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COUPLED FIXED POINT THEOREM IN G – METRIC SPACES

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ABSTRACT

In this manuscript, we shall prove a coupled fixed point theorem in which the mapping satisfies a contractive condition. In our result we use the concept of g-monotone mapping in G-metric space. Our result is modified conversion of Rajesh Shrivastava et al. [1] into G-metric space.

Keywords: *G-metric space, coupled fixed point, mixed monotone, coupled common fixed point, mixed g-monotone mapping.*

INTRODUCTION

Fixed point theory has a wide application in almost all fields of sciences, such as Chemistry, Economics and many branches of engineering. Banach contraction principle is one of the care subject that has been studied.

The notion of φ-contraction was given by Boyd and Wong [2] in 1969. The definition of coupled fixed point was given by Gnana-Bhaskar and Lakshmikantham [3] in 2006. Many results were obtained on coupled fixed point. The notion of G-metric space was introduced by Mustafa and Sims [4] in 2006. They proved some fixed point theorems under certain conditions in this space.

PRELIMINARIES

Definition: 1.1 (See [4]) Let X be a nonempty set, and let $G: X\times X\times X\to R^+$, be a function satisfying:
(G1) $G(x, y, z) = 0$ if $x = y = z$
(G2) $0 < G(x, x, y)$, for all $x, y \in X$; with $x\neq y$,
(G3) $G(x, x, y)\leq G(x, y, z)$, for all $x, y, z \in X$ with $z\neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, (symmetry in all three variables), and
(G5) $G(x, y, z)\leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality),
then the function G is called a generalized metric, or, more specifically a G – metric on X, and the pair (X,G) is a G-metric space.

Definition: 1.2 Let $(X, \leq)$ be a partially ordered set and $F: X\times X\to X$ be a mapping. F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$, $x_1\leq x_2 \Rightarrow F(x_1, y)\leq F(x_2, y)$, for $x_1, x_2\in X$ and $y_1\leq y_2 \Rightarrow F(x, y_2)\leq F(x, y_1)$, for $y_1, y_2\in X$.

Definition: 1.3 (See [3]). An element $(x, y)\in X\times X$ is said to be a coupled fixed point of the mapping $F: X\times X\to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition: 1.4 (See [5]) An element $x, y \in X$ is called a coupled coincidence point of the mapping $F: X \times X\to X$ and $g: X\to X$ if, $F(x, y) = g(x)$, $F(y, x) = g(y)$

Definition: 1.5 Let $(X, \leq)$ be an ordered set and $F: X \times X\to X$ and $g: X \times X$. The mapping F is said to has the mixed g- monotone property if F is g- monotone non-decreasing in its first argument and is g- monotone non-increasing in its second argument, that is, for any $x, y\in X$, $x_1, x_2\in X$, $g(x_1)\leq g(x_2)\Rightarrow F(x_1,y)\leq F(x_2,y)$ and $y_1, y_2\in X$, $g(y_1)\leq g(y_2)\Rightarrow F(y_2,x)\leq F(y_1,x)$

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Definition: 1.6 (See [5]) Let X be non empty set and F: $X\times X\to X$ and g: $X\to X$ one says F and g are commutative if, $g(F(x, y)) = F(g(x), g(y))$, for all $x, y\in X$.

MAIN RESULTS

In this paper we extend the result of Rajesh Shrivastava *et.al.* [3] in G-Metric space, which is given as below,

Theorem: 1.7 Let $(X, \le)$ be a partially ordered set and suppose there is a metric G on X such that (X, G) is a complete G- metric space. Suppose F: $X\times X\to X$ and g: $X\to X$ are such that F has the mixed g- monotone property and

$$G(F(x, y), F(u, v), F(u, v)) \le p \max \{G(g(x), F(x, y), F(x, y)), G(g(u), F(u, v), F(u, v))\} + q \max \{G(g(x), F(u, v), F(u, v)), G(g(u), F(x, y), F(x, y))$$

for all $x, y, u, v\in X$ and $p, q\in [0,1)$ such that $0\le p + q < 1$ with $g(x) \ge g(u)$ and $g(y) \le g(v)$.

Suppose that $F(X\times X) \subseteq g(X)$, g is continuous and also suppose either
i. If a non decreasing sequence $x_n \to x$, then $x_n \le x$, for all n,
ii. If a non increasing sequence $y_n\to y$, then $y \le y_n$, for all n.

If there exist $x_0, y_0\in X$ such that $g(x_0) \le F(x_0, y_0)$ and $g(y_0) \ge F(x_0,y_0)$. Then there exist $x, y\in X$ such that

$g(x_0) \le F(x_0, y_0)$ and $g(y_0) \ge F(x_0,y_0)$.

Since $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof: Let $x_0, y_0 \in X$ be such that $g(x_0) \le F (x_0, y_0)$ and $g(y_0) \ge F (x_0,y_0)$. Since $F (X\times X) \subseteq g(X)$, we can choose $x_1, y_1\in X$ such that $g (x_1) = F (x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Again from $F(X\times X) \subseteq g(X)$, we can choose $x_2, y_2\in X$ such that $g(x_2) = F (x_1, y_1)$ and $g(y_2) = F (y_1, x_1)$. Continuing the process we can construct sequence x_n and y_n in X such that

$$g(x_{n+1}) = F (x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \text{ for all } n \ge 0. \tag{2}$$

We shall show that $g(x_n) \le g(x_{n+1})$ for all $n \ge 0$. (3)

and $g(y_n) \ge g(y_{n+1})$ for all $n \ge 0$ (4)

For this we shall use the mathematical induction. Let n = 0. Since $g(x_0) \le F (x_0, y_0)$ and $g(y_0) \ge F (x_0,y_0)$ and as $g(x_1) = F (x_0,y_0)$ and $g(y_1) = F (y_0, x_0)$, we have $g(x_0) \le g(x_1)$ and $g(y_0) \ge g(y_1)$. Thus (3) and (4) hold for n = 0.

Suppose now (3) and (4) holds for some fixed $n\ge 0$. Then, since $g(x_n) \le g(x_{n+1})$ and $g(y_{n+1}) \ge g(y_n)$, and as F has the mixed g- monotone property, then we have

$$g(x_{n+1}) = F (x_n, y_n) \le F(x_{n+1},y_n), \tag{5}$$

$$F(y_{n+1}, x_n) \le F(y_n, x_n) = g(y_{n+1}) \tag{6}$$
and
$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \ge F(x_{n+1}, y_n) \tag{7}$$

$$\text{and } F(y_{n+2}, x_n) \ge F(x_{n+1}, y_n) = g(y_{n+2}) \tag{8}$$

$$\text{Now from (5) – (8) we get } g(x_{n+1}) \le g(x_{n+2}) \tag{9}$$

$$\text{and } g(y_{n+1}) \ge g(y_{n+2}) \tag{10}$$

Thus by the mathematical induction, we conclude that (5) – (8) holds for all $n\ge 0$. Therefore,
$g(x_0) \le g(x_1) \le g(x_2) \le g(x_3) \le ....\le g(x_n) \le g(x_{n+1}) \le........$ and

$g(y_0) \ge g(y_1) \ge g(y_2) \ge g(y_3) \ge ....\ge g(y_n) \ge g(y_{n+1}) \ge........$

Since, $g(x_{n-1}) \le g(x_n)$ and $g(y_{n-1}) \ge g(y_n)$
$G(g(x_n), g(x_{n+1}), g(x_{n+1})) = G(F(x_{n-1},y_{n-1}), F (x_n, y_n), F (x_n, y_n))$ by using, (1) and (2) we have,

$$G(F (x_{n-1}, y_{n-1}), F (x_n, y_n), F (x_n, y_n)) \le p \max\{G(g(x_{n-1}), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1})), G(g(x_n), F (x_n, y_n), F (x_n, y_n))\} + q \max\{G(g(x_{n-1}), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1})), G(g(x_n), F(x_n, y_n), F (x_n, y_n))\}.$$

This gives,

$G(g(x_n), g(x_{n+1}), g(x_{n+1})) \le p \max \{G(g(x_{n-1}), g(x_n), g(x_n)), G(g(x_n), g(x_{n+1}), g(x_{n+1}))\}$
$\qquad + q \max\{ G(g(x_n), g(x_n), g(x_n)), G(g(x_{n-1}), g(x_{n+1}), g(x_{n+1}))\}$

$$G(g(x_n) , g(x_{n+1}), g(x_{n+1})) \le \frac{p+q}{1-q} G(g(x_{n-1}), g(x_n), g(x_n)) \tag{11}$$

Similarly, from (1) and (2), as $g(y_n) \le g(y_{n-1})$ and $g(x_n) \ge g(x_{n-1})$

$G(g(y_{n+1}), g(y_n), g(y_n)) = G(F(y_n, x_n), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}))$

$G(F(y_n, x_n), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1})) \le p \max\{G\, g(y_n), F(y_n, x_n), F(y_n, x_n)), G(g(y_{n-1}), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}))\}$
$\qquad + q \max\{G\, (g(y_n), F(y_{n-1},x_{n-1}), F(y_{n-1},x_{n-1})), G\, g(y_{n-1}), F(y_n, x_n), F(y_n, x_n))\}$

$G(g(y_{n+1}), g(y_n), g(y_n)) \le p \max\{G(g(y_n), g(y_{n+1}), g(y_{n+1}))\, , G(g(y_{n-1}), g(y_n), g(y_n))\}$
$\qquad + q \max\{ G(g(y_n), g(y_n), g(y_n)), G(g(y_{n-1}), g(y_{n+1}), g(y_{n+1}))\}$

$$G(g(y_{n+1}), g(y_n), g(y_n)) \le \frac{p+q}{1-q} G(g(y_{n-1}), g(y_n), g(y_n)) \tag{12}$$

Let us denote $\frac{p+q}{1-q}$ = h and, $G(g(x_n), g(x_{n+1}), g(x_{n+1})) + G(g(y_{n+1}), g(y_n), g(y_n)) = d_n$ then by adding (11) and (12),

we get $d_n \le hd_{n-1} \le h2d_{n-2} \le h3d_{n-3} \le \ldots\ldots \le hn\, d_0$, which implies that,

$\lim_{n\to\infty} d_n = 0$ Thus, $\lim_{n\to\infty} G(g(x_n), g(x_{n+1}), g(x_{n+1})) = \lim_{n\to\infty} G(g(y_{n+1}), g(y_n), g(y_n)) = 0$

For each $m \ge n$ we have,

$G(g(x_n), g(x_m), g(x_m)) \le G(g(x_n),g(x_{n+1}),g(x_{n+1})) + G(g(x_n), g(x_{n+2}),g(x_{n+2})) + \ldots\ldots + G(g(x_{m-1}),g(x_m), g(x_m))$ and

$G(g(y_n), g(y_m), g(y_m)) \le G(g(y_n), g(y_{n+1}), g(y_{n+1})) + G(g(y_n), g(y_{n+2}), g(y_{n+2})) + \ldots\ldots + G(g(y_{m-1}), g(y_m), g(y_m))$

By adding the both of above, we get

$G(g(x_n), g(x_m), g(x_m)) + G(g(y_n), g(y_m), g(y_m)) \le \frac{h^n}{1-h} d_0$ which implies,

$\lim_{n\to\infty} G(g(x_n), g(x_m), g(x_m)) + G(g(y_n), g(y_m), g(y_m)) = 0$

Therefore, $g(x_n)$ and $g(y_n)$ are Cauchy sequence in X. since X is complete metric space, there exist x, y$\in$ X such that

$\lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} g(y_n) = y$.

Thus by taking limit as $n\to\infty$ in (2), we get

$x = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} F(x_{n-1},y_{n-1}) = F(x, y)$

$y = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = F(y, x)$

Therefore, F and g have a coupled coincidence fixed point.

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