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ON THE OSCILLATORY BEHAVIOR FOR A CERTAIN CLASS OF SECOND ORDER DELAY DIFFERENCE EQUATIONS

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ABSTRACT

In this paper, we study the oscillatory behavior for a certain class of second order delay difference equation of the form

$$\Delta\left(\frac{1}{a_n}\Delta u_n\right)+q_n u_{\sigma(n)}=0 \qquad (1.1)$$

Where $\{a_n\},\{q_n\}$ *are real sequence and* $\{a_n\}>0$. *Examples are inserted to illustrate the results.*

Keywords: *Oscillatory, Second order, Double Sequence, Delay Difference equations.*

AMS Classification: *39A21.*

INTRODUCTION

We consider the second order delay difference equation of the form

$$\Delta\left(\frac{1}{a_n}\Delta u_n\right)+q_n u_{\sigma(n)}=0 \qquad (1.1)$$

Where Δ is the forward difference operator is defined by $\Delta u_n = u_{n+1}-u_n$ and $\{a_n\},\{q_n\}$ are real sequence. With respected to the difference equations (1.1) throughout we shall assume that the following conditions holds.

(C1): $\{a_n\},\{q_n\}$ are real sequence and $\{a_n\}>0$

(C2): $\sigma(n)>0$ is an integer such that $\lim\limits_{n\to\infty}\sigma(n)=\infty$

(C3): $R_n=\sum\limits_{s=n_0}^{n-1} a_s \to\infty$ as $n\to\infty$

By a solution of equation (1.1) we mean a real sequence $\{u_n\}$ satisfying (1.1) for $n\ge n_0$. A solution $\{u_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory. For more details on oscillatory behavior difference equation we refer [1-23].

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MAIN RESULT

In this section, we present some sufficient conditions for the oscillation of all the solutions of equations (1.1)

Theorem: 1 Assume the (C3) hold $\Delta\sigma(n) \geq 0$

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n-1}\left[R_{\sigma(n)}p_s + \frac{\left(\Delta R_{\sigma(s)}\right)^2}{4a_{s+1}R_{\sigma(n)}}\right] = \infty \tag{1.2}$$

Then every solution of equations (1.1) is oscillatory.

Proof: Let $\{u_n\}$ be non-oscillatory solution of equation (1.1) without loss of generality, we suppose that $u_n > 0, u_{\sigma(n)} > 0$ and for $n \geq n_1$ from the equation $\Delta\left(\frac{1}{a_n}\Delta u_n\right) < 0$ for $n \geq n_1$. Since $\Delta\left(\frac{1}{a_n}\Delta u_n\right)$ is non-increasing there exist a non-negative constant k and $n_2 \geq n_1$ $\frac{1}{a_n}\Delta u_n \leq -k$ for $n \geq n_2, k > 0$

$\Delta u_n \leq -ka_n$, $n \geq n_2, k > 0$

Summing the inequality for $n_2 \; to \; n-1$

$$u_n \leq u_{n_2} - k\sum_{s=n_2}^{n-1} a_s$$

Letting $n \to \infty$ we have $u_n \to \infty$, which is contradiction to the fact that u_n is positive.

Then $\Delta\left(\frac{1}{a_n}\Delta u_n\right) > 0 \quad and \quad \frac{1}{a_n}\Delta u_n > 0$

Define

$$\omega_n = \frac{R_{\sigma(n)}\Delta u_n}{a_n u_{\sigma(n)}} \tag{A}$$

$$\Delta\omega_n = \frac{R_{\sigma(n)}}{u_{\sigma(n)}}\Delta\left(\frac{1}{a_n}\Delta u_n\right) + \frac{\Delta u_{n+1}}{a_{n+1}}\left[\Delta\left(\frac{R_{\sigma(n)}}{u_{\sigma(n)}}\right)\right]$$

$$\Delta\omega_n = \frac{R_{\sigma(n)}}{u_{\sigma(n)}}\Delta\left(\frac{1}{a_n}\Delta u_n\right) + \frac{\Delta u_{n+1}}{a_{n+1}}\left[\frac{\Delta R_{\sigma(n)}u_{\sigma(n)} - \Delta u_{\sigma(n)}R_{\sigma(n)}}{u_{\sigma(n)}u_{\sigma(n+1)}}\right]$$

$$\Delta\omega_n = \frac{R_{\sigma(n)}}{u_{\sigma(n)}}\Delta\left(\frac{1}{a_n}\Delta u_n\right) + \frac{\Delta u_{n+1}}{a_{n+1}}\frac{\Delta R_{\sigma(n)}}{u_{\sigma(n+1)}} - \frac{\Delta u_{n+1}}{a_{n+1}}\frac{\Delta u_{\sigma(n)}R_{\sigma(n)}}{u_{\sigma(n)}u_{\sigma(n+1)}}$$

$$\Delta\omega_n = -R_{\sigma(n)}q_n + \frac{\Delta R_{\sigma(n)}}{R_{\sigma(n+1)}}\omega_{n+1} - \frac{R_{\sigma(n)}\Delta u_{n+1}\Delta u_{\sigma(n)}}{a_{n+1}u_{\sigma(n)}u_{\sigma(n+1)}}$$

In the view of (C2) and (1.1)

$$\Delta\omega_n = -R_{\sigma(n)}q_n + \frac{\Delta R_{\sigma(n)}}{R_{\sigma(n+1)}}\omega_{n+1} - \frac{R_{\sigma(n)}\left(\Delta u_{n+1}\right)^2}{a_{n+1}(u_{\sigma(n+1)})^2}$$

$$\Delta\omega_n = -R_{\sigma(n)}q_n + \frac{\Delta R_{\sigma(n)}}{R_{\sigma(n+1)}}\omega_{n+1} - \frac{a_{n+1}R_{\sigma(n)}\left(\omega_{n+1}\right)^2}{(\mathrm{R}_{\sigma(n+1)})^2}$$ That is

$$\Delta\omega_n = -R_{\sigma(n)}q_n + \frac{(\Delta R_{\sigma(n)})^2}{4a_{n+1}R_{\sigma(n)}} - \left[\frac{\sqrt{a_{n+1}R_{\sigma(n)}}}{R_{\sigma(n+1)}}\omega_{n+1} - \frac{\Delta R_{\sigma(n)}}{2\sqrt{a_{n+1}R_{\sigma(n)}}}\right]^2$$

This implies that

$$\Delta\omega_n < -\left(R_{\sigma(n)}q_n - \frac{(\Delta R_{\sigma(n)})^2}{4a_{n+1}R_{\sigma(n)}}\right) \tag{1.3}$$

Summing the inequality for $n_2\ to\ n-1$ we have

$$\omega_n \le \omega_{n_1} - \sum_{s=n_2}^{n-1}\left(R_{\sigma(n)}q_n - \frac{(\Delta R_{\sigma(n)})^2}{4a_{n+1}R_{\sigma(n)}}\right)$$

Letting $n\to\infty$, we have, in view of (1.2) that $\omega_n \to \infty\ \ as\ \ n\to\infty,$ which contradicts $\omega_n > 0$ and the proof is complete.

Theorem: 2 Let all the assumption of Theorem 1 holds except the condition (1.2) which changed to

$$\lim_{n\to\infty}\sup\frac{1}{n^r}\sum_{s=n_0}^{n-1}(n-s)^r\left(q_nR_{\sigma(n)} - \frac{(\Delta R_{\sigma(n)})^2}{4a_{n+1}R_{\sigma(n)}}\right) = \infty \tag{1.4}$$

Then every solutions $\{u_n\}$ of equation (1.1) is oscillatory.

Proof: Proceeding as in the proof of Theorem 1, we assume that equations (1.1) non-oscillatory solution, say $u_n > 0, u_{\sigma(n)} > 0$ and for $n \ge n_1$ from the equation (1.3) we have $n \ge n_1$.

$$\sum_{s=n_1}^{n-1}(n-s)^r\left(R_{\sigma(s)}q_s - \frac{(\Delta R_{\sigma(s)})^2}{4a_{s+1}R_{\sigma(s)}}\right) < -\sum_{s=n_1}^{n-1}(n-s)^r\Delta\omega_s \tag{1.5}$$

Since

$$\sum_{s=n_1}^{n-1}(n-s)^r\Delta\omega_s = r\sum_{s=n_1}^{n-1}(n-s)^{r-1}\omega_s - \omega_{n_1}(n-n_1)^r \tag{1.6}$$

We get

$$\frac{1}{n^r}\sum_{s=n_1}^{n-1}(n-s)^rM_s \le \omega_{n_1}\left(\frac{n-n_1}{n}\right)^r - \frac{r}{n^r}\sum_{s=n_1}^{n-1}(n-s)^{r-1}\omega_s \tag{1.7}$$

Where

$$M_s = R_{\sigma(s)}q_s - \frac{(\Delta R_{\sigma(s)})^2}{4a_{s+1}R_{\sigma(s)}}$$

Letting $n\to\infty$,

we have

$$\frac{1}{n^r}\sum_{s=n_1}^{n-1}(n-s)^rM_s \le \omega_{n_1}\left(\frac{n-n_1}{n}\right)^r \tag{1.8}$$

Then

$$\lim_{n\to\infty}\sup\frac{1}{n^r}\sum_{s=n_1}^{n-1}(n-s)^rM_s \to \omega_{n_1}$$

Which contradicts the condition (1.4)

This completes the proof.

Next, we present some new oscillation results for equation (1.1) we introduce a double sequence $\left\{H\left(m,n\right)/\, m \ge n \ge 0\right\}$

Such that

$(i)\ \ H(m,n) = 0 \ \ for \ \ m \ge o$

$(ii)\ \ H(m,n) > 0 \ \text{ for } \ \mathrm{m > n \ge o} \ \text{ and}$

$(ii)\ \ -L_2 H(m,n) = \mathrm{h(m,n)}\sqrt{H(m,n)};\ \ for \ \ m \ge n \ge o$

Theorem: 3 Assume that (C1)-(C3) holds and let $\{u_n\}$ be a positive sequence and assume that there exist a double sequence $\left\{H\left(m,n\right)/\, m \ge n \ge 0\right\}$ such that

$$\lim_{n\to\infty}\sup \frac{1}{H(n,\mathrm{n}_1)} \sum_{s=n_0}^{n-1} H(n,s)\left(q_n R_{\sigma(n)} - \frac{(\Delta R_{\sigma(n)})^2}{4a_{n+1}R_{\sigma(n)}} \right) = \infty \tag{1.9}$$

$$where \quad M(n,s) = \frac{h(n,s)}{\sqrt{H(n,s)}} - \frac{\Delta R_{\sigma(n)}}{R_{\sigma(n+1)}} \tag{1.10}$$

Then every solutions $\{u_n\}$ of equation (1.1) is oscillatory

Proof: Let $\{u_n\}$ be non-oscillatory solution of equation (1.1). Let us first assume the $\{u_n\}$ is eventually positive and that $u_n > 0, u_{\sigma(n)} > 0$ and for $n \ge n_1$

In the view of (A) and (B)

Let us denote $\gamma_s = \dfrac{\Delta R_{\sigma(n)}}{R_{\sigma(n+1)}} \quad and \quad B_s = \dfrac{a_{s+1}R_{\sigma(s)}}{R^2_{\sigma(n+1)}}$

$$\Delta\omega_n = -R_{\sigma(n)}q_n + \frac{\Delta R_{\sigma(n)}}{R_{\sigma(n+1)}}\omega_{n+1} - \frac{a_{n+1}R_{\sigma(n)}\left(\omega_{n+1}\right)^2}{(\mathrm{R}_{\sigma(n+1)})^2}$$

$$R_{\sigma(n)}q_n = -\Delta\omega_n + \frac{\Delta R_{\sigma(n)}}{R_{\sigma(n+1)}}\omega_{n+1} - \frac{a_{n+1}R_{\sigma(n)}\left(\omega_{n+1}\right)^2}{(\mathrm{R}_{\sigma(n+1)})^2}$$

$$\sum_{s=n_1}^{n-1} H(n,s)R_{\sigma(\mathrm{s})}q_s \le \sum_{s=n_1}^{n-1} H(\mathrm{n,s})\left[-\Delta\omega_s + \gamma_s\omega_{s+1} - B_s\left(\omega_s\right)^2\right]$$

$$\begin{aligned}\sum_{s=n_1}^{n-1} H(n,s)R_{\sigma(\mathrm{s})}q_s &\le [H(n,s)\omega_s]_{s=n_1}^{n} - \sum_{s=n_1}^{n-1}\left\{L_2 H(\mathrm{n\ s})\omega_{s+1} + H(n,s)\left[\gamma_s\omega_{s+1} - B_s\left(\omega_s\right)^2\right]\right\} \\ &= H(n,\mathrm{n}_1)\omega_{n_1} - \sum_{s=n_1}^{n-1}\left[\sqrt{H(\mathrm{n\ s})}\left(h(n,s) - \sqrt{(H(n,s)}\gamma_s\right)\omega_{s+1} + H(n,s)B_s\left(\omega_{s+1}\right)^2\right] \\ &= H(n,\mathrm{n}_1)\omega_{n_1} - \sum_{s=n_1}^{n-1} H(\mathrm{n,s})\left[\sqrt{B_s}\,\omega_{s+1} + \frac{1}{2}\frac{M(n,s)}{\sqrt{B_s}}\right]^2 + \sum_{s=n_1}^{n-1}\frac{M^2(n,s)H(n,s)}{4B_s}\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\sup \frac{1}{H(n,\mathrm{n}_1)} \sum_{s=n_0}^{n-1} H(n,s)\left(q_n R_{\sigma(n)} - \frac{(\Delta R_{\sigma(n)})^2}{4a_{n+1}R_{\sigma(n)}} \right) \le \omega_{n_1}$$

Which clearly contradicts (1.9). This contradiction completes our proof

Remarks: By choosing various specific double sequences $\{H(m,n)\}$ we can derive several oscillation criteria for (1.1)

Let us consider the double sequence $\{H(m,n)\}$ defined by

$H(m,n)=(m-n)^{\mu}, \quad m \ge n \ge o,$ Where $\mu \ge 1$ is a constant.

Then $H(m,n)=0 \quad for \quad m \ge 0, \; H(m,n)>0$ for $m>n\ge 0 \quad and \quad L_2H(m,n)\le 0$ for $m>n\ge 0$

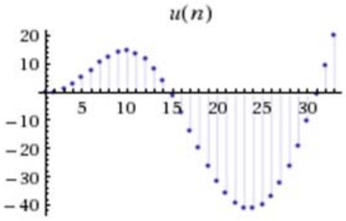
Hence, we have the following corollary

Corollary: 1 If $\limsup_{n\to\infty} \frac{1}{m^{\mu}} \sum_{s=n_0}^{n-1} (m-n)^{\mu} \left(q_n R_{\sigma(n)} - \frac{(\Delta R_{\sigma(n)})^2}{4a_{n+1}R_{\sigma(n)}} \right) = \infty \; for \; some \; \mu \ge 1,$ Then every solutions $\{u_n\}$ of equation (1.1) is oscillatory

Example: 1 Consider the delay difference equations

$$\Delta\left(\frac{1}{n-1}\Delta u_n\right)+\frac{\frac{1}{n}}{(n-1)}u_{n-1}=0 \qquad (E1)$$

$$where \;\; \lambda \ge 0 \quad \mu = 2 \quad R_{\sigma(n)}=1, the\, equation \;\; (E1) \;\; is \;\; Oscillatory.$$



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