}_{gg}$-closed sets and U be a g-open set such that $A \cup B \subset U$. Since $A_g{}^\star \subset U$, $B_g{}^\star \subset U$ and by [1, Theorem 3.7(h)], $(A \cup B)_g{}^\star = A_g{}^\star \cup B_g{}^\star \subset U$ which implies that $A \cup B$ is an $\mathbf{I}_{gg}$-closed set.

(c) Let A be an $\mathbf{I}_{gg}$-closed set and B be a closed set in (X, τ, **I**). Suppose that $A \cap B \subset U$ and U is g-open in X. Then $A \subset U \cup (X- B)$. Now X-B is open and hence $U \cup (X-B)$ is g-open. Since A is $\mathbf{I}_{gg}$-closed, $cl_g{}^\star(A) \subset U \cup (X - B)$. Therefore, $cl_g{}^\star(A) \cap B \subset U$ which implies that $cl_g{}^\star(A \cap B) \subset U$. So $A \cap B$ is an $\mathbf{I}_{gg}$ -closed set.

(d) The proof follows from (a) and (b).

The following example shows that the converse of Theorem 2.10(a) is not true.

Example: 2.11 Let X = {a, b, c}, τ = {∅, X,{a},{a, b}} and **I**={∅, {a}}. If A = {a} and $A \subset U$ where U = {a} is open, then $cl(A) = X \not\subset U$ but $A_g{}^\star = \emptyset \subset U$ which implies that A is g-closed.

The following example shows that the concept of $\mathbf{I}_g$-closed sets and $\mathbf{I}_{gg}$ -closed sets are independent.

Example: 2.12 In Example 2.11, if A ={a} and U={a}∈τ, then $cl^\star(A) = X \not\subset U$ but $A_g{}^\star = \emptyset \subset U$ which implies that A is $\mathbf{I}_{gg}$-closed but not $\mathbf{I}_g$-closed.

Theorem: 2.13 Let (X, τ, **I**) be an ideal topological space and $A \subset X$. If $A \subset A_g{}^\star$ and A is $\mathbf{I}_{gg}$-closed, then A is $\mathbf{I}_g$-closed.

Proof: Let $A \subset U$ where $A \subset X$ and $U \in \tau$. Since U is g-open, $A_g{}^\star \subset U$ by hypothesis. Since $A \subset A_g{}^\star$ and by [1, Theorem 3.10], $A^\star = A_g{}^\star \subset U$ which implies that A is $\mathbf{I}_g$-closed.

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