ON \(g^*-g\)-CLOSED SETS

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ABSTRACT

In this paper, we introduced and study the notions of \(g^*-g\)-closed sets and study some of their properties.

1. INTRODUCTION

In 1970, Levine [8] first introduced the concept of generalized closed (briefly, g-closed) sets were defined and investigated. The idea of grill on a topological space was first introduced by Choquet [4] in 1947. It is observed from literature that the concept of grills is a powerful supporting tool, like nets and filters, in dealing with many topological concept quite effectively. In [15], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. The aim of this paper is to introduce \(g^*-g\)-closed sets and investigate the relations of \(g^*-g\)-closed sets between such sets.

2. PRELIMINARIES

Throughout this paper, \((X, \tau)\) (or \(X\)) represent a topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset \(A\) of a space \(X\), \(cl(A)\) and \(int(A)\) denote the closure of \(A\) and the interior of \(A\), respectively. The power set of \(X\) will be denoted by \(\mathcal{P}(X)\). A collection \(\mathcal{G}\) of a nonempty subsets of a space \(X\) is called a grill [1] on \(X\) if

1. \(A \in \mathcal{G}\) and \(A \subseteq B \Rightarrow B \in \mathcal{G}\).
2. \(A, B \subseteq X\) and \(A \cup B \in \mathcal{G}\) \(\Rightarrow A \in \mathcal{G}\) or \(B \in \mathcal{G}\).

For any point \(x\) of a topological space \((X, \tau)\), \(\tau(x)\) denote the collection of all open neighbourhoods of \(x\). We recall the following results which are useful in the sequel.

Definition: 2.1 [15] Let \((X, \tau)\) be a topological space and \(\mathcal{G}\) be a grill on \(X\). The mapping \(\phi: \mathcal{G}(X) \rightarrow \mathcal{G}(X)\), denoted by \(\phi(A, \tau)\) for \(A \in \mathcal{G}(X)\) or simply \(\phi(A)\) called the operator associated with the grill \(\mathcal{G}\) and the topology \(\tau\) and is defined by \(\phi(A)=\{x \in X \mid A \cap U \neq \emptyset, \forall U \in \tau(x)\}\). Let \(\mathcal{G}\) be a grill on a space \(X\). Then a map \(\Psi: \mathcal{G}(X) \rightarrow \mathcal{G}(X)\) is defined by \(\Psi(A)=A \cup \phi(A)\), for all \(A \in \mathcal{G}(X)\). The map \(\Psi\) satisfies Kuratowski closure axioms. Corresponding to a grill \(\mathcal{G}\) on a topological space \((X, \tau)\), there exits a unique topology \(\tau_0\) on \(X\) given by \(\tau_0=\{U \subseteq X / \Psi(X-U)=X-U\}\), where for any \(A \subseteq X\), \(\Psi(A)=A \cup \phi(A)=\tau_0-\text{cl}(A)\). For any grill \(\mathcal{G}\) on a topological space by \((X, \tau, \mathcal{G})\).

Definition: 2.2 A subset \(A\) of a topological space \((X, \tau)\) is called

1. a pre-open set [12] if \(A \subseteq \text{int(}\text{cl}(A))\) and a pre-closed set if \(\text{cl(}\text{int}(A)) \subseteq A\).
2. a semi-open set [7] if \(A \subseteq \text{cl}(\text{int}(A))\) and a semi-closed set if \(\text{int}(\text{cl}(A)) \subseteq A\).
3. a semi-preopen set [11] if \(A \subseteq \text{cl}(\text{int}(\text{cl}(A)))\) and a semi-pre-closed set [2] if \(\text{int}(\text{cl}(A)) \subseteq A\).

Definition: 2.3 A subset \(A\) of a topological space \((X, \tau)\) is called

1. a generalized closed set (briefly g-closed) [8] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
2. a generalized semi-closed set (briefly gs-closed) [2] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
3. an g-generalized closed set (briefly gg-closed) [9] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
4. a generalized semi-pre-closed set (briefly gsp-closed) [5] if \(\text{spcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
5. a generalized preclosed set (briefly gp-closed) [11] if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
6. a generalized grill closed set (briefly g-g-closed) [6] if \(\Psi(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
7. a generalized closed set (briefly g-closed) [17] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
Theorem: 2.4 [15]
1) If $\mathcal{g}_1$ and $\mathcal{g}_2$ are two grills on a space $X$ with $\mathcal{g}_1 \subset \mathcal{g}_2$, then $\tau_{\mathcal{g}_1} \subset \tau_{\mathcal{g}_2}$.
2) If $\mathcal{g}$ is a grill on a space $X$ and $B \notin \mathcal{g}$, then $B$ is closed in $(X, \tau, \mathcal{g})$.
3) For any subset $A$ of a space $X$ and any grill $\mathcal{g}$ on $X$, $\mathcal{g}(A)$ is $\tau_{\mathcal{g}}$-closed.

Theorem: 2.5 [15] Let $(X, \tau)$ be a topological space and $\mathcal{g}$ be any grill on $X$. Then
1) $A \subseteq B (\subseteq X) \Rightarrow \mathcal{g}(A) \subseteq \mathcal{g}(B)$;
2) $A \subseteq X$ and $A \notin \mathcal{g} \Rightarrow \mathcal{g}(A) = \Phi$;
3) $\mathcal{g}(\mathcal{g}(A)) \subseteq \mathcal{g}(A) \subseteq \mathcal{g}(\mathcal{g}(A))$, for any $A \subseteq X$;
4) $\mathcal{g}(A \cup B) = \mathcal{g}(A) \cup \mathcal{g}(B)$ for any $A,B \subseteq X$;
5) $A \subseteq \mathcal{g}(A) \Rightarrow \mathcal{g}(A) = \tau_{\mathcal{g}} - \mathcal{g}(\mathcal{g}(A)) = \tau_{\mathcal{g}}$;
6) $U \in \tau$ and $\tau \subseteq \{ \Phi \} \subset \mathcal{g} \Rightarrow U \subseteq \mathcal{g}(U)$;
7) If $U \in \tau$ then $U \cap \Phi = U \cap \mathcal{g}(U)$, for any $A \subseteq X$.

Theorem: 2.6 Let $(X, \tau)$ be a topological space and $\mathcal{g}$ be any grill on $X$. Then, for any $A, B \subseteq X$.
1) $A \subseteq \mathcal{g}(A)$ [15];
2) $\mathcal{g}(\Phi) = \Phi$ [15];
3) $\mathcal{g}(A \cup B) = \mathcal{g}(A) \cup \mathcal{g}(B)$ [15];
4) $\mathcal{g}(\mathcal{g}(A)) = \mathcal{g}(A)$ [15];
5) int $(A) \subseteq \text{int} (\mathcal{g}(A))$;
6) int $(\mathcal{g}(A \cap B)) \subseteq \text{int} (\mathcal{g}(A))$;
7) int $(\mathcal{g}(A \cap B)) \subseteq \text{int} (\mathcal{g}(B))$;
8) int $(\mathcal{g}(A)) \subseteq \mathcal{g}(A)$;
9) $A \subseteq B \Rightarrow \mathcal{g}(A) \subseteq \mathcal{g}(B)$.

Theorem: 2.7 [16] Let $(X, \tau)$ be a topological space and $\mathcal{g}$ be any grill on $X$. Then, for any $A, B \subseteq X$,
(1) $\mathcal{g}(A) \subseteq \mathcal{g}(A) = \tau_{\mathcal{g}} - \mathcal{g}(\mathcal{g}(A)) \subseteq \mathcal{g}(A)$;
(2) $A \cup \mathcal{g}(\text{int}(A)) \subseteq \mathcal{g}(A)$;
(3) $A \subseteq \mathcal{g}(A)$ and $B \subseteq \mathcal{g}(B) \Rightarrow \mathcal{g}(A \cap B) \subseteq \mathcal{g}(A) \cap \mathcal{g}(B)$.

3. $g^{*}-\mathcal{g}$-CLOSED SETS

Definition: 3.1 A subset $A$ of $(X, \tau)$ is called a $g^{*}-\mathcal{g}$-closed set if $\mathcal{g}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g-open in $(X, \tau)$.

Theorem: 3.2 Every closed set is a $g^{*}-\mathcal{g}$-closed. But not conversely.

Proof: Let $A$ be a closed set. Then $\mathcal{g}(A) = A$. Let $U$ be any g-open set such that $A \subseteq U$. Then $\mathcal{g}(A) \subseteq U$. We know that $\mathcal{g}(A) = A \cup \mathcal{g}(\text{int}(A)) = \tau_{\mathcal{g}} - \mathcal{g}(\mathcal{g}(A)) \subseteq \mathcal{g}(A)$.

Example: 3.3 Let $X = \{a, b, c\}$, $\tau = \{ \Phi \}, \{ b \}, \{ X \}$ and $\mathcal{g} = \{ \{a\}, \{a, b\}, \{X\} \}$. Let $A = \{c\}$. $A$ is a $g^{*}-\mathcal{g}$-closed set but not closed set.

Theorem: 3.4 Every g-closed set is a $g^{*}-\mathcal{g}$-closed. But not conversely.

Proof: Let $A$ be a g-closed set. Then $\mathcal{g}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g-open. We know that $\mathcal{g}(A) = A \cup \mathcal{g}(\text{int}(A)) = \tau_{\mathcal{g}} - \mathcal{g}(\mathcal{g}(A)) \subseteq \mathcal{g}(A)$.

Example: 3.5 Let $X = \{a, b, c\}$, $\tau = \{ \Phi \}, \{a\}, \{a, b\}, \{X\}$ and $\mathcal{g} = \{ \{a\}, \{a, b\}, \{X\} \}$. Let $A = \{a\}$. $A$ is a $g^{*}-\mathcal{g}$-closed set but not g-closed set of $(X, \tau)$.

Theorem: 3.6 Every $g^{*}-\mathcal{g}$-closed set is a gsp closed. But not conversely.

Proof: Let $A$ be a $g^{*}-\mathcal{g}$-closed set in $(X, \tau)$. Then $\mathcal{g}(A) \subseteq U$. Whenever $A \subseteq U$ and $U$ is g-open. From the above theorem $\mathcal{g}(A) \subseteq U$. But every closed set is a semi-pre-closed set, we have $\text{spcl}(A) \subseteq U$ and also every open set is g-open. Therefore $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

Example: 3.7 Let $X = \{a, b, c\}$, $\tau = \{ \Phi \}, \{b\}, \{X\}$ and $\mathcal{g} = \{ \{a\}, \{a, b\}, \{X\} \}$. Let $A = \{b\}$. $A$ is a $g^{*}-\mathcal{g}$-closed set but not gsp-closed set of $(X, \tau)$.

Theorem: 3.8 Every $g^{*}-\mathcal{g}$-closed set is a gp-closed. But not conversely.
Proof: Let A be a \( g^* \)-closed set in \((X, \tau)\). Then \( \Psi(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( g \)-open. From the above \( cl(A) \subseteq U \). But every closed set is \( a \)-closed set, we have \( pcl(A) \subseteq U \) and also every open set is \( g \)-open. Therefore \( pcl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open.

Example: 3.9 Let \( X = \{a, b, c\} \), \( \tau = \{\{\phi\}, \{b\}, \{X\}\} \) and \( g = \{\{a\}, \{a, b\}, \{X\}\} \). Let \( A = \{b\} \). Then \( A \) is gp-closed set but not \( g^* \)-closed set of \((X, \tau)\).

Remark: 3.10 If \( A \) and \( B \) are \( g^* \)-closed set, then \( A \cup B \) is also a \( g^* \)-closed set.

Theorem: 3.11 Every \( g^* \)-closed set is a \( a \)-closed set. But not conversely.

Proof: Let \( A \) be a \( g^* \)-closed set in \((X, \tau)\). Then \( \Psi(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( g \)-open. We know \( cl(A) \subseteq U \). From the above \( cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open.

Example: 3.12 Let \( X = \{a, b, c\} \), \( \tau = \{\{\phi\}, \{b\}, \{X\}\} \) and \( g = \{\{a\}, \{a, b\}, \{X\}\} \). Let \( A = \{b\} \). Then \( A \) is a \( a \)-closed set but not \( g^* \)-closed set of \((X, \tau)\).

Theorem: 3.13 Every \( g^* \)-closed set is a \( gs \)-closed set. But not conversely.

Proof: Let \( A \) be \( g^* \)-closed set in \((X, \tau)\). Then \( \Psi(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( g \)-open. We know \( scl(A) \subseteq U \) and also every open set is \( g \)-open. Therefore \( scl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open.

Example: 3.14 Let \( X = \{a, b, c\} \), \( \tau = \{\{\phi\}, \{b\}, \{X\}\} \) and \( g = \{\{a\}, \{a, b\}, \{X\}\} \). Let \( A = \{b, c\} \). Then \( A \) is a \( gs \)-closed set but not \( g^* \)-closed set of \((X, \tau)\).

Theorem: 3.15 A subset \( A \) of \((X, \tau)\) if a \( g^* \)-closed set if and only if \( \Psi(A) - A \) does not contain any non-empty \( g \)-closed set.

Proof: Necessity: Let \( F \) be a \( g \)-closed set of \((X, \tau)\) such that \( F \subseteq \Psi(A) - A \). Then \( A \subseteq X - F \).

Since \( A \) is \( g^* \)-closed and \( X - F \) is \( g \)-open, \( \Psi(A) \subseteq X - F \). This implies \( F \subseteq X - \Psi(A) \).

So \( F \subseteq (X - \Psi(A)) \cap (\Psi(A) - A) \subseteq (X - \Psi(A)) \cap \Psi(A) = \phi \). Therefore \( F = \phi \).

Sufficiency: Suppose \( A \) is a subset of \((X, \tau)\) such that \( \Psi(A) - A \) does not contain any non-empty \( g \)-closed set. Let \( U \) be a \( g \)-closed set of \((X, \tau)\) such that \( A \subseteq U \). If \( \Psi(A) \not\subseteq U \), then \( \Psi(A) \cap U \not= \phi \). Since \( \Psi(A) \) is a closed set. Then we have \( \phi \not\subseteq \Psi(A) \cap U \) is a \( g \)-closed set of \((X, \tau)\). Then \( \phi \not\subseteq \Psi(A) \cap U \subseteq \Psi(A) - A \). So \( \Psi(A) - A \) contains a non-empty \( g \)-closed set. This contradicts the hypothesis. Therefore \( A \) is a \( g^* \)-closed set.

Theorem: 3.16 If \( A \) is a \( g^* \)-closed set of \((X, \tau)\) such that \( A \subseteq B \subseteq \Psi(A) \), then \( B \) is also a \( g^* \)-closed set of \((X, \tau)\).

Proof: Let \( B \) be a \( g \)-closed set of \((X, \tau)\) such that \( B \subseteq U \). Then \( A \subseteq B \). Since \( A \) is \( g^* \)-closed, \( \Psi(A) \subseteq U \). Now \( \Psi(B) \subseteq \Psi(\Psi(A)) = \Psi(A) \subseteq U \). Therefore \( B \) is also a \( g^* \)-closed set of \((X, \tau)\).

Theorem: 3.17 Every \( g^* \)-closed set is a \( g \)-closed. But not conversely.

Example: 3.18 Let \( X = \{a, b, c\} \), \( \tau = \{\{\phi\}, \{a\}, \{c\}, \{a, c\}, \{X\}\} \) and \( g = \{\{a\}, \{a, b\}, \{X\}\} \). Let \( A = \{a, c\} \). Then \( A \) is a \( g \)-closed but not \( g^* \)-closed set of \((X, \tau)\).

Theorem: 3.19 Every \( g^* \)-closed set is a \( g \)-closed set. But not conversely.

Example: 3.20 Let \( X = \{a, b, c\} \), \( \tau = \{\{\phi\}, \{a\}, \{c\}, \{a, c\}, \{X\}\} \) and \( g = \{\{a\}, \{a, b\}, \{X\}\} \). Let \( A = \{c\} \). Then \( A \) is a \( g \)-closed set but not \( g^* \)-closed set of \((X, \tau)\).
4. $g^*\mathcal{g}$-CONTINUOUS

**Definition:** 4.1 A function $f: (X, \tau, \mathcal{g}) \to (Y, \sigma)$ is called $g^*\mathcal{g}$-continuous if $f^{-1}(V)$ is a $g^*\mathcal{g}$-closed set of $(X, \tau, \mathcal{g})$ for every closed set $V$ of $(Y, \sigma)$.

**Theorem:** 4.2 Every continuous map is $g^*\mathcal{g}$-continuous. But not conversely.

**Proof:** Let $f: (X, \tau, \mathcal{g}) \to (Y, \sigma)$ be a continuous map. Let $V$ be a closed set in $Y$. Then $f^{-1}(V)$ is closed. Since every closed set is $g\mathcal{g}$-closed set. We have $f^{-1}(V)$ is a $g^*\mathcal{g}$-closed set. Therefore $f$ is $g^*\mathcal{g}$-continuous.

**Example:** 4.3 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{X\}\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, \{Y\}\}$ and $\mathcal{g} = \{\{a\}, \{a, b\}, \{X\}\}$. Define $f: (X, \tau, \mathcal{g}) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. $f$ is not continuous since $\{c\}$ is a closed set of $(Y, \sigma)$. But $f^{-1}(c) = \{c\}$ is not closed set of $(X, \tau, \mathcal{g})$. However $f$ is $g^*\mathcal{g}$-continuous.

**Theorem:** 4.4 Every $g\mathcal{g}$-continuous map is $g^*\mathcal{g}$-continuous map. But not conversely.

**Proof:** Let $f: (X, \tau, \mathcal{g}) \to (Y, \sigma)$ be a $g\mathcal{g}$-continuous map. Let $V$ be a $g\mathcal{g}$-closed set in $Y$. Then $f^{-1}(V)$ is $g\mathcal{g}$-closed. Since every $g\mathcal{g}$-closed is $g^*\mathcal{g}$-closed set. We have $f^{-1}(V)$ is a $g^*\mathcal{g}$-closed set. Therefore $f$ is $g^*\mathcal{g}$-continuous.

**Example:** 4.5 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{a, b\}, \{x\}\}$ and $\sigma = \{\emptyset, \{a\}, \{a, c\}, \{x\}\}$. Define $f: (X, \tau, \mathcal{g}) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. $f$ is not $g\mathcal{g}$-continuous. Since $\{c\}$ is a closed set in $(Y, \sigma)$. But $f^{-1}(c) = \{c\}$ is not a $g\mathcal{g}$-closed set of $(X, \tau, \mathcal{g})$. However $f$ is $g^*\mathcal{g}$-continuous.

**Theorem:** 4.6 Every $g^*\mathcal{g}$-continuous map is gsp continuous and hence an $ag\mathcal{g}$-continuous, $gp\mathcal{g}$-continuous and $gs\mathcal{g}$-continuous. But not conversely.

**Proof:** Let $f: (X, \tau, \mathcal{g}) \to (Y, \sigma)$ be a $g^*\mathcal{g}$-continuous map. Let $V$ be a closed set of $(Y, \sigma)$. Since $f$ is $g^*\mathcal{g}$-continuous, $f^{-1}(V)$ is a $g^*\mathcal{g}$-closed set of $(X, \tau, \mathcal{g})$. By the theorems 3.6, 3.8, 3.11 and 3.13, $f^{-1}(V)$ is gsp-closed, gp-closed, $ag\mathcal{g}$-closed and gs-closed set of $(X, \tau)$.

**Example:** 4.7 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, \{x\}\}$, $\sigma = \{\emptyset, \{c\}, \{x\}\}$ and $\mathcal{g} = \{\{a\}, \{a, c\}, \{x\}\}$. Define $g: (X, \tau, \mathcal{g}) \to (Y, \sigma)$ by $g(a) = a$, $g(b) = b$ and $g(c) = c$. $f$ is not $g^*\mathcal{g}$-continuous. Since $\{a, b\}$ is a closed set in $(Y, \sigma)$, but $f^{-1}(a, b) = \{a, b\}$ is not $g^*\mathcal{g}$-closed. However $f$ is gsp continuous, gp-continuous, $ag\mathcal{g}$-continuous and gs-continuous.

**REFERENCES**


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