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ON A DIFFERENCE OF GENERALIZED f- DIVERGENCES

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ABSTRACT

In this paper, we establish new information inequalities in terms of difference of generalized divergence measures. Further obtain new divergences corresponding to that difference and get the bounds of new divergence in terms of other standard divergence measures. Lastly, obtain the relation among new divergences and Renyi's entropy.

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***Key Words**: Difference of generalized ϕ- divergence measures, convex and normalized function, new information inequalities, new divergence measures, bounds of new divergence measures.*

1. INTRODUCTION:

Let $\Gamma_n = \left\{ P = (p_1, p_2, p_3..., p_n) : p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2$ be the set of all complete finite discrete probabili -ty distributions. If we take $p_i \geq 0$ for some $i = 1, 2, 3, ..., n$, then we have to suppose that $0\, f(0) = 0\, f\left(\frac{0}{0}\right) = 0$.

Csiszar's ϕ- divergence [1] is a generalized information divergence measure, which is given by:

$$I_\phi(P,Q) = \sum_{i=1}^{n} q_i\, \phi\left(\frac{p_i}{q_i}\right) \tag{1.1}$$

Similarly **[6]** introduced a generalized measure of information given by

$$S_\phi(P,Q) = \sum_{i=1}^{n} q_i\, \phi\left(\frac{p_i + q_i}{2\, q_i}\right) \tag{1.2}$$

where ϕ: $(0,\infty) \to$ R (set of real no.) is a real continuous convex function and $P = (p_1, p_2, p_3..., p_n)$, $Q = (q_1, q_2, q_3..., q_n) \in \Gamma_n$, where p_i and q_i are probability mass functions. Many known divergences can be obtained from these generalized measures by suitably defining the convex function f. These are as follows:

The following measures (1.3), (1.4) and (1.5) are one parameter generalized divergence measures [11], where "$s \in R$" is a parameter.

$$\psi_s(P,Q) = \left[s(s-1)\right]^{-1}\left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right], s \neq 0,1 = \textbf{Relative Information of type "s".} \tag{1.3}$$

$$\Omega_s(P,Q) = \left[s(s-1)\right]^{-1}\left[\sum_{i=1}^{n} p_i \left(\frac{p_i+q_i}{2p_i}\right)^s - 1\right], s \neq 0, 1 = \textbf{Unified Relative JS and AG Divergence of type "s".} \tag{1.4}$$

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$$\tau_s(P,Q)=(s-1)^{-1}\sum_{i=1}^{n}\frac{p_i-q_i}{2}\left(\frac{p_i+q_i}{2q_i}\right)^{s-1},\ s\neq 1 = \textbf{Relative J- Divergence of type “s”.} \tag{1.5}$$

$$K(P,Q)=\sum_{i=1}^{n}p_i\log\left(\frac{p_i}{q_i}\right) = \textbf{Relative Entropy [7].} \tag{1.6}$$

$$\chi^2(P,Q)=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i} = \textbf{Chi- Square Measure [8].} \tag{1.7}$$

$$F(P,Q)=\sum_{i=1}^{n}p_i\log\left(\frac{2\,p_i}{p_i+q_i}\right) = \textbf{Relative JS Divergence [10].} \tag{1.8}$$

$$G(P,Q)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2\,p_i}\right) = \textbf{Relative AG Divergence [12].} \tag{1.9}$$

$$R_a(P,Q)=\sum_{i=1}^{n}\frac{p_i^{\,a}}{q_i^{\,a-1}},\ a>1 = \textbf{Renyi's "}a\textbf{" order entropy [9].} \tag{1.10}$$

$$\Delta(P,Q)=2\left[1-H(P,Q)\right]=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{p_i+q_i} = \textbf{Triangular Discrimination [2].} \tag{1.11}$$

where $H(P,Q)=\sum_{i=1}^{n}\frac{2\,p_i\,q_i}{p_i+q_i}$ is Harmonic Mean Divergence.

$$J_R(P,Q)=\sum_{i=1}^{n}(p_i-q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) = \textbf{Relative J- Divergence [3].} \tag{1.12}$$

where $F(P,Q)$ and $G(P,Q)$ are given by (1.8) and (1.9) respectively.

2. DIFFERENCE OF GENERALIZED f- DIVERGENCES

The following theorem is well known in literature **[1].**

Theorem: 2.1 If the function ϕ is convex and normalized, i.e., $\phi(1)=0$, then $I_\phi(P,Q)$ and its adjoint $I_\phi(Q,P)$ are both non-negative and convex in the pair of probability distribution $(P,Q)\in\Gamma_n\times\Gamma_n$.

Now, we obtain some new information inequalities in terms of difference of generalized divergence measures and results are on similar lines to the results presented by **[11].**

Theorem: 2.2 Let $\phi_1, \phi_2 : I\subset \mathbb{R}_+\to\mathbb{R}$ be two normalized convex mappings, i.e. $\phi_1(1)=\phi_2(1)=0$ and suppose the assumptions:

(i) ϕ_1 and ϕ_2 are twice differentiable on (α, β) where $0<\alpha\le 1\le\beta<\infty, \alpha\neq\beta$.

(ii) There exists the real constants m, M such that m < M and

$$m\le\frac{\phi_1''(t)}{\phi_2''(t)}\le M,\ \ \phi_2''(t)>0,\ \ \forall\ \ t\in(\alpha,\beta). \tag{2.1}$$

then we have the inequalities:

$$m\left[I_{\phi_2}(P,Q)-S_{\phi_2}(P,Q)\right]\le I_{\phi_1}(P,Q)-S_{\phi_1}(P,Q)\le M\left[I_{\phi_2}(P,Q)-S_{\phi_2}(P,Q)\right] \tag{2.2}$$

$$m\left[E_{S_{\phi_2}}(P,Q)-S_{\phi_2}(P,Q)\right]\le E_{S_{\phi_1}}(P,Q)-S_{\phi_1}(P,Q)\le M\left[E_{S_{\phi_2}}(P,Q)-S_{\phi_2}(P,Q)\right] \tag{2.3}$$

$$m\left[A_{S_{\phi_2}}(\alpha,\beta)-S_{\phi_2}(P,Q)\right]\le A_{S_{\phi_1}}(\alpha,\beta)-S_{\phi_1}(P,Q)\le M\left[A_{S_{\phi_2}}(\alpha,\beta)-S_{\phi_2}(P,Q)\right] \tag{2.4}$$

where $$E_{S_\phi}(P,Q)=\frac{1}{2}\sum_{i=1}^{n}(p_i-q_i)\,\phi'\left(\frac{p_i+q_i}{2\,q_i}\right), \tag{2.5}$$

$$A_{S_\phi}(\alpha,\beta)=\frac{1}{4}(\beta-\alpha)\left\{\phi'(\beta)-\phi'(\alpha)\right\} \tag{2.6}$$

and $I_\phi(P,Q),\ S_\phi(P,Q)$ are given by (1.1) and (1.2) respectively.

Proof: Let us consider two functions

$$\Phi_m(t)=\phi_1(t)-m\,\phi_2(t) \tag{2.7}$$

and $$\Phi_M(t)=M\,\phi_2(t)-\phi_1(t) \tag{2.8}$$

where m and M are the max. and min. values of the function $\frac{\phi_1''(t)}{\phi_2''(t)}$, $\forall\ t\in(\alpha,\beta)$.

Since $$\phi_1(1)=\phi_2(1)=0\Rightarrow\Phi_m(1)=\Phi_M(1)=0 \tag{2.9}$$

and the functions $\phi_1(t)$ and $\phi_2(t)$ are twice differentiable. Then in view of (2.1), we have

$$\Phi_m''(t)=\phi_1''(t)-m\,\phi_2''(t)=\phi_2''(t)\left(\frac{\phi_1''(t)}{\phi_2''(t)}-m\right)\ge 0 \tag{2.10}$$

$$\Phi_M''(t)=M\,\phi_2''(t)-\phi_1''(t)=\phi_2''(t)\left(M-\frac{\phi_1''(t)}{\phi_2''(t)}\right)\ge 0 \tag{2.11}$$

For all t ∈(α, β).

In view 2.9, 2.10 and 2.11, we can say that the functions $\Phi_m(t)$ and $\Phi_M(t)$ are convex and normalized on (α, β).

Now, with the help of linearity property, we have the following cases:

Case I: $$I_{\Phi_m}(P,Q)-S_{\Phi_m}(P,Q)=I_{\phi_1-m\phi_2}(P,Q)-S_{\phi_1-m\phi_2}(P,Q)$$
$$=I_{\phi_1}(P,Q)-mI_{\phi_2}(P,Q)-S_{\phi_1}(P,Q)+mS_{\phi_2}(P,Q) \tag{2.12}$$

and $$I_{\Phi_M}(P,Q)-S_{\Phi_M}(P,Q)=I_{M\phi_2-\phi_1}(P,Q)-S_{M\phi_2-\phi_1}(P,Q)$$
$$=MI_{\phi_2}(P,Q)-I_{\phi_1}(P,Q)-MS_{\phi_2}(P,Q)+S_{\phi_1}(P,Q) \tag{2.13}$$

{Since $I_\phi(P,Q)\ge S_\phi(P,Q)$ in [6]}.

Therefore (2.12) and (2.13) can be written as the followings:

$$\left[I_{\phi_1}(P,Q)-S_{\phi_1}(P,Q)\right]-m\left[I_{\phi_2}(P,Q)-S_{\phi_2}(P,Q)\right]\ge 0$$

and $$M\left[I_{\phi_2}(P,Q)-S_{\phi_2}(P,Q)\right]-\left[I_{\phi_1}(P,Q)-S_{\phi_1}(P,Q)\right]\ge 0$$

or $$\left[I_{\phi_1}(P,Q)-S_{\phi_1}(P,Q)\right]\ge m\left[I_{\phi_2}(P,Q)-S_{\phi_2}(P,Q)\right] \tag{2.14}$$

and $$M\left[I_{\phi_2}(P,Q)-S_{\phi_2}(P,Q)\right]\ge\left[I_{\phi_1}(P,Q)-S_{\phi_1}(P,Q)\right] \tag{2.15}$$

(2.14) and (2.15), together give the result (2.2).

Case II: $E_{S_{\Phi_m}}(P,Q)-S_{\Phi_m}(P,Q)=E_{S_{\phi_1-m\phi_2}}(P,Q)-S_{\phi_1-m\phi_2}(P,Q)$

$$=E_{S_{\phi_1}}(P,Q)-mE_{S_{\phi_2}}(P,Q)-S_{\phi_1}(P,Q)+mS_{\phi_2}(P,Q) \tag{2.16}$$

and $E_{S_{\Phi_M}}(P,Q)-S_{\Phi_M}(P,Q)=E_{S_{M\phi_2-\phi_1}}(P,Q)-S_{M\phi_2-\phi_1}(P,Q)$

$$=ME_{S_{\phi_2}}(P,Q)-E_{S_{\phi_1}}(P,Q)-MS_{\phi_2}(P,Q)+S_{\phi_1}(P,Q) \tag{2.17}$$

{Since $E_{S_\phi}(P,Q)\geq S_\phi(P,Q)$ **in [5]}**.

Therefore (2.16) and (2.17) can be written as the followings:

$$\left[E_{S_{\phi_1}}(P,Q)-S_{\phi_1}(P,Q)\right]-m\left[E_{S_{\phi_2}}(P,Q)-S_{\phi_2}(P,Q)\right]\geq 0$$

and $$M\left[E_{S_{\phi_2}}(P,Q)-S_{\phi_2}(P,Q)\right]-\left[E_{S_{\phi_1}}(P,Q)-S_{\phi_1}(P,Q)\right]\geq 0$$

or $$\left[E_{S_{\phi_1}}(P,Q)-S_{\phi_1}(P,Q)\right]\geq m\left[E_{S_{\phi_2}}(P,Q)-S_{\phi_2}(P,Q)\right] \tag{2.18}$$

And $$M\left[E_{S_{\phi_2}}(P,Q)-S_{\phi_2}(P,Q)\right]\geq\left[E_{S_{\phi_1}}(P,Q)-S_{\phi_1}(P,Q)\right] \tag{2.19}$$

(2.18) and (2.19), together give the result (2.3).

Case III: $A_{S_{\Phi_m}}(\alpha,\beta)-S_{\Phi_m}(P,Q)=A_{S_{\phi_1-m\phi_2}}(\alpha,\beta)-S_{\phi_1-m\phi_2}(P,Q)$

$$=A_{S_{\phi_1}}(\alpha,\beta)-mA_{S_{\phi_2}}(\alpha,\beta)-S_{\phi_1}(P,Q)+mS_{\phi_2}(P,Q) \tag{2.20}$$

and $A_{S_{\Phi_M}}(\alpha,\beta)-S_{\Phi_M}(P,Q)=A_{S_{M\phi_2-\phi_1}}(\alpha,\beta)-S_{M\phi_2-\phi_1}(P,Q)$

$$=MA_{S_{\phi_2}}(\alpha,\beta)-A_{S_{\phi_1}}(\alpha,\beta)-MS_{\phi_2}(P,Q)+S_{\phi_1}(P,Q) \tag{2.21}$$

{Since $A_{S_\phi}(\alpha,\beta)\geq S_\phi(P,Q)$ **in [5]}**.

Therefore (2.20) and (2.21) can be written as the followings:

$$\left[A_{S_{\phi_1}}(\alpha,\beta)-S_{\phi_1}(P,Q)\right]-m\left[A_{S_{\phi_2}}(\alpha,\beta)-S_{\phi_2}(P,Q)\right]\geq 0$$

and $$M\left[A_{S_{\phi_2}}(\alpha,\beta)-S_{\phi_2}(P,Q)\right]-\left[A_{S_{\phi_1}}(\alpha,\beta)-S_{\phi_1}(P,Q)\right]\geq 0$$

or $$\left[A_{S_{\phi_1}}(\alpha,\beta)-S_{\phi_1}(P,Q)\right]\geq m\left[A_{S_{\phi_2}}(\alpha,\beta)-S_{\phi_2}(P,Q)\right] \tag{2.22}$$

and $$M\left[A_{S_{\phi_2}}(\alpha,\beta)-S_{\phi_2}(P,Q)\right]\geq\left[A_{S_{\phi_1}}(\alpha,\beta)-S_{\phi_1}(P,Q)\right] \tag{2.23}$$

(2.22) and (2.23), together give the result (2.4).

3. NEW DIVERGENCE MEASURES AND PROPERTIES:

In this section, we shall derive some new divergence measures, for this:

Let $\phi:(0,\infty)\to R$ (set of real no.) be a function defined as

$$\phi(t)=\phi_1(t)=\frac{1}{t^2}-1,\ \forall\ t\in(0,\infty)\ \text{and}\ \phi_1(1)=0 \tag{3.1}$$

and $$\phi_1'(t) = -\frac{2}{t^3},\ \phi_1''(t) = \frac{6}{t^4} \tag{3.2}$$

Since $\phi_1''(t) > 0\ \forall\ t \in (0,\infty)$ and $\phi_1(1) = 0$, therefore $\phi_1(t)$ is convex and normalized function respectively.

Now, put $\phi_1(t)$ in (1.1) and (1.2) and put $\phi_1'(t)$ in (2.5) and (2.6), we get the following new divergence measures.

$$I_{\phi_1}(P,Q) - S_{\phi_1}(P,Q) = IS(P,Q) = \sum_{i=1}^{n} \frac{q_i^3 (3p_i + q_i)(q_i - p_i)}{p_i^2 (p_i + q_i)^2} \geq 0\ \forall P, Q \in \Gamma_n, \tag{3.3}$$

$$E_{S_{\phi_1}}(P,Q) - S_{\phi_1}(P,Q) = ES(P,Q) = \sum_{i=1}^{n} \frac{q_i (p_i + 5q_i)(p_i - q_i)^2}{(p_i + q_i)^3} \geq 0\ \forall P, Q \in \Gamma_n \tag{3.4}$$

$$S_{\phi_1}(P,Q) = J^*(P,Q) = \sum_{i=1}^{n} \frac{q_i (p_i + 3q_i)(q_i - p_i)}{(p_i + q_i)^2} \geq 0\ \forall P, Q \in \Gamma_n \tag{3.5}$$

$$A_{S_{\phi_1}}(\alpha,\beta) = \frac{1}{2} \frac{(\beta-\alpha)^2 (\alpha^2 + \alpha\,\beta + \beta^2)}{\alpha^3\ \beta^3},\ 0 < \alpha \leq 1 \leq \beta < \infty,\ \alpha \neq \beta \tag{3.6}$$

Here, measures $IS(P,Q)$, $ES(P,Q)$ and $J^*(P,Q)$ are non- negative and convex in the pair of probability distribution P, Q $\in \Gamma_n$ and $IS(P,Q) = 0,\ ES(P,Q), = 0\ J^*(P,Q) = 0$ iff $p_i = q_i\ \forall\ i = 1, 2, 3..., n$.

Since $IS(P,Q) \neq IS(Q,P)$, $ES(P,Q) \neq ES(Q,P)$ and $J^*(P,Q) \neq J^*(Q,P)$

Therefore $IS(P,Q)$, $ES(P,Q)$ and $J^*(P,Q)$ are non- symmetric divergence measures.

4. BOUNDS OF NEW DIVERGENCE MEASURES

Now in this section, we shall obtain bounds of $IS(P,Q)$, $ES(P,Q)$ and $J^*(P,Q)$ in terms of other divergences by using theorem 2.2.

Proposition 4: Let $P, Q \in \Gamma_n, 0 < \alpha \leq 1 \leq \beta < \infty, \alpha \neq \beta$ and $s \geq -2\ (s \in R)$, then we have:

$$\frac{6}{\beta^{s+2}}\left[\psi_s(P,Q) - \Omega_s(Q,P)\right] \leq IS(P,Q) \leq \frac{6}{\alpha^{s+2}}\left[\psi_s(P,Q) - \Omega_s(Q,P)\right] \tag{4.1}$$

$$\frac{6}{\beta^{s+2}}\left[\tau_s(P,Q) - \Omega_s(Q,P)\right] \leq ES(P,Q) \leq \frac{6}{\alpha^{s+2}}\left[\tau_s(P,Q) - \Omega_s(Q,P)\right] \tag{4.2}$$

$$\frac{6}{\beta^{s+2}}\left[A_{S_{\phi_2}}(\alpha,\beta) - \Omega_s(Q,P)\right] \leq A_{S_{\phi_1}}(\alpha,\beta) - J^*(P,Q) \leq \frac{6}{\alpha^{s+2}}\left[A_{S_{\phi_2}}(\alpha,\beta) - \Omega_s(Q,P)\right] \tag{4.3}$$

where $\psi_s(P,Q)$, $\Omega_s(P,Q)$, $\tau_s(P,Q)$, $IS(P,Q)$, $ES(P,Q)$, $J^*(P,Q)$, $A_{S_{\phi_1}}(\alpha,\beta)$ and $A_{S_{\phi_2}}(\alpha,\beta)$ are given by (1.3), (1.4), (1.5), (3.3), (3.4), (3.5), (3.6) and (4.9) respectively.

Proof: let us consider

$$\phi_2(t) = \left[s(s-1)\right]^{-1}\left[t^s - 1\right],\ s \neq 0, 1,\ t \in (0,\infty), \phi_2(1) = 0\ and\ s \in R \tag{4.4}$$

and $\phi_2'(t) = (s-1)^{-1} t^{s-1},\ s \neq 1$ and $\phi_2''(t) = t^{s-2} > 0\ \forall\ t > 0\ and\ s \in R$. (4.5)

Since $\phi_2''(t) > 0\ \forall\ t \in (0,\infty)$ and $\phi_2(1) = 0$, therefore $\phi_2(t)$ is convex and normalized function respectively.

Now, put $\phi_2(t)$ in (1.1) and (1.2) and put $\phi_2'(t)$ in (2.5) and (2.6), we get the followings:

$$I_{\phi_2}(P,Q) = \sum_{i=1}^{n} q_i \phi_2\left(\frac{p_i}{q_i}\right) = \left[s(s-1)\right]^{-1}\left[\sum_{i=1}^{n} p_i^{\,s} q_i^{1-s} - 1\right] = \psi_s(P,Q),\ s \neq 0, 1 \tag{4.6}$$

$$S_{\phi_2}(P,Q) = \sum_{i=1}^{n} q_i \phi_2\left(\frac{p_i+q_i}{2\,q_i}\right) = \left[s(s-1)\right]^{-1}\left[\sum_{i=1}^{n} q_i \left(\frac{p_i+q_i}{2q_i}\right)^{s} - 1\right] = \Omega_s(Q,P),\ s \neq 0, 1 \tag{4.7}$$

$$E_{S_{\phi_2}}(P,Q) = \sum_{i=1}^{n} \frac{p_i - q_i}{2}\,\phi_2'\left(\frac{p_i+q_i}{2q_i}\right) = \left[\frac{(s-1)^{-1}}{2^s}\right]\sum_{i=1}^{n}(p_i - q_i)\left(1+\frac{p_i}{q_i}\right)^{s-1} = \tau_s(P,Q), s \neq 1 \tag{4.8}$$

$$A_{S_{\phi_2}}(\alpha,\beta) = \frac{1}{4}(\beta-\alpha)\left[\phi_2'(\beta) - \phi_2'(\alpha)\right] = \frac{(s-1)^{-1}(\beta-\alpha)}{4}\left[\beta^{s-1} - \alpha^{s-1}\right], s \neq 1 \tag{4.9}$$

Now, let us consider the following function.

$$f_s(t) = \frac{\phi_1''(t)}{\phi_2''(t)} = \frac{6}{t^{s+2}}, t \in (0,\infty)\ and\ s \in R \tag{4.10}$$

Where $\phi_1''(t)\ and\ \phi_2''(t)$ are given by 3.2 and 4.5 respectively.

and $f_s'(t) = -\dfrac{6(s+2)}{t^{s+3}},\ s \in R.$ (4.11)

We can check that, $f_s'(t) = \begin{Bmatrix} \leq 0, if\ s \geq -2 \\ \geq 0, if\ s \leq -2 \end{Bmatrix}$

i.e. $f_s(t)$ **is monotonically decreasing in** $s \geq -2$ **and monotonically increasing in** $s \leq -2$**.**

Therefore, at $s \geq -2$**,** we have

$$M = \max_{t\in(\alpha,\beta)} \frac{\phi_1''(t)}{\phi_2''(t)} = \max_{t\in(\alpha,\beta)} f_s(t) = f_s(\alpha) = \frac{6}{\alpha^{s+2}} \quad \text{and} \tag{4.12}$$

$$m = \min_{t\in(\alpha,\beta)} \frac{\phi_1''(t)}{\phi_2''(t)} = \min_{t\in(\alpha,\beta)} f_s(t) = f_s(\beta) = \frac{6}{\beta^{s+2}} \tag{4.13}$$

The inequalities 4.1, 4.2 and 4.3 are obtained by using (3.3), (3.4), (3.5), (3.6), (4.6), (4.7), (4.8), (4.9), (4.12) and (4.13) in (2.2), (2.3) and (2.4).

Now, we shall consider some special cases at $s = 0,\ s = 1\ and\ s = 2$**.**

Result 4.1 (at s=0).

Let $P, Q \in \Gamma_n, 0 < \alpha \leq 1 \leq \beta < \infty,\ \alpha \neq \beta$ and $s = 0$, then we have:

$$\frac{6}{\beta^2}\left[K(Q,P) - F(Q,P)\right] \leq IS(P,Q) \leq \frac{6}{\alpha^2}\left[K(Q,P) - F(Q,P)\right] \tag{4.14}$$

$$\frac{6}{\beta^2}\left[\frac{1}{2}\Delta(P,Q)-F(Q,P)\right]\le ES(P,Q)\le\frac{6}{\alpha^2}\left[\frac{1}{2}\Delta(P,Q)-F(Q,P)\right] \tag{4.15}$$

$$\frac{6}{\beta^2}\left[\frac{1}{4}\frac{(\beta-\alpha)^2}{\alpha\,\beta}-F(Q,P)\right]\le A_{S_{\phi_1}}(\alpha,\beta)-J^*(P,Q)\le\frac{6}{\alpha^2}\left[\frac{1}{4}\frac{(\beta-\alpha)^2}{\alpha\,\beta}-F(Q,P)\right] \tag{4.16}$$

Proof: Put s=0 in (4.6), (4.7), (4.8) and (4.9), we get the followings:

$$\psi_0(P,Q)=\lim_{s\to 0}\psi_s(P,Q)=\lim_{s\to 0}\left[s(s-1)\right]^{-1}\left[\sum_{i=1}^n p_i^{\,s}q_i^{1-s}-1\right]=\sum_{i=1}^n q_i\log\left(\frac{q_i}{p_i}\right)=K(Q,P) \tag{4.17}$$

$$\Omega_0(Q,P)=\lim_{s\to 0}\Omega_s(Q,P)=\lim_{s\to 0}\left[s(s-1)\right]^{-1}\left[\sum_{i=1}^n q_i\left(\frac{p_i+q_i}{2q_i}\right)^s-1\right]$$

$$=\sum_{i=1}^n q_i\log\left(\frac{2q_i}{p_i+q_i}\right)=F(Q,P) \tag{4.18}$$

$$\tau_0(P,Q)=-\sum_{i=1}^n(p_i-q_i)\left(\frac{p_i+q_i}{q_i}\right)^{-1}=\sum_{i=1}^n(q_i-p_i)\left(\frac{q_i}{p_i+q_i}\right)$$

$$=\sum_{i=1}^n\frac{\left(q_i^2+p_iq_i\right)-2\,p_iq_i}{p_i+q_i}=\sum_{i=1}^n q_i-\sum_{i=1}^n\frac{2\,p_iq_i}{p_i+q_i}=1-H(P,Q)=\frac{1}{2}\Delta(P,Q) \tag{4.19}$$

$$A_{S_{\phi_2}}(\alpha,\beta)=-\frac{(\beta-\alpha)}{4}\left[\frac{1}{\beta}-\frac{1}{\alpha}\right]=\frac{1}{4}\frac{(\beta-\alpha)^2}{\alpha\,\beta} \tag{4.20}$$

Put (4.17), (4.18), (4.19) and (4.20) in (4.1), (4.2) and (4.3) at s=0, and get the results (4.14), (4.15) and (4.16).

Result 4.2 (at s=1).

Let $P,Q\in\Gamma_n, 0<\alpha\le 1\le\beta<\infty, \alpha\ne\beta$ and $s=1$, then we have:

$$\frac{6}{\beta^3}\left[K(P,Q)-G(Q,P)\right]\le IS(P,Q)\le\frac{6}{\alpha^3}\left[K(P,Q)-G(Q,P)\right] \tag{4.21}$$

$$\frac{6}{\beta^3}\left[F(Q,P)\right]\le ES(P,Q)\le\frac{6}{\alpha^3}\left[F(Q,P)\right] \tag{4.22}$$

$$\frac{6}{\beta^3}\left[\frac{1}{4}(\beta-\alpha)\log\left(\frac{\beta}{\alpha}\right)-G(Q,P)\right]\le A_{S_{\phi_1}}(\alpha,\beta)-J^*(P,Q)\le\frac{6}{\alpha^3}\left[\frac{1}{4}(\beta-\alpha)\log\left(\frac{\beta}{\alpha}\right)-G(Q,P)\right] \tag{4.23}$$

Proof: Put s=1 in (4.6), (4.7), (4.8) and (4.9), we get the followings:

$$\psi_1(P,Q)=\lim_{s\to 1}\psi_s(P,Q)=\lim_{s\to 1}\left[s(s-1)\right]^{-1}\left[\sum_{i=1}^n p_i^{\,s}q_i^{1-s}-1\right]=\sum_{i=1}^n p_i\log\left(\frac{p_i}{q_i}\right)=K(P,Q) \tag{4.24}$$

$$\Omega_1(Q,P)=\lim_{s\to 1}\Omega_s(Q,P)=\lim_{s\to 1}\left[s(s-1)\right]^{-1}\left[\sum_{i=1}^n q_i\left(\frac{p_i+q_i}{2q_i}\right)^s-1\right]$$

$$=\sum_{i=1}^n\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right)=G(Q,P) \tag{4.25}$$

$$\tau_1(P,Q)=\lim_{s\to 1}\tau_s(P,Q)=\lim_{s\to 1}\left[\frac{(s-1)^{-1}}{2^s}\right]\sum_{i=1}^{n}(p_i-q_i)\left(1+\frac{p_i}{q_i}\right)^{s-1}$$

$$\frac{1}{2}\sum_{i=1}^{n}(p_i-q_i)\log\left(\frac{p_i+q_i}{2q_i}\right)=\frac{1}{2}J_R(P,Q)=F(Q,P)+G(Q,P) \tag{4.26}$$

$$A_{S_{\phi_2}}(\alpha,\beta)=\lim_{s\to 1}\frac{(s-1)^{-1}(\beta-\alpha)}{4}\left[\beta^{s-1}-\alpha^{s-1}\right]=\frac{1}{4}(\beta-\alpha)\log\left(\frac{\beta}{\alpha}\right) \tag{4.27}$$

Put (4.24), (4.25), (4.26) and (4.27) in (4.1), (4.2) and (4.3) at s=1, and get the results (4.21), (4.22) and (4.23).

Result 4.3 (at s=2).

Let $P, Q\in\Gamma_n, 0<\alpha\le 1\le\beta<\infty, \alpha\ne\beta$ and $s=2$, then we have:

$$\frac{9}{4\beta^4}\left[\chi^2(P,Q)\right]\le IS(P,Q)\le\frac{9}{4\alpha^4}\left[\chi^2(P,Q)\right] \tag{4.28}$$

$$\frac{3}{4\beta^4}\left[\chi^2(P,Q)\right]\le ES(P,Q)\le\frac{3}{4\alpha^4}\left[\chi^2(P,Q)\right] \tag{4.29}$$

$$\frac{3}{2\beta^4}\left[(\beta-\alpha)^2-\frac{1}{2}\chi^2(P,Q)\right]\le A_{S_{\phi_1}}(\alpha,\beta)-J^*(P,Q)\le\frac{3}{2\alpha^4}\left[(\beta-\alpha)^2-\frac{1}{2}\chi^2(P,Q)\right] \tag{4.30}$$

Proof: Put s=2 in (4.6), (4.7), (4.8) and (4.9), we get the followings:

$$\psi_2(P,Q)=\frac{1}{2}\left[\sum_{i=1}^{n}\frac{p_i^2}{q_i}-1\right]=\frac{1}{2}\left[\sum_{i=1}^{n}\frac{p_i^2}{q_i}-2p_i+q_i\right]=\frac{1}{2}\left[\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i}\right]=\frac{1}{2}\chi^2(P,Q) \tag{4.31}$$

$$\Omega_2(Q,P)=\frac{1}{2}\left[\sum_{i=1}^{n}q_i\left(\frac{p_i+q_i}{2q_i}\right)^2-1\right]=\frac{1}{2}\left[\sum_{i=1}^{n}\frac{(p_i+q_i)^2}{4q_i}-p_i\right]$$
$$=\frac{1}{2}\left[\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{4q_i}\right]=\frac{1}{8}\chi^2(P,Q) \tag{4.32}$$

$$\tau_2(P,Q)=\frac{1}{4}\sum_{i=1}^{n}(p_i-q_i)\left(1+\frac{p_i}{q_i}\right)=\frac{1}{4}\left[\sum_{i=1}^{n}(p_i-q_i)+\sum_{i=1}^{n}(p_i-q_i)\frac{p_i}{q_i}\right]=\frac{1}{4}\left[\sum_{i=1}^{n}\frac{p_i^2}{q_i}-1\right]$$
$$=\frac{1}{4}\left[\sum_{i=1}^{n}\frac{p_i^2}{q_i}-2p_i+q_i\right]=\frac{1}{4}\left[\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i}\right]=\frac{1}{4}\chi^2(P,Q) \tag{4.33}$$

$$A_{S_{\phi_2}}(\alpha,\beta)=\frac{(\beta-\alpha)}{4}(\beta-\alpha)=\frac{1}{4}(\beta-\alpha)^2 \tag{4.34}$$

Put (4.31), (4.32), (4.33) and (4.34) in (4.1), (4.2) and (4.3) at s=2, and get the results (4.28), (4.29) and (4.30).

5. RELATION AMONG $IS(P,Q), J^*(P,Q)$ *and* $R_a(P,Q)$

Since, $\frac{x+y}{2}\ge\sqrt{x\,y}\ \forall\ x,\ y\ge 0$ (A.M $\ge$ G.M) (5.1) i.e. $$\left(x+y\right)^2 \geq 4xy \tag{5.2}$$

put $x = p_i$ and $y = q_i$, in (5.2), we get

$$\left(p_i+q_i\right)^2 \geq 4p_iq_i \tag{5.3}$$

Proposition 5: Let $(P,Q)\in\Gamma_n\times\Gamma_n$, then we have the following relations:

$$IS\left(P,Q\right) \leq \frac{1}{2}R_3\left(Q,P\right) - \frac{3}{4}R_2\left(Q,P\right) + \frac{1}{4}R_4\left(Q,P\right) \tag{5.4}$$

$$J^*\left(P,Q\right) \leq \frac{3}{4}R_2\left(Q,P\right) \tag{5.5}$$

$$J^*\left(P,Q\right) + IS\left(P,Q\right) \leq \frac{1}{2}R_3\left(Q,P\right) + \frac{1}{4}R_4\left(Q,P\right) \tag{5.6}$$

$$3\left[J_R\left(P,Q\right) + A\left(P,Q\right)\right] + 4\ IS\left(P,Q\right) \leq R_4\left(Q,P\right) + 2\ R_3\left(Q,P\right) \tag{5.7}$$

where, $IS\left(P,Q\right)$ and $J^*\left(P,Q\right)$ are given by (3.3) and (3.5) respectively.

Proof: multiply (5.3) by $\dfrac{p_i^2}{q_i^3\left(3p_i+q_i\right)\left(q_i-p_i\right)}$ and sum over all $i = 1,2,3...,n$, we get

$$\sum_{i=1}^{n}\frac{p_i^2\left(p_i+q_i\right)^2}{q_i^3\left(3p_i+q_i\right)\left(q_i-p_i\right)} \geq \sum_{i=1}^{n}\frac{4q_ip_i^3}{q_i^3\left(3p_i+q_i\right)\left(q_i-p_i\right)}$$

or $$\sum_{i=1}^{n}\frac{q_i^3\left(3p_i+q_i\right)\left(q_i-p_i\right)}{p_i^2\left(p_i+q_i\right)^2} \leq \sum_{i=1}^{n}\frac{q_i^3\left(3p_i+q_i\right)\left(q_i-p_i\right)}{4p_i^3q_i}$$

or $$IS\left(P,Q\right) \leq \frac{1}{2}\sum_{i=1}^{n}\frac{q_i^3}{p_i^2} - \frac{3}{4}\sum_{i=1}^{n}\frac{q_i^2}{p_i} + \frac{1}{4}\sum_{i=1}^{n}\frac{q_i^4}{p_i^3}$$

or $$IS\left(P,Q\right) \leq \frac{1}{2}R_3\left(Q,P\right) - \frac{3}{4}R_2\left(Q,P\right) + \frac{1}{4}R_4\left(Q,P\right).$$ **Hence the relation (5.4)**

or $$\frac{3}{4}R_2\left(Q,P\right) \leq \frac{1}{2}R_3\left(Q,P\right) + \frac{1}{4}R_4\left(Q,P\right) - IS\left(P,Q\right) \tag{5.8}$$

Again, multiply (5.3) by $\dfrac{1}{q_i\left(3q_i+p_i\right)\left(q_i-p_i\right)}$ and sum over all $i = 1,2,3...,n$, we get

$$\sum_{i=1}^{n}\frac{\left(p_i+q_i\right)^2}{q_i\left(3q_i+p_i\right)\left(q_i-p_i\right)} \geq \sum_{i=1}^{n}\frac{4p_iq_i}{q_i\left(3q_i+p_i\right)\left(q_i-p_i\right)}$$

or $$\sum_{i=1}^{n}\frac{q_i\left(3q_i+p_i\right)\left(q_i-p_i\right)}{\left(p_i+q_i\right)^2} \leq \sum_{i=1}^{n}\frac{q_i\left(3q_i+p_i\right)\left(q_i-p_i\right)}{4p_iq_i}$$

or $$J^*\left(P,Q\right) \leq \frac{3}{4}\sum_{i=1}^{n}\frac{q_i^2}{p_i} - \frac{1}{2}\sum_{i=1}^{n}q_i - \frac{1}{4}\sum_{i=1}^{n}p_i \qquad \left\{\because \sum_{i=1}^{n}p_i = \sum_{i=1}^{n}q_i = 1\right\}$$

or $$J^*(P,Q) + \frac{3}{4} \le \frac{3}{4} R_2(Q,P) \tag{5.9}$$

From (5.9), we can easily see that

$J^*(P,Q) \le \frac{3}{4} R_2(Q,P)$. **Hence the relation (5.5)**

Now, by taking (5.4) and (5.8) together, we get

$$J^*(P,Q) \le \frac{3}{4} R_2(Q,P) \le \frac{1}{2} R_3(Q,P) + \frac{1}{4} R_4(Q,P) - IS(P,Q) \tag{5.10}$$

By taking first and third part of the relation (5.10), we get the inequalities (5.6).

Now $$J_R(P,Q) + A(P,Q) \le R_2(Q,P) \text{ [4]} \tag{5.11}$$

By taking (5.8) and (5.11) together, we get

$$\frac{3}{4} J_R(P,Q) + \frac{3}{4} A(P,Q) \le \frac{3}{4} R_2(Q,P) \le \frac{1}{2} R_3(Q,P) + \frac{1}{4} R_4(Q,P) - IS(P,Q) \tag{5.12}$$

By taking first and third part of the relation (5.12), we get

$$\frac{3}{4} J_R(P,Q) + \frac{3}{4} A(P,Q) + IS(P,Q) \le \frac{1}{2} R_3(Q,P) + \frac{1}{4} R_4(Q,P)$$

$3\left[J_R(P,Q) + A(P,Q)\right] + 4\, IS(P,Q) \le R_4(Q,P) + 2\, R_3(Q,P)$. **Hence the relation (5.7)**

where, $A(P,Q)$ = Arithmetic mean divergence = $\sum_{i=1}^{n} \frac{p_i + q_i}{2} = 1$.

Note 1: $K(P,Q), \chi^2(P,Q), F(P,Q), G(P,Q), \Delta(P,Q), J_R(P,Q), a\ n\ d R_a(P,Q)$ have been taken from (1.6), (1.7), (1.8), (1.9), (1.11), (1.12) and (1.8) respectively in result 4.1, 4.2 and 4.3 and proposition 5.

Note 2: In result 4.1 and 4.2, limits have been evaluated by using L' Hospital Rule.

Note 3: For $s \le -2$, the function $f_s(t)$ is increasing (discussed in proof of proposition 4), so we get the following inequalities:

$$\frac{6}{\alpha^{s+2}}\left[\psi_s(P,Q) - \Omega_s(Q,P)\right] \le IS(P,Q) \le \frac{6}{\beta^{s+2}}\left[\psi_s(P,Q) - \Omega_s(Q,P)\right] \tag{A}$$

$$\frac{6}{\alpha^{s+2}}\left[\tau_s(P,Q) - \Omega_s(Q,P)\right] \le ES(P,Q) \le \frac{6}{\beta^{s+2}}\left[\tau_s(P,Q) - \Omega_s(Q,P)\right] \tag{B}$$

$$\frac{6}{\alpha^{s+2}}\left[A_{S_{\phi_2}}(\alpha,\beta) - \Omega_s(Q,P)\right] \le A_{S_{\phi_1}}(\alpha,\beta) - J^*(P,Q) \le \frac{6}{\beta^{s+2}}\left[A_{S_{\phi_2}}(\alpha,\beta) - \Omega_s(Q,P)\right] \tag{C}$$

{We are omitting the proof and special cases of these inequalities at different values of "s".}

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