International Journal of Mathematical Archive-5(1), 2014, 278-287
Available online through www.ijma.info ISSN 2229 – 5046

α-COMPATIBLE MAPPINGS OF TYPE (P) AND COMMON α-FIXED POINT THEOREM

Renuka Rathore*
Deptt. of Math., Medi-caps Institute of Science and Technology, Indore-(M.P.), India.

(Received on: 04-03-13; Revised & Accepted on: 20-01-14)

ABSTRACT

In this paper we introduce the concept of α-compatible mappings of type (P), which is equivalent to the notion of α-compatible mappings as well as compatible mappings of type (P) under certain conditions. We prove a common α-fixed point theorem for four α- compatible mappings of type (P).

1. INTRODUCTION

The theory of fixed points is a quite popular and attractive area of researches in Mathematics. It has equally drawn attention of people working both in Pure as well as Applied Mathematics. Fixed points have long been used in Analysis to solve various kinds of differential and integral equations. It has wider applications to the theory of positive matrices.

The theory of fixed points took its proper shape with the landmark result of polish mathematician S. Banach popularly known as Banach Contraction Principle. Till then many workers including S. Brouwer, J. Schauder, G. D. Birkoff, O. D. Kellog, M. Balanzat, Y. J. Cho, G. Jungck, B. Fisher, S. M. Kang, R. Kannan, R. P. Pant, etc. have contributed and given the present shape to the theory.

G. Jungck [1] has given a generalization of the Banach's contraction theorem by using the concept of commuting mappings. S. Sessa [9] generalized the concept of commuting mapping by using the concept of weakly commuting mappings.

Further G. Jungck [2] generalized weak commutativity by introducing the concept of compatible mappings. Jungck and others proved common fixed point theorems using this concept ([2]-[4], [7], [8]).

In [5] Jungck introduced the notion of weakly compatible maps. In [6] Jungck, introduced the concept of compatible mappings of type (A) and proved common fixed point theorems for compatible mappings of type (A) on a complete metric space.

Recently Pathak-Chang-Cho-Kang [11] introduced the concept of compatible mappings of type (P) in metric space (X, d) and compare it with the compatible and compatible mappings of type (A).

In a paper [14,15] author introduced the concept of α-fixed point, α-commuting mappings, weakly α-commuting mappings, α-compatible mappings, weakly α-compatible mappings and α-compatible mappings of type (A) and proved some common α-fixed point theorems.

In this paper, we introduce the concept of α-compatible mappings of type (P) in metric space (X, d), which is equivalent to the concept of α-compatible mappings and as well as α-compatible mappings of type (A) under certain conditions. We prove a common α-fixed point theorem for four α-compatible mappings of type (P) on a complete metric spaces.

2. α-COMPATIBLE MAPPINGS OF TYPE (P)

In this section, we introduce the concept of α-compatible mappings of type (P) in metric space (X, d) and show that the concept of α-compatible mappings, α-compatible mappings of type (A) and α-compatible mappings of type (P) are equivalent under some conditions and give some properties of α-compatible mappings of type (P) for our main result. Throughout this paper (X, d) denotes a metric space.

*Corresponding author: Renuka Rathore**
Deptt. of Math., Medi-caps Institute of Science and Technology, Indore-(M.P.), India.

We also recall the following definitions and properties of α-compatible mappings and α-compatible mappings of type (A).[14,15]

Definition: 2.1 Let α, S and T be self maps of a metric space (X, d). Then S and T are called *α-compatible* if (αoS) and (αoT) are compatible if

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oS)(x_n)) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} (\alpha oS)(x_n) = \lim_{n\to\infty} (\alpha oT)(x_n) = t \quad \text{for some } t \text{ in } X.$$

Definition: 2.2 Let α, S and T be self maps of a metric space (X, d). Then S and T are said to be *α-compatible mappings of type (A)* if (αoS) and (αoT) are compatible mappings of type (A) if

$$\lim_{n\to\infty} d((\alpha oT)(\alpha oS)(x_n), (\alpha oS)(\alpha oS)(x_n)) = 0$$
$$\text{and} \quad \lim_{n\to\infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oT)(x_n)) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} (\alpha oS)(x_n) = \lim_{n\to\infty} (\alpha oT)(x_n) = t \text{ for some } t \text{ in } X.$$

Definition: 2.3 Let α, S and T be self maps of a metric space (X, d). Then S and T are said to be *α-compatible mappings of type (P)* if (αoS) and (αoT) are compatible mappings of type (P) if

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} (\alpha oS)(x_n) = \lim_{n\to\infty} (\alpha oT)(x_n) = t \text{ for some } t \text{ in } X.$$

Definition: 2.4 Let α, S: (X, d) $\to$ (X, d) be mappings. Then (αoS) is said to be sequentially continuous at a point $t \in X$ if for every sequence $\{x_n\}$ in X, such that

$$\lim_{n\to\infty} d(x_n, t) = 0, \text{ we have } \lim_{n\to\infty} d((\alpha oS)x_n, (\alpha oS)) = 0.$$

The following propositions show that Definitions 2.1 and 2.2 are equivalent under some conditions:

Proposition: 2.1 Let α, S, T : (X, d) $\to$ (X, d) be mappings such that (αoS) and (αoT) are continuous. If S and T are α-compatible, then they are α-compatible of type (A).

Proposition: 2.2 Let α, S, T: (X, d) $\to$ (X, d) be mappings such that one of (αoS) and (αoT) is continuous. If S and T are α-compatible mappings of type (A), then they are α-compatible.

Remark: 2.1 In [15] we can find two examples that Proposition 2.1 and 2.2 are not true if S and T are not continuous on X.

We can also show that (αoS) and (αoT) are continuous, then S and T are α-compatible if and only if they are α-compatible mappings of type (P), as follows:

Proposition: 2.3 Let α, S, T: (X, d) $\to$ (X, d) be mappings such that (αoS) and (αoT) are continuous. Then S and T are α-compatible mappings if and only if they are α-compatible mappings of type (P).

Proof: Let $\{x_n\}$ be a sequence in X such that $(\alpha oS)(x_n)$, $(\alpha oT)(x_n) \to t$ for some $t \in X$. Since (αoS) and (αoT) are continuous, then we have

$$\lim_{n\to\infty}(\alpha oS)(\alpha oS)(x_n) = \lim_{n\to\infty}(\alpha oS)(\alpha oT)(x_n) = (\alpha oS)t$$
$$\lim_{n\to\infty}(\alpha oT)(\alpha oS)(x_n) = \lim_{n\to\infty}(\alpha oT)(\alpha oT)(x_n) = (\alpha oT)t$$

Suppose that S and T are α-compatible mappings. Then

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oS))(x_n)) = 0$$

By the triangle inequality of the metric d, we have

$$\begin{aligned} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) &\le d((\alpha oS)(\alpha oS)(x_n), (\alpha oS)(\alpha oT)(x_n)) + d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oT)(x_n)) \\ &\le d((\alpha oS)(\alpha oS)(x_n), (\alpha oS)(\alpha oT)(x_n)) + d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oS)(x_n)) \\ &\quad + d((\alpha oT)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) \end{aligned}$$

Taking $n \to \infty$, since S and T are α-compatible and (αoS) and (αoT) are continuous, then

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) = 0$$

Therefore, S and T are α-compatible mappings of type (P).

Conversely, Suppose that S and T are α-compatible mappings of type (P).That is,

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT))(x_n)) = 0$$

By the triangle inequality of the metric d, we have

$$\begin{aligned} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oS)(x_n)) &\le d((\alpha oS)(\alpha oT)(x_n), (\alpha oS)(\alpha oS)(x_n)) + d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oS)(x_n)) \\ &\le d((\alpha oS)(\alpha oT)(x_n), (\alpha oS)(\alpha oS)(x_n)) + d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) \\ &\quad + d((\alpha oT)(\alpha oT)(x_n), (\alpha oT)(\alpha oS)(x_n)) \end{aligned}$$

Taking $n \to \infty$, since S and T are α-compatible of type (P) and (αoS) and (αoT) are continuous, then we have

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oS)(x_n)) = 0$$

Therefore, S and T are α-compatible mappings. This completes the proof **.**

Proposition: 2.4 Let α, S, T: (X, d) $\to$ (X, d) be mappings such that one of (αoS) and (αoT) is continuous. Then S and T are α-compatible mappings of type (A) if and only if they are α-compatible mappings of type (P).

Proof: Let $\{x_n\}$ be a sequence in X such that $(\alpha oS)(x_n)$, $(\alpha oT)(x_n) \to t$ for some $t \in X$. Assume without loss of generality, that (αoT) is continuous, then we have

$$\lim_{n\to\infty}(\alpha oT)(\alpha oS)(x_n) = \lim_{n\to\infty}(\alpha oT)(\alpha oT)(x_n) = (\alpha oT)t$$

Suppose that S and T are α-compatible mappings of type (A), that is,

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oT))(x_n)) = 0$$

and

$$\lim_{n\to\infty} d((\alpha oT)(\alpha oS)(x_n), (\alpha oS)(\alpha oS))(x_n)) = 0$$

By the triangle inequality of the metric d, we have

$$d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) \le d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oS)(x_n)) + d((\alpha oT)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n))$$

Taking $n \to \infty$, since S and T are α-compatible of type (A) and (αoT) is continuous, then

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) = 0$$

Therefore, S and T are α-compatible mappings of type (P).

Conversely, Suppose that S and T are α-compatible mappings of type (P), that is,

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT))(x_n)) = 0$$

By the triangle inequality of the metric d, we have

$$d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oS)(x_n)) \le d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) + d((\alpha oT)(\alpha oT)(x_n), (\alpha oT)(\alpha oS)(x_n))$$

Taking $n \to \infty$, since S and T are α-compatible of type (P) and (αoT) is continuous, then

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oS)(x_n) = 0$$

Similarly, we have

$$\lim_{n\to\infty} d((\alpha oT)(\alpha oT)(x_n), (\alpha oS)(\alpha oT)(x_n)) = 0$$

Therefore, S and T are α-compatible mappings of type (A). This completes the proof.

We give following examples to show that the results of Proposition 2.3 and 2.4 are not true if (αoS) and (αoT) are not continuous.

Example: 2.1 Let X = R, the set of reals with usual metric d(x, y) = | x − y |. Define α, S, T: R $\to$ R such that

$$\alpha(x) = \begin{cases} x^2 & if \quad x \ne 0 \\ 1 & if \quad x = 0, \end{cases} \quad S(x) = \begin{cases} 1/x & if \quad x \ne 0 \\ 2 & if \quad x = 0, \end{cases} \text{ and } T(x) = \begin{cases} 1/x^2 & if \quad x \ne 0 \\ 2 & if \quad x = 0, \end{cases}$$

Then (αoS) and (αoT) are not continuous at $x = 0$. Consider a sequence $\{x_n\}$ in X defined by $x_n = n^3$, $n = 1, 2 \ldots$ Then we have, as $n \to \infty$

$$(\alpha oS)x_n = 1/n^6 \to 0, \quad (\alpha oT)x_n = 1/n^{12} \to 0$$

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oS)(x_n)) = \lim_{n\to\infty} d(n^{24}, n^{24}) = 0$$

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) = \lim_{n\to\infty} d(n^{12}, n^{48}) = |n^{12} - n^{48}| = \infty$$

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oT)(x_n)) = \lim_{n\to\infty} d(n^{24}, n^{48}) = |n^{24} - n^{48}| = \infty$$

$$\lim_{n\to\infty} d((\alpha oT)(\alpha oS)(x_n), (\alpha oS)(\alpha oS)(x_n)) = \lim_{n\to\infty} d(n^{24}, n^{12}) = |n^{24} - n^{12}| = \infty$$

Thus S and T are α-compatible mappings but neither α-compatible mappings of type (A) nor α-compatible mappings of type (P).

Example: 2.2 Let $X = [0, 2]$ with the usual metric $d(x, y) = |x - y|$. Define α, S, T: $X \to X$ such that

$$\alpha(x) = \begin{cases} \sqrt{x} & if \quad x \in [0,1) \\ 2\sqrt{\ } & if \quad x \in [1,2], \end{cases} \quad S(x) = \begin{cases} x^2 & if \quad x \in [0,1) \\ 2 & if \quad x \in [1,2], \end{cases} \text{ and } T(x) = \begin{cases} (2-x)^2 & if \quad x \in [0,1) \\ 2 & if \quad x \in [1,2] \end{cases}$$

Then (αoS) and (αoT) are not continuous at $x = 1$. Let $\{x_n\} \subseteq [0, 2]$ be a sequence such that $x_n \to 1$ and assume that $x_n < 1$ for all n, then

$(\alpha oT)x_n = 2 - x_n \to 1$from right hand side and $(\alpha oS)x_n = x_n \to 1$ from right hand side.

Since $2 - x_n > 1$, for all n, thus

$(\alpha oS)(\alpha oT)x_n = (\alpha oS)(2 - x_n) = 2$ and $(\alpha oT)(\alpha oS)x_n = (\alpha oT)(x_n) = 2 - x_n$

Consequently,

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oS)(x_n)) = \lim_{n\to\infty} |2 - (2 - x_n)| \to 1$$

Also,

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) = \lim_{n\to\infty} d((\alpha oS)(x_n), (\alpha oT)(2 - x_n)) = \lim_{n\to\infty} d(x_n, 2) = \lim_{n\to\infty} |x_n - 2| \to 1$$

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oT)(x_n)) = \lim_{n\to\infty} |2 - (\alpha oT)(2 - x_n)| = |2 - 2| \to 0 \text{ as } x_n \to 1 \text{ and}$$

$$\lim_{n\to\infty} d((\alpha oT)(\alpha oS)(x_n), (\alpha oS)(\alpha oS)(x_n)) = \lim_{n\to\infty} |(2 - x_n) - x_n| = \lim_{n\to\infty} |1 - 2x_n| \to 0 \text{ as } x_n \to 1$$

Thus S and T are α-compatible mappings of type (A) but they are neither α-compatible nor α-compatible mappings of type (P).

Example: 2.3 Let $X = R$, the set of reals with usual metric $d(x, y) = |x - y|$. Define α, S, T: $R \to R$ such that

$$\alpha(x) = \begin{cases} 1/x & if \quad x \neq 0 \\ 1 & if \quad x = 0, \end{cases} \quad S(x) = \begin{cases} x/2 & if \quad x \neq 0 \\ 2 & if \quad x = 0 \end{cases} \text{ and } T(x) = \begin{cases} x/3 & if \quad x \neq 0 \\ 1 & if \quad x = 0, \end{cases}$$

Then (αoS) and (αoT) are not sequentially continuous at $x = 0$. Consider a sequence $\{x_n\}$ in X defined by $x_n = n^2$, $n = 1, 2 \ldots$ Then we have, as $n \to \infty$

$$(\alpha oS)x_n = 2/n^2 \to 0, \quad (\alpha oT)x_n = 3/n^2 \to 0$$

$$\lim_{n\to\infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oS)(x_n)) = \lim_{n\to\infty} d(2n^2/3, 3n^2/2) = |2n^2/3 - 3n^2/2| = \infty$$

$\lim_{n \to \infty} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) = \lim_{n \to \infty} d(n^2, n^2) = 0$

$$\lim_{n \to \infty} d((\alpha oS)(\alpha oT)(x_n), (\alpha oT)(\alpha oT)(x_n)) = \lim_{n \to \infty} d(2n^2 / 3, n^2) = | 2n^2 / 3 - n^2 | = \infty$$

$$\lim_{n \to \infty} d((\alpha oT)(\alpha oS)(x_n), (\alpha oS)(\alpha oS)(x_n)) = \lim_{n \to \infty} d(3n^2 / 2, n^2) = | 3n^2 / 2 - n^2 | = \infty$$

Thus S and T are α-compatible mappings of type (P) but neither α-compatible of type (A) nor α-compatible.

Next we give several properties of α-compatible mappings of type (P) for our main theorems.

Proposition: 2.5 Let α, S, T: $(X, d) \to (X, d)$ be mappings. If S and T are α-compatible mappings of type (P) and $(\alpha oS)(t) = (\alpha oT)(t)$ for some $t \in X$, then

$$(\alpha oS)(\alpha oT)(t) = (\alpha oS)(\alpha oS)(t) = (\alpha oT)(\alpha oT)(t) = (\alpha oT)(\alpha oS)(t).$$

Proof: Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = t$, for $n = 1, 2, 3, \ldots$, and $(\alpha oS)(t) = (\alpha oT)(t)$. Then we have,

$$(\alpha oS)(x_n), (\alpha oT)(x_n) \to (\alpha oS)(t) \text{ as } n \to \infty.$$

Since S and T are α-compatible of type (P), then we have

$d((\alpha oS)(\alpha oS)(t), (\alpha oT)(\alpha oT)(t)) = \lim_{n\to\infty} d((\alpha oS)(\alpha oS)(x_n), (\alpha oT)(\alpha oT)(x_n)) = 0$
and so $(\alpha oS)(\alpha oS)(t) = (\alpha oT)(\alpha oT)(t)$.

Since $(\alpha oT)(t) = (\alpha oS)(t)$

therefore $(\alpha oT)(\alpha oT)(t) = (\alpha oT)(\alpha oS)(t)$

and $(\alpha oS)(\alpha oT)(t) = (\alpha oS)(\alpha oS)(t)$

So, $(\alpha oS)(\alpha oT)(t) = (\alpha oS)(\alpha oS)(t) = (\alpha oT)(\alpha oT)(t) = (\alpha oT)(\alpha oS)(t)$.

Proposition: 2.6 Let S and T be α-compatible mappings of type (P) from a metric space (X, d) into itself. Suppose $\lim_{n\to\infty} (\alpha oS)x_n = \lim_{n\to\infty} (\alpha oT)x_n = t$ for some t in X. Then

(1) $\lim_{n\to\infty}(\alpha oT)(\alpha oT)x_n = (\alpha oS)t$ if (αoS) is sequentially continuous.
(2) $\lim_{n\to\infty}(\alpha oS)(\alpha oS)x_n = (\alpha oT)t$ if (αoT) is sequentially continuous.
(3) $(\alpha oS)(\alpha oT)(t) = (\alpha oS)(\alpha oS)(t)$ and $(\alpha oS)t = (\alpha oT)t$ if (αoS) and (αoT) are sequentially continuous at t.

Proof: Suppose that $\lim_{n\to\infty}(\alpha oS)x_n = \lim_{n\to\infty}(\alpha oT)x_n = t$ for some $t \in X$.

(1) Since, (αoS) is sequentially continuous, then we have

$$\lim_{n\to\infty}(\alpha oS)(\alpha oS)x_n = (\alpha oS)t.$$

By triangle inequality, we have

$$d((\alpha oT)(\alpha oT)x_n, (\alpha oS)t) \le d((\alpha oT)(\alpha oT)x_n, (\alpha oS)(\alpha oS)x_n) + d((\alpha oS)(\alpha oS)x_n, (\alpha oS)t)$$

Letting $n\to\infty$, since S and T are α-compatible mappings of type (P), then we have

$$\lim_{n\to\infty} d((\alpha oT)(\alpha oT)x_n, (\alpha oS)t) = 0$$

so $\lim_{n\to\infty} (\alpha oT)(\alpha oT)x_n = (\alpha oS)t$.

(2) The proof of $\lim_{n\to\infty} (\alpha oS)(\alpha oS)x_n = (\alpha oT)t$ follows on the similar lines as argued in(1).

(3) Since, (αoT) is sequentially continuous at t, we have

$$\lim_{n\to\infty} (\alpha oT)(\alpha oT)x_n = (\alpha oT)t$$

Since, (αoS) is sequentially continuous at t, by (1) also we have

$$\lim_{n\to\infty} (\alpha oT)(\alpha oT)x_n = (\alpha oS)t$$

Hence, by the uniqueness of the limit, we have $(\alpha oS)t = (\alpha oT)t$

By Proposition 2.5, we have

$$(\alpha o S)(\alpha o T)t = (\alpha o T)(\alpha o S)t.$$

This completes the proof.

Throughout this section, suppose that a function $\phi : [0,\infty)^{14} \to [0,\infty)$ satisfies the followings:

(i) ϕ is a upper semi-continuous and non-decreasing in each co-ordinate variable.

(ii) $\phi(t) = \max\{\phi(t, 0, 2t, 0, t, 0, 2t, t, 0, t, 2t, t, t, t), \phi(t, 0, 0, 2t, t, 2t, 0, t, 2t, t, 0, t, t, t), \phi(0, t, 0, 0, t, 0, 0, 0, t, 0, t, t, 0, 0)\} < t$, for some $t > 0$.

Lemma: 2.1 Let α, A, B, S and T be self maps of complete metric space (X, d) into itself such that

(2.1) $(\alpha oA)(X) \subseteq (\alpha oT)(X)$ and $(\alpha oB)(X) \subseteq (\alpha oS)(X)$

(2.2)
$$\begin{aligned}[d((\alpha oA)x, (\alpha oB)y)]^2 \le \phi(&d((\alpha oS)x, (\alpha oA)x)\, d((\alpha oT)y, (\alpha oB)y),\\
&d((\alpha oS)x, (\alpha oB)y)\, d((\alpha oT)y, (\alpha oA)x),\\
&d((\alpha oS)x, (\alpha oA)x)\, d((\alpha oS)x, (\alpha oB)y),\\
&d((\alpha oT)y, (\alpha oA)x)\, d((\alpha oT)y, (\alpha oB)y),\\
&[d((\alpha oS)x, (\alpha oT)y)]^2,\\
&d((\alpha oS)x, (\alpha oA)x)\, d((\alpha oT)y, (\alpha oA)x),\\
&d((\alpha oT)y, (\alpha oB)y)\, d((\alpha oS)x, (\alpha oB)y),\\
&d((\alpha oS)x, (\alpha oT)y)\, d((\alpha oS)x, (\alpha oA)x),\\
&d((\alpha oS)x, (\alpha oT)y)\, d((\alpha oT)y, (\alpha oA)x),\\
&d((\alpha oS)x, (\alpha oT)y)\, d((\alpha oT)y, (\alpha oB)y),\\
&d((\alpha oS)x, (\alpha oT)y)\, d((\alpha oS)x, (\alpha oB)y),\\
&d((\alpha oS)x, (\alpha oT)y)\, d((\alpha oA)x, (\alpha oB)y),\\
&d((\alpha oS)x, (\alpha oA)x)\, d((\alpha oA)x, (\alpha oB)y),\\
&d((\alpha oT)y, (\alpha oB)y)\, d((\alpha oA)x, (\alpha oB)y))\end{aligned}$$

for all x, y $\in$ X, where ϕ satisfies (i) and (ii), then there is a Cauchy sequence $\{y_n\}$ in X, defined by,

$$y_{2n-1} = (\alpha oT)x_{2n-1} = (\alpha oA)x_{2n-2} \text{ and } y_{2n} = (\alpha oS)x_{2n} = (\alpha oB)x_{2n-1} \text{ for } n = 1, 2, 3, \ldots$$

Proof: Let $x_0 \in X$ be arbitrary since $(\alpha oA)(X) \subseteq (\alpha oT)(X)$ and $(\alpha oB)(X) \subseteq (\alpha oS)(X)$, we can choose x_1, x_2 in X, such that

$$y_1 = (\alpha oT)x_1 = (\alpha oA)x_0 \text{ and } y_2 = (\alpha oS)x_2 = (\alpha oB)x_1$$

In general we can choose x_{2n-1} and x_{2n} in X, such that

$$y_{2n-1} = (\alpha oT)x_{2n-1} = (\alpha oA)x_{2n-2} \text{ and } y_{2n} = (\alpha oS)x_{2n} = (\alpha oB)x_{2n-1} \quad (2.3)$$

Thus the indicated sequence $\{y_n\}$ exists. To show that $\{y_n\}$ is Cauchy, by (2.2) and (2.3) imply that

$$\begin{aligned}[d(y_{2n+1}, y_{2n+2})]^2 &= [d((\alpha oA)x_{2n}, (\alpha oB)x_{2n+1})]^2\\
&\le \phi(d((\alpha oS)x_{2n}, (\alpha oA)x_{2n})\, d((\alpha oT)x_{2n+1}, (\alpha oB)x_{2n+1}), d((\alpha oS)x_{2n}, (\alpha oB)x_{2n+1})\, d((\alpha oT)x_{2n+1}, (\alpha oA)x_{2n}),\\
&\quad d((\alpha oS)x_{2n}, (\alpha oA)x_{2n})\, d((\alpha oS)x_{2n}, (\alpha oB)x_{2n+1}), d((\alpha oT)x_{2n+1}, (\alpha oA)x_{2n})\, d((\alpha oT)x_{2n+1}, (\alpha oB)x_{2n+1}),\\
&\quad [d((\alpha oS)x_{2n}, (\alpha oT)x_{2n+1})]^2, d((\alpha oS)x_{2n}, (\alpha oA)x_{2n})\, d((\alpha oT)x_{2n+1}, (\alpha oA)x_{2n}),\\
&\quad d((\alpha oT)x_{2n+1}, (\alpha oB)x_{2n+1})\, d((\alpha oS)x_{2n}, (\alpha oB)x_{2n+1}), d((\alpha oS)x_{2n}, (\alpha oT)x_{2n+1})\, d((\alpha oS)x_{2n}, (\alpha oA)x_{2n}),\\
&\quad d((\alpha oS)x_{2n}, (\alpha oT)x_{2n+1})\, d((\alpha oT)x_{2n+1}, (\alpha oA)x_{2n}), d((\alpha oS)x_{2n}, (\alpha oT)x_{2n+1})\, d((\alpha oT)x_{2n+1}, (\alpha oB)x_{2n+1}),\\
&\quad d((\alpha oS)x_{2n}, (\alpha oT)x_{2n+1})\, d((\alpha oS)x_{2n}, (\alpha oB)x_{2n+1}), d((\alpha oS)x_{2n}, (\alpha oT)x_{2n+1})\, d((\alpha oA)x_{2n}, (\alpha oB)x_{2n+1}),\\
&\quad d((\alpha oS)x_{2n}, (\alpha oA)x_{2n})\, d((\alpha oA)x_{2n}, (\alpha oB)x_{2n+1}), d((\alpha oT)x_{2n+1}, (\alpha oB)x_{2n+1})\, d((\alpha oA)x_{2n}, (\alpha oB)x_{2n+1})\end{aligned}$$

or

$$\begin{aligned}[d(y_{2n+1}, y_{2n+2})]^2 \le \phi(&d(y_{2n}, y_{2n+1})\, d(y_{2n+1}, y_{2n+2}), 0, d(y_{2n}, y_{2n+1})\, d(y_{2n}, y_{2n+2}),\\
&0, [d(y_{2n}, y_{2n+1})]^2, 0, d(y_{2n+1}, y_{2n+2})\, d(y_{2n}, y_{2n+2}),\\
&d(y_{2n}, y_{2n+1})\, d(y_{2n}, y_{2n+1}), 0, d(y_{2n}, y_{2n+1})\, d(y_{2n+1}, y_{2n+2}),\\
&d(y_{2n}, y_{2n+1})\, d(y_{2n}, y_{2n+2}), d(y_{2n}, y_{2n+1})\, d(y_{2n+1}, y_{2n+2}),\\
&d(y_{2n}, y_{2n+1})\, d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2})\, d(y_{2n+1}, y_{2n+2}))\end{aligned}$$

Let $d(y_n, y_{n+1}) = d_n$ then we get

$$\begin{aligned}d^2_{2n+1} \le \phi(&d_{2n}\, d_{2n+1}, 0, d_{2n}(d_{2n} + d_{2n+1}), 0, d^2_{2n}, 0, d_{2n+1}(d_{2n} + d_{2n+1}), d^2_{2n},\\
&0, d_{2n}\, d_{2n+1}, d_{2n}(d_{2n} + d_{2n+1}), d_{2n}\, d_{2n+1}, d_{2n}\, d_{2n+1}, d^2_{2n+1}) \quad (2.4)\end{aligned}$$

We want to show that $d_{2n+1} \le d_{2n}$. For this assume that for some n, $d_{2n+1} > d_{2n}$. Now by (2.4), we have

$$d^2_{2n+1} \le \phi(d^2_{2n+1}, 0, 2d^2_{2n+1}, 0, d^2_{2n+1}, 0, 2d^2_{2n+1}, d^2_{2n+1}, 0, d^2_{2n+1}, 2d^2_{2n+1}, d^2_{2n+1}, d^2_{2n+1}, d^2_{2n+1}) < d^2_{2n+1}$$

Which is a contradiction. Therefore $d_{2n+1} \le d_{2n}$, similarly again using (2.2) we obtain

$$d^2_{2n} \le \phi(d_{2n}\, d_{2n-1}, 0, 0, d_{2n-1}(d_{2n-1} + d_{2n}), d^2_{2n-1}, d_{2n}(d_{2n-1} + d_{2n}), 0, d_{2n-1}d_{2n}, d_{2n-1}(d_{2n-1} + d_{2n}), d^2_{2n-1}, 0, d_{2n-1}\, d_{2n}, d^2_{2n}, d_{2n-1}\, d_{2n}) \tag{2.5}$$

If possible let $d_{2n} > d_{2n-1}$ for some n, then (2.5) gives

$$d^2_{2n} \le \phi(d^2_{2n}, 0, 0, 2d^2_{2n}, d^2_{2n}, 2d^2_{2n}, 0, d^2_{2n}, 2d^2_{2n}, d^2_{2n}, 0, d^2_{2n}, d^2_{2n}, d^2_{2n}) < d^2_{2n}$$

Which is a contradiction. Therefore $d_{2n} \le d_{2n-1}$. Thus $\{d_n\}$ is non-increasing sequence in R^+ let $\{d_n\} \to r \in R^+$. If $r \ne 0$, then since ϕ is upper semi-continuous, we have from (2.4) and (2.5) as $n \to \infty$

$$r^2 \le \phi(r^2, 0, 2r^2, 0, r^2, 0, 2r^2, r^2, 0, r^2, 2r^2, r^2, r^2, r^2) < r^2$$

and $$r^2 \le \phi(r^2, 0, 0, 2r^2, r^2, 2r^2, 0, r^2, r^2, 2r^2, 0, r^2, r^2, r^2) < r^2$$

Which is a contradiction, therefore $d_n \to 0$. Now we shall show that sequence $\{y_n\}$ is Cauchy. If not so then there exist an $\varepsilon > 0$ and sequence of positive integer $\{m(k)\}$ and $\{n(k)\}$ with $k \le n(k) < m(k)$ such that

$$c_k = d(y_{m(k)}, y_{n(k)}) \ge \varepsilon; \quad k = 1, 2, \ldots \tag{2.6}$$

Let m(k) be the least integer exceeding n(k) for which (2.6) is true then by the well ordering principle,

$d(y_{m(k)-1}, y_{n(k)}) > \varepsilon$. Now

$$\varepsilon \le c_k \le d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)}) < d_{m(k)-1} + \varepsilon \to \varepsilon \text{ as } k \to \infty$$

and thus $c_k \to \varepsilon$. Further c_k can have different values under the following four cases, viz.,

(a) m is even and n is odd; (b) m and n are odd;
(c) m is odd and n is even; (d) m and n are even.

Now, in case (a), we have

$$c_k = d(y_{2m}, y_{2n-1}) \le d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2n}) + d(y_{2n}, y_{2n-1})$$

letting $n \to \infty$ we get

$$\varepsilon \le 0 + 0 + \lim_{n\to\infty} d(y_{2m+1}, y_{2n}) \tag{2.7}$$

Now using (2.2) and (2.3), we obtain

$$[d(y_{2m+1}, y_{2n})]^2 = [d((\alpha o A)x_{2m}, (\alpha o B)x_{2n-1})]^2$$

$$\le \phi (d_{2m}\, d_{2n-1}, (c_k + d_{2n-1})(c_k + d_{2m}), d_{2m}(c_k + d_{2n-1}), d_{2n-1} (c_k + d_{2m}), c_k^2, d_{2m} (c_k + d_{2m}), d_{2n-1}(c_k + d_{2n-1}), c_k\, d_{2m}, c_k (c_k + d_{2m}), c_k\, d_{2n-1}, c_k (c_k + d_{2n-1}), c_k (d_{2m} + c_k + d_{2n-1})\, d_{2m} (c_k + d_{2n-1}), d_{2n-1}(d_{2m} + c_k + d_{2n-1})).$$

Since ϕ is upper semi-continuous so letting $n \to \infty$ and using (ii), we get

$$\lim_{n\to\infty}[d(y_{2m+1}, y_{2n})]^2 \le \phi(0, \varepsilon^2, 0, 0, \varepsilon^2, 0, 0, 0, \varepsilon^2, 0, \varepsilon^2, \varepsilon^2, 0, 0) < \varepsilon^2$$

So we get a contradiction and hence $\varepsilon = 0$. Similarly, other cases also give $\varepsilon = 0$. Therefore, the sequence $\{y_n\}$ is Cauchy sequence.

Now we prove the our main result.

Theorem: 2.1 Let α, A, B, S, T be self maps of complete metric space (X, d) satisfying (2.1), (2.2) and (2.8). The pairs A, S and B, T are α-compatible of type (P).

One of (αoA), (αoB), (αoS) and (αoT) is sequentially continuous at their coincidence point.

Then A, B, S and T have a unique common α-fixed point in X.

Proof: By Lemma 2.1, the sequence $\{y_n\}$ as defined by (2.2) is Cauchy, so it converges to a point z in X. Consequently the subsequence $\{(\alpha oA)x_{2n}\}$, $\{(\alpha oS)x_{2n}\}$, $\{(\alpha oB)x_{2n-1}\}$ and $\{(\alpha oT)x_{2n-1}\}$ converges to z.

Now suppose (αoA) is sequentially continuous, since A and S are α-compatible of type (P), then by Proposition 2.6 we have

$(\alpha oA)(\alpha oS)x_{2n} \to (\alpha oA)z$ and $(\alpha oS)(\alpha oS)x_{2n} \to (\alpha oA)z$ as $n \to \infty$. By (2.2),

$$\begin{aligned}
[d((\alpha oA)(\alpha oS)x_{2n}, (\alpha oB)x_{2n-1})]^2 \le \phi\, (&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oA)(\alpha oS)x_{2n})\, d((\alpha oT)x_{2n-1}, (\alpha oB)x_{2n-1}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oB)x_{2n-1})\, d((\alpha oT)x_{2n-1}, (\alpha oA)(\alpha oS)x_{2n}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oA)(\alpha oS)x_{2n})\, d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oB)x_{2n-1}),\\
&d((\alpha oT)x_{2n-1}, (\alpha oA)(\alpha oS)x_{2n})\, d((\alpha oT)x_{2n-1}, (\alpha oB)x_{2n-1}),\\
&[d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)x_{2n-1})]^2,\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oA)(\alpha oS)x_{2n})\, d((\alpha oT)x_{2n-1}, (\alpha oA)(\alpha oS)x_{2n}),\\
&d((\alpha oT)x_{2n-1}, (\alpha oB)x_{2n-1})\, d((\alpha oS)(\alpha oS)x_{2n},((\alpha oB)x_{2n-1}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)x_{2n-1})\, d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oA)(\alpha oS)x_{2n}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)x_{2n-1})\, d((\alpha oT)x_{2n-1}, (\alpha oA)(\alpha oS)x_{2n}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)x_{2n-1})\, d((\alpha oT)x_{2n-1}, (\alpha oB)x_{2n-1}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)x_{2n-1})\, d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oB)x_{2n-1}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)x_{2n-1})\, d((\alpha oA)(\alpha oS)x_{2n}, (\alpha oB)x_{2n-1}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oA)(\alpha oS)x_{2n})\, d((\alpha oA)(\alpha oS)x_{2n}, (\alpha oB)x_{2n-1}),\\
&d((\alpha oT)x_{2n-1}, (\alpha oB)x_{2n-1})\, d((\alpha oA)(\alpha oS)x_{2n}, (\alpha oB)x_{2n-1}))
\end{aligned}$$

Since ϕ is upper semi-continuous, so letting $n \to \infty$, we obtain

$$\begin{aligned}
d((\alpha oA)z , z)]^2 \le \phi\, (&0, d((\alpha oA)z, z)]^2, 0, 0, d((\alpha oA)z, z)]^2, 0, 0, 0, d((\alpha oA)z, z)]^2, 0,\\
&d((\alpha oA)z, z)]^2, d((\alpha oA)z, z)]^2, 0, 0)
\end{aligned}$$

So that $(\alpha oA)z = z$, since $(\alpha oA)(X) \subseteq (\alpha oT)(X)$, there exists a point v in X such that $z = (\alpha oA)z = (\alpha oT)v$. Again using (2.2), we get

$$\begin{aligned}
[d((\alpha oA)(\alpha oS)x_{2n}, (\alpha oB)v)]^2 \le \phi\, (&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oA)(\alpha oS)x_{2n})\, d((\alpha oT)v, (\alpha oB)v),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oB)v)\, d((\alpha oT)v, (\alpha oA)(\alpha oS)x_{2n}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oA)(\alpha oS)x_{2n})\, d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oB)v),\\
&d(\alpha oT)v, (\alpha oA)(\alpha oS)x_{2n})\, d((\alpha oT)v, (\alpha oB)v), [d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)v)]^2,\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oA)(\alpha oS)x_{2n})\, d((\alpha oT)v, (\alpha oA)(\alpha oS)x_{2n}),\\
&d((\alpha oT)v, (\alpha oB)v)\, d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oB)v),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)v)\, d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oA)(\alpha oS)x_{2n}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)v)\, d((\alpha oT)v, (\alpha oA)(\alpha oS)x_{2n}),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)v)d((\alpha oT)v, (\alpha oB)v),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)v)d((\alpha oT)v, (\alpha oB)v),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)v)\, d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oB)v),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oT)v)\, d((\alpha oA)(\alpha oS)x_{2n}, (\alpha oB)v),\\
&d((\alpha oS)(\alpha oS)x_{2n}, (\alpha oA)(\alpha oS)x_{2n})\, d((\alpha oA)(\alpha oS)x_{2n}, (\alpha oB)v),\\
&d((\alpha oT)v, (\alpha oB)v)\, d((\alpha oA)(\alpha oS)x_{2n}, (\alpha oB)v))
\end{aligned}$$

letting $n \to \infty$, we have

$$\begin{aligned}
[d(z, (\alpha oB)v)]^2 &\le \phi\, (0, 0, 0, 0, 0, 0,[d(z, (\alpha oB)v)]^2, 0, 0, 0, 0, 0, 0, [d(z, (\alpha oB)v)]^2)\\
&< [d(z, (\alpha oB)v)]^2
\end{aligned}$$

Which is a contradiction, therefore $z = (\alpha oB)v$.

Now B and T are α-compatible mappings of type (P) and $(\alpha oT)v = (\alpha oB)v = z$, therefore by Proposition 2.5, we have

$$(\alpha oT)(\alpha oB)v = (\alpha oB)(\alpha oT)v$$

Hence $(\alpha oT)z = (\alpha oB)z$.

Now using (2.2) again, we get

$$\begin{aligned}[d((\alpha oA)x_{2n}, (\alpha oB)z)]^2 \leq\ & \phi(d((\alpha oS)x_{2n}, (\alpha oA)x_{2n})\, d((\alpha oT)z, (\alpha oB)z), d((\alpha oS)x_{2n}, (\alpha oB)z)\, d((\alpha oT)z, (\alpha oA)x_{2n}),\\ & d((\alpha oS)x_{2n}, (\alpha oA)x_{2n})\, d((\alpha oS)x_{2n}, (\alpha oB)z), d((\alpha oT)z, (\alpha oA)x_{2n})\, d((\alpha oT)z, (\alpha oB)z),\\ & [d((\alpha oS)x_{2n}, (\alpha oT)z)]^2, d((\alpha oS)x_{2n}, (\alpha oA)x_{2n})\, d((\alpha oT)z, (\alpha oA)x_{2n}),\\ & d((\alpha oT)z, (\alpha oB)z)\, d((\alpha oS)x_{2n}, (\alpha oB)z), d((\alpha oS)x_{2n}, (\alpha oT)z)\, d((\alpha oS)x_{2n}, (\alpha oA)x_{2n}),\\ & d((\alpha oS)x_{2n}, (\alpha oT)z)\, d((\alpha oT)z, (\alpha oA)x_{2n}), d((\alpha oS)x_{2n}, (\alpha oT)z)\, d((\alpha oT)z, (\alpha oB)z),\\ & d((\alpha oS)x_{2n}, (\alpha oT)z)\, d((\alpha oS)x_{2n}, (\alpha oB)z), d((\alpha oS)x_{2n}, (\alpha oT)z)\, d((\alpha oA)x_{2n}, (\alpha oB)z),\\ & d((\alpha oS)x_{2n}, (\alpha oA)x_{2n})\, d((\alpha oA)x_{2n}, (\alpha oB)z), d((\alpha oT)z, (\alpha oB)z)\, d((\alpha oA)x_{2n}, (\alpha oB)z))\end{aligned}$$

letting $n \to \infty$, above inequality reduces to

$$\begin{aligned}[d(z, (\alpha oB)z)]^2 &\leq \phi(\,0, [d(z, (\alpha oB)z)]^2, 0, 0, [d(z, (\alpha oB)z)]^2, 0, 0, 0, [d(z, (\alpha oB)z)]^2, 0,\\ &\qquad [d(z, (\alpha oB)z)]^2, [d(z, (\alpha oB)z)]^2, 0, 0)\\ &< [d(z, (\alpha oB)z)]^2\end{aligned}$$

so that $z = (\alpha oB)z = (\alpha oT)v = (\alpha oA)z$.

Since $(\alpha oB)(X) \subseteq (\alpha oS)(X)$ there exists a point w in X such that $z = (\alpha oB)z = (\alpha oS)w$.

Again condition (2.2) imply that

$$\begin{aligned}[d((\alpha oA)w, z)]^2 &= [d((\alpha oA)w, (\alpha oB)z)]^2\\ &\leq \phi(d((\alpha oS)w, (\alpha oA)w)\, d((\alpha oT)z, (\alpha oB)z), d((\alpha oS)w, (\alpha oB)z)\, d((\alpha oT)z, (\alpha oA)w),\\ &\quad d((\alpha oS)w, (\alpha oA)w)\, d((\alpha oS)w, (\alpha oB)z), d((\alpha oT)z, (\alpha oA)w)\, d((\alpha oT)z, (\alpha oB)z),\\ &\quad [d((\alpha oS)w, (\alpha oT)z)]^2, d((\alpha oS)w, (\alpha oA)w)\, d((\alpha oT)z, (\alpha oA)w),\\ &\quad d((\alpha oT)z, (\alpha oB)z)\, d((\alpha oS)w, (\alpha oB)z), d((\alpha oS)w, (\alpha oT)z)\, d((\alpha oS)w, (\alpha oA)w),\\ &\quad d((\alpha oS)w, (\alpha oT)z)\, d((\alpha oT)z, (\alpha oA)w), d((\alpha oS)w, (\alpha oT)z)\, d((\alpha oT)z, (\alpha oB)z),\\ &\quad d((\alpha oS)w, (\alpha oT)z)\, d((\alpha oS)w, (\alpha oB)z), d((\alpha oS)w, (\alpha oT)z)\, d((\alpha oA)w, (\alpha oB)z)\\ &\quad d((\alpha oS)w, (\alpha oA)w)\, d((\alpha oA)w, (\alpha oB)z), d((\alpha oT)z, (\alpha oB)z)\, d((\alpha oA)w, (\alpha oB)z))\end{aligned}$$

or

$$\begin{aligned}[d((\alpha oA)w, z)]^2 &\leq \phi(\,0, 0, 0, 0, 0, [d((\alpha oA)w, z)]^2, 0, 0, 0, 0, 0, 0, [d((\alpha oA)w, z)]^2, 0)\\ &< [d((\alpha oA)w, z)]^2\end{aligned}$$

so that $(\alpha oA)w = z$, since A and S are α-compatible mappings of type (P) and $(\alpha oA)w = (\alpha oS)w = z$ we obtain using Proposition 2.5

$(\alpha oS)(\alpha oA)w = (\alpha oA)(\alpha oS)w$ and hence $(\alpha oS)z = (\alpha oA)z = z$.

Therefore z is a common α-fixed point of A, B, S and T.

Similarly, we can complete the result by taking (αoB) or (αoS) or (αoT) as sequentially continuous .Uniqueness follows easily from (2.2).

Now we present the following example to prove the validity of Theorem 2.1.

Example: 2.4 Let X = [0, 1] with usual metric in real line. Define A, B, S, T and α by $A(x) = x/8$, $B(x) = x/6$, $S(x) = x/2$, $T(x) = 2x/3$ and $\alpha(x) = x/3$ for all $x \in [0,1]$. Then clearly the function (αoA), (αoB), (αoS) and (αoT) are sequentially continuous and satisfy

$$(\alpha oA)(x) = [0, 1/24] \subseteq [0, 2/9] = (\alpha oT)(x)$$

$$(\alpha oB)(x) = [0, 1/18] \subseteq [0, 1/6] = (\alpha oS)(x)$$

Moreover

$$|(\alpha oA)(x) - (\alpha oS)(x)| = |x/24 - x/6| = x/8 \to 0 \qquad \text{if and only if } x \to 0$$

$$|(\alpha oA)(\alpha oA)(x) - (\alpha oS)(\alpha oS)(x)| = |x/576 - x/36| = 5x/192 \to 0 \qquad \text{if and only if } x \to 0$$

Therefore A and S are α-compatible mappings of type (P). Similarly,

$$|(\alpha oB)(x) - (\alpha oT)(x)| = |x/18 - 2x/9| = x/6 \to 0 \qquad \text{if and only if } x \to 0$$

$$|(\alpha oB)(\alpha oB)(x) - (\alpha oT)(\alpha oT)(x)| = |x/324 - 4x/81| = 5x/108 \to 0 \quad \text{if and only if } x \to 0$$

Thus B and T are also α-compatible mappings of type (P). Let us define the function ϕ as $\phi(t_1, t_2, \ldots, t_{14}) = h \max\{t_1, t_2, \ldots, t_{14}\}$ for all $t_1 \in R^+$; $i = 1, 2, \ldots, 14$; $1/16 \leq h < 1/2$, then ϕ satisfies condition (i) and (ii).

Also we obtain

$$|(\alpha oA)(x) - (\alpha oB)(y)| = |x/24 - y/18| = (1/72)\,|3x - 4y|$$

$$|(\alpha oS)(x) - (\alpha oT)(y)| = |x/6 - 2y/9| = (1/18)\,|3x - 4y|$$

$$|(\alpha oA)(x) - (\alpha oB)(y)|^2 = (1/16)\,|(\alpha oS)(x) - (\alpha oT)(y)|^2$$
$$\leq \phi(d((\alpha oS)(x), (\alpha oA)(x))d((\alpha oT)(y), (\alpha oB)(y)), \ldots, [d((\alpha oS)(x), (\alpha oT)(y))]^2, \ldots)$$

So that condition (2) is satisfied and thus the hypothesis of Theorem 2.1 is satisfied and clearly x = 0 is the unique common α-fixed point of A, B, S and T.

REFERENCES

1. G. Jungck; Commuting maps and fixed points, *Amer. Math. Monthly*, 83 (1979), 261-263.

2. G. Jungck ; Compatible mappings and common fixed points, *Internat. J. Math. And Math. Sci.*, 9 (1986), 771-779.

3. G. Jungck; Compatible mappings and common fixed points (2), *Internat. J. Math. and Math. Sci.*, 11 (1988), 285-288.

4. G. Jungck; Common fixed points of commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.*, 103 (1988), 977-983.

5. G. Jungck; Common fixed points for non continuous nonself maps on non metric spaces, *Far East J. Math. Sci.*, 4 (1996), 199-215.

6. G. Jungck, P. P. Murthy and Y. J. Cho; Compatible mapping of type (A) and common fixed points, *Math. Japonica*, 38 (2) (1993), 381-390.

7. H. Kaneko and S. Sessa; Fixed point theorems for compatible multi-valued and single-valued mappings, *Internat. J. Math. and Math. Sci.*, 12 (1989), 257-262.

8. S. M. Kang, Y. J. Cho and G. Jungck; Common fixed points of compatible mapping, *Internat. J. Math. and Math. Sci.*, 13 (1990), 61-66.

9. Ramesh Kumar and Renu Chugh; Compatible mappings of type (P) and common fixed point theorem, Vikram Mathematical Journal, 15 (1995), 1-16.

10. H. K. Pathak, S. S. Chang and Y. J. Cho; Fixed point theorems for compatible mapping of type (P), Indian J. Math. , 36(2) (1994), 151-166.

11. H. K. Pathak, Y. J. Cho, S. S. Chang and S. M. Kang; Compatible mappings of type (P) and fixed point theorems in metric spaces and probabilistics metric spaces, Novi Sad J. Math., 26(2) (1996), 87-109.

12. S. Sessa; On weak a commutativity conditions of mappings in fixed point considerations, *Publ. Inst. Math. (Beograd)*, 32 (1982), 149-153.

13. S. P. Singh and B. A. Meade; On common fixed point theorems, *Bull. Austral. Math.*, 16 (1977), 49-53.

14. Anjali Srivastava and Renuka Rathore; α-Reciprocal continuity and α-fixed point, Bull. Cal. Math. Soc. Vol. (2006)

15. Anjali Srivastava and Renuka Rathore;α-Compatible mappings of type (A) and common α-fixed point. (Communicated).

Source of support: Nil, Conflict of interest: None Declared