WEIGHTED SHARING OF A SMALL FUNCTION
OF A MEROMORPHIC FUNCTION AND ITS DERIVATIVES

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ABSTRACT

In this paper, we study the relationship between a meromorphic function and its k-th order derivative which share a small function with weight (multiplicities) l (a positive integer) ignoring multiplicity.

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INTRODUCTION AND RESULTS

In this paper we shall use the standard notations of Nevanlinna theory such as \( T(r, f), N(r, f), m(r, f) \) and so on (see[3]), where \( f \) is a meromorphic function defined on the whole complex plane. The quantity \( S(r, f) \) is defined by \( S(r, f) = \theta(T(r, f)) \) as \( r \to \infty \) possibly outside a set of finite linear measure. A meromorphic function \( a(z) \) is called a small function with respect to \( f \) provided \( T(r, a) = S(r, f) \) holds.

Suppose that \( f \) and \( g \) are two non-constant meromorphic functions, \( a \) is a small function with respect to \( f \) and \( g \) and \( k \) be a positive integer. Now \( f \) and \( g \) share 'a' ignoring multiplicities or IM (counting multiplicities or CM) if \( f - a \) and \( g - a \) have the same zeros ignoring (counting) multiplicities. We denote by \( N_k\left(r, \frac{1}{f - a}\right) \), the counting function for zeros of \( f - a \) with multiplicity \( \leq k \) (counting multiplicity), and by \( \overline{N}_k\left(r, \frac{1}{f - a}\right) \), the corresponding one for which the multiplicity is not counted. Similarly by \( N_{\ell k}\left(r, \frac{1}{f - a}\right) \), we mean the counting function for zeros of \( f - a \) with multiplicity at least \( k \) (counting multiplicity) and by \( \overline{N}_{\ell k}\left(r, \frac{1}{f - a}\right) \), we mean the corresponding one for which the multiplicity is not counted.

We denote \( N_k\left(r, \frac{1}{f - a}\right) = \overline{N}\left(r, \frac{1}{f - a}\right) + N_{\ell 2}\left(r, \frac{1}{f - a}\right) + \ldots + \overline{N}_{\ell k}\left(r, \frac{1}{f - a}\right) \)

where \( \overline{N}_i\left(r, \frac{1}{f - a}\right) = \overline{N}\left(r, \frac{1}{f - a}\right) \) and \( \overline{\delta}_p(a, f) = 1 - \limsup_{r \to \infty} \frac{N_p\left(r, \frac{1}{f - a}\right)}{T(r, f)} \), where \( p \) is a positive integer; then clearly \( 0 \leq \overline{\delta}(a, f) \leq \overline{\delta}_i(a, f) \leq \Theta(a, f) \leq 1 \),

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where \( \delta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{T(r, f)})}{T(r, f)} \) and \( \Theta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{T(r, f)})}{T(r, f)} \).

In [6], Q.C. Zhang proved the following theorem about a meromorphic function and its k-th order derivative.

**Theorem: A** Let \( f \) be a non-constant meromorphic function and let \( k \) be a positive integer. Suppose that \( f \) and \( f^{(k)} \) share 1 CM and \( 2N(r, f) + N\left(r, \frac{1}{f^{(k)}}\right) \leq (\lambda + o(1))T(r, f^{(k)}) \) for \( r \in I \), where \( I \) is a set of infinite linear measure and \( \lambda \) satisfies \( 0 < \lambda < 1 \) then \( \frac{f^{(k)} - 1}{f - 1} \equiv c \) for some constant \( c \in C - \{0\} \).

In 2003, Kit-wing [5] discussed the problem of a meromorphic function and its \( k \)-th derivative sharing one small function and proved the following result.

**Theorem: B** Let \( k \geq 1 \). Let \( f \) be a non-constant non-entire meromorphic function, \( a \in \sigma(f) \) and \( a(\not= 0, \infty) \) and \( f \) do not have any common pole. If \( f, f^{(k)} \) share a CM and \( 4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k \), then \( f = f^{(k)} \).

Two years latter, in 2005, Q.C. Zhang [2] proved the following theorem.

**Theorem: C** Let \( f \) be a non-constant meromorphic function and \( k(\geq 1), l(\geq 0) \) be integers. Also let \( a \equiv a(z) \) (not \( \equiv 0, \infty \)) be a meromorphic function such that \( T(r, a) = S(r, f) \). Suppose that \( f - a \) and \( f^{(k)} - a \) share \( (0, I) \). If \( l \geq 2 \) and \( (3 + k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4 \) or, if \( l = 1 \) and \( (4 + k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6 \), or, if \( l = 0 \), i.e., \( f - a \) and \( f^{(k)} - a \) share the value 0 IM and \( (6 + k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10 \) then \( f = f^{(k)} \).


**Theorem: D** Let \( k(\geq 1), n(\geq 1) \) be integers and \( f \) be a non-constant meromorphic function. Also let \( a(z) \) (not \( \equiv 0, \infty \)) be a small function with respect to \( f \). If \( f \) and \( (f^n)^{(k)} \) share a(z) IM and

\[
4N(r, f) + 2N\left(r, \frac{1}{f^n}\right) + \frac{1}{2}N\left(r, \frac{1}{(f^n)^{(k)}}\right) \leq (\lambda + o(1))T\left(r, \left(f^n\right)^{(k)}\right),
\]

or, if \( f \) and \( (f^n)^{(k)} \) share a(z) CM and

\[
\left(4 + k\right)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > 2k + 10,
\]

where \( r \in I \) and \( I \) is a set of infinite linear measure, then \( \frac{f^n - a}{(f^n)^{(k)} - a} = c \), for some constant \( c \in C - \{0\} \).

**Theorem: E** Let \( k(\geq 1), n(\geq 1) \) be integers and let \( f \) be a non-constant meromorphic function. Also let \( a(z) \) (not \( \equiv 0, \infty \)) be a small function with respect to \( f \). If \( f \) and \( (f^n)^{(k)} \) share a(z) IM and \( (k + 3)\Theta(\infty, f) + \delta_{2+k}(0, f) > k + 4 \) then \( f \equiv (f^n)^{(k)} \).

In this paper, we will prove the following two theorems which will include the behavior of a meromorphic function and its \( k \) th derivative sharing a small function with multiplicity not greater than \( l \), a positive integer.
Theorem: 1 Let $k$, $m$ and $n$ are three positive integers with $m \leq n$ and let $f$ be a non-constant meromorphic function. Also let $a(z)$ (not $\equiv 0, \infty$) be a small function with respect to $f$. If $\overline{E}(a, f^m(z)) = \overline{E}(a, f^n(z))$, where $l$ is a positive integer and

$$\overline{N}(r, f) + 2N_z \left( r, \frac{1}{f} \right) + 2N_z \left( r, \frac{1}{(f^n)^{(k)}} \right) + N \left( r, \frac{1}{(f^n)^{(k)}} \right) \leq (\lambda + 0(1))\overline{T}(r, (f^n)^{(k)}), \text{ for } 0 < \lambda < 1,$$

where $r \in I$ and $I$ is a set of infinite linear measure, then $\frac{(f^n)^{(k)} - a}{f^m - a} = c$ for some constant $a \in \mathbb{C} - \{0\}$ where $\mathbb{C}$ is the set of complex numbers.

Theorem: 2 Let $f$ be a non-constant meromorphic function and let $k$ and $n$ be two positive integers. If $\overline{E}(a, f) \equiv \overline{E}(a, (f^n)^{(k)})$, where $l$ is a positive integer and $a(z)$ (not $\equiv 0, \infty$) be a small function of $f$ and $(2k + 6)\Theta(\infty, f) + \Theta(0, f) + 2\delta(0, f) + 2\delta_{k+2}(0, f) > 2k + 10$ then $f = (f^n)^{(k)}$.

2. LEMMAS

Here we mention some existing lemmas of the literature which will be frequently used to prove the aforementioned theorems.

Lemma 2.1 (see[7]): Let $f$ be a non-constant meromorphic function and $k,p$ be two positive integers. Then

$$N_p \left( r, \frac{1}{(f^n)^{(k)}} \right) \leq N_{p+k} \left( r, \frac{1}{f} \right) + kN(r, f) + S(r, f).$$

Lemma: 2.2 (see[4]) Let $f$ be a non-constant meromorphic function and let $n$ be a positive integer.

$$P(f) = a_{n-1}f^{n-1} + \ldots + a_{1}f$$

where $a_{i}$ is a meromorphic function such that $T(r, a_{i}) = S(r, f)(i = 1,2,\ldots, n)$. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

3. PROOF OF THE THEOREMS

Proof of Theorem: 1 Let $F = \frac{f^m}{a}$ and $G = \frac{(f^n)^{(k)}}{a}$. Therefore, $F - 1 - \frac{f^m - a}{a}$ and $G - 1 = \frac{(f^n)^{(k)} - c}{a}$.

Now, $\overline{E}(a, f^m) = \overline{E}(a, (f^n)^{(k)})$ except the zeros and poles of $a(z)$. Define,

$$H = \left( \frac{F^n}{F} - \frac{2F^n}{F - 1} \right) \left( \frac{G^n}{G} - \frac{2G^n}{G - 1} \right).$$

We now consider two cases:

Case: I Suppose $H \not\equiv 0$. Then $m(r, H) = S(r, f)$. Now if $z_0$ is a common simple zero of $F$-1 and $G$-1 (except the zeros and poles of $a(z)$), then after simple calculation, we get $H(z_0) = 0$.

So, $\overline{N} \left( r, \frac{1}{G - 1} \right) \leq N \left( r, \frac{1}{H} \right) + S(r, f) \leq T(r, H) + S(r, f) \leq N(r, H) + S(r, f)$

Again by analysis, we can deduce that,

$$N(r, H) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + \overline{N}_L \left( r, \frac{1}{F - 1} \right) + \overline{N}_L \left( r, \frac{1}{G - 1} \right) + \overline{N}_* \left( r, \frac{1}{F - 1} \right) + \overline{N}_* \left( r, \frac{1}{G - 1} \right) + S(r, f).$$

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Also, \( N\left( r, \frac{1}{G-1} \right) = \mathcal{N}^{(1)}_{E} + \mathcal{N}^{(2)}_{E} \left( r, \frac{1}{G-1} \right) + N_L \left( r, \frac{1}{G-1} \right) + N_L \left( r, \frac{1}{F-1} \right) + \mathcal{N}_{*} \left( r, \frac{1}{G-1} \right) + S(r, f) \).

Therefore,
\[
\mathcal{N}\left( r, \frac{1}{G-1} \right) \leq \mathcal{N}(r, f) + \mathcal{N}_E \left( r, \frac{1}{F} \right) + \mathcal{N}_E \left( r, \frac{1}{G} \right) + 2N_L \left( r, \frac{1}{F-1} \right) + 2N_L \left( r, \frac{1}{G-1} \right) \\
+ 2N_L \left( r, \frac{1}{G-1} \right) + \mathcal{N}^{(2)}_E \left( r, \frac{1}{G-1} \right) + 2N_0 \left( r, \frac{1}{G-1} \right) + \mathcal{N}_* \left( r, \frac{1}{F-1} \right) + \mathcal{N}_0 \left( r, \frac{1}{G} \right) + S(r, f) \tag{1}
\]

Since, \( \mathcal{E}_1(1, F) = \mathcal{E}_1(1, G) \),

Therefore,
\[
2N_L \left( r, \frac{1}{G-1} \right) + 2N_0 \left( r, \frac{1}{G} \right) + \mathcal{N}_0 \left( r, \frac{1}{G-1} \right) \leq 2 \mathcal{N}_0 \left( r, \frac{1}{G-1} \right).
\]

From (1), we have,
\[
\mathcal{N}\left( r, \frac{1}{G-1} \right) \leq \mathcal{N}(r, f) + \mathcal{N}_E \left( r, \frac{1}{F} \right) + \mathcal{N}_0 \left( r, \frac{1}{G} \right) + 2N_L \left( r, \frac{1}{F-1} \right) + 2N_L \left( r, \frac{1}{G-1} \right) \tag{2}
\]

We also have,
\[
N_2 \left( r, \frac{1}{F} \right) + 2N_L \left( r, \frac{1}{F-1} \right) + \mathcal{N}_0 \left( r, \frac{1}{G} \right) + \mathcal{N}_0 \left( r, \frac{1}{G} \right) \leq 2 \mathcal{N}_0 \left( r, \frac{1}{G} \right) \tag{3}
\]

Now by the second fundamental theorem we get,
\[
T(r, G) \leq \mathcal{N}(r, G) + \mathcal{N}_0 \left( r, \frac{1}{G} \right) - N_0 \left( r, \frac{1}{G} \right) + S(r, G) \tag{4}
\]

From (4) using (2) and (3) we get,
\[
T(r, G) \leq 2 \mathcal{N}(r, f) + 2N_2 \left( r, \frac{1}{F} \right) + 2N_2 \left( r, \frac{1}{G} \right) + \mathcal{N}_0 \left( r, \frac{1}{G} \right) + S(r, f) \tag{5}
\]

By lemma(2.1) we have,
\[
T(r, f^n)^{(k)} \leq 6 \mathcal{N}(r, f) + 2N_2 \left( r, \frac{1}{f^n} \right) + 2N_2 \left( r, \frac{1}{(f^n)^k} \right) + \mathcal{N}_0 \left( r, \frac{1}{(f^n)^k} \right) + S(r, f)
\]

which contradicts the given conditions of the theorem.

**Case: II** Suppose \( H(z) \equiv 0 \) i.e., \( \frac{F^n}{F^n} - \frac{2F^n}{F-1} = \frac{G^n}{G} - \frac{2G^n}{G-1} \). Integrating we get,
\[
\log F' - 2 \log (F-1) = \log G' - \log (G-1) + \log A. \quad \text{where} \ A \ is \ a \ constant \neq 0.
\]

That is, \( \log \frac{F'}{(F-1)^2} = \log \frac{AG'}{(G-1)^2} \).
Again integrating we get,
\[ \frac{1}{F-1} = \frac{A}{G-1} + B \]  
(6)

Now if \( z_0 \) is a pole of \( f \) with multiplicity \( p \) which is not the poles and the zeros of \( a(z) \), then \( z_0 \) is the pole of \( F \) with multiplicity \( mp \) and the pole of \( G \) with multiplicity \( np+k(\neq mp) \). This contradicts (6). This implies \( f \) has no pole, that is \( f \) is an entire function.

So, \( \overline{N}(r,F) = S(r,f) \) and \( \overline{N}(r,G) = S(r,f) \). Now we prove that \( B = 0 \).

We first assume that \( B \neq 0 \), then
\[ \frac{1}{F-1} = \frac{B \left( G-1+\frac{A}{B} \right)}{G-1} \]

Therefore, \( \overline{N} \left( r, \frac{1}{G-1+\frac{A}{B}} \right) = \overline{N}(r,F) = S(r,f) \)

Now we assume \( \frac{A}{B} \neq 1 \).

By the second fundamental theorem,
\[ T(r,G) \leq \overline{N}(r,G) + \overline{N} \left( r, \frac{1}{G} \right) + \overline{N} \left( r, \frac{1}{G-1+\frac{A}{B}} \right) + S(r,G) \]
\[ \leq \overline{N} \left( r, \frac{1}{G} \right) + S(r,f) \]
\[ \leq T(r,G) + S(r,f) \]

Hence \( T(r,G) = \overline{N} \left( r, \frac{1}{G} \right) + S(r,f) \) i.e., \( T(r,(f^n)^{(k)}) = \overline{N} \left( r, \frac{1}{(f^n)^{(k)}} \right) + S(r,f) \)

This contradicts the given condition of the theorem.

Next, we assume \( \frac{A}{B} = 1 \). Then, \( (AF - A - 1) G = -1 \).

So,
\[ \frac{a^2}{f^n (Af^m - Aa - A)} = \frac{(f^n)^{(4)}}{f^n} \]

Now by lemma (2.1) and (2.2), we get,
\[ (n+m)T(r,f) = T \left( r, \frac{(f^n)^{(4)}}{f^n} \right) + S(r,f) \]
\[ \leq n \overline{N} \left( r, \frac{1}{f^n} \right) + k \overline{N}(r,f) + S(r,f) \]
\[ \leq nT(r,f) + S(r,f) \]
i.e., \( T(r,f) = S(r,f) \). This is not true.

Hence our assumption is not true and therefore \( B = 0 \). So, \( \frac{G-1}{F-1} = A \)

This proves the theorem.
Proof of the theorem: 2

Let \( F = \frac{f(z)}{a(z)} \) and \( G = \frac{(f^n)^{(k)}}{a(z)} \). So, \( \bar{E}_{ij}(a, f) = \bar{E}_{ij}(a, (f^n)^{(k)}) \) implies, \( \bar{E}_{ij}(1, F) = \bar{E}_{ij}(1, G) \), except the zeros and poles of \( a(z) \).

We define, \( H = \left( \frac{F''}{F' - 2F''} \right) - \left( \frac{G''}{G' - 2G''} \right) \).

Now we consider two cases:

**Case: I** Suppose \( H \neq 0 \).

Then (5) of the proof in theorem 1 still holds. Writing (5) for the function \( F \), we get,

\[
T(r, F) \leq 2 \bar{N}(r, F) + 2 \bar{N}\left( r, \frac{1}{G} \right) + 2 \bar{N}\left( r, \frac{1}{F'} \right) + \bar{N}\left( r, \frac{1}{G} \right) + S(r, f)
\]

i.e. \( T(r, f) \leq 2 \bar{N}(r, f) + 2 \bar{N}_2\left( r, \frac{1}{(f^n)^{(k)}} \right) + 2 \bar{N}(r, f) + \bar{N}_2\left( r, \frac{1}{f} \right) + 2 \bar{N}(r, f) + \bar{N}\left( r, \frac{1}{f} \right) + S(r, f) \)

\[
\leq (2k + 6) \bar{N}(r, f) + 2N_{k+2}\left( r, \frac{1}{f} \right) + 2 \bar{N}_2\left( r, \frac{1}{f} \right) + \bar{N}\left( r, \frac{1}{f} \right) + S(r, f)
\]

i.e., \( (2k + 6) \Theta(a, f) + \Theta(0, f) + 2 \delta_2(0, f) + 2 \delta_{k+2}(0, f) \leq 2k + 10 \)

This contradicts the given condition of the theorem.

**Case: II** Suppose \( H \equiv 0 \).

So \( \frac{1}{F - 1} = \frac{A}{G - 1} + B \), where \( A \neq 0 \), \( B \) are constant. By the same argument of the proof of theorem 1, we get,

\( \bar{N}(r, F) = S(r, f) \) and \( \bar{N}(r, G) = S(r, f) \).

So, \( \Theta(\infty, f) = 1 \).

Assume that, \( B \neq 0 \), then

\[
\frac{B\left( F - 1 - \frac{1}{B} \right)}{F - 1} = - \frac{A}{G - 1}
\]

So, \( \bar{N}\left( r, \frac{1}{F - 1 + \frac{1}{B}} \right) = \bar{N}(r, G) = S(r, f) \).

If \( B \neq -1 \), then by the second fundamental theorem for \( F \), we have

\[
T(r, F) \leq \bar{N}(r, F) + \bar{N}\left( r, \frac{1}{F} \right) + \bar{N}\left( r, \frac{1}{F - 1 + \frac{A}{B}} \right) + S(r, f)
\]

\[
\leq \bar{N}\left( r, \frac{1}{F} \right) + S(r, f)
\]

\[
\leq T(r, F) + S(r, f)
\]

So \( T(r, F) \leq \bar{N}\left( r, \frac{1}{F} \right) + S(r, f) \) i.e., \( T(r, f) \leq \bar{N}\left( r, \frac{1}{f} \right) \). Hence, \( \Theta(0, f) = 0 \).
Putting $\Theta(x, f) = 1, \Theta(0, f) = 0$ and also $\delta(0, f) \leq \Theta(0, f) = 0$ in the given condition of the theorem we have, $\delta_{k+2}(0, f) > 2$, which is not true. Hence $B = -1$.

So $\overline{N} \left( r, \frac{1}{F} \right) = S(r, f)$, i.e., $\overline{N} \left( r, \frac{1}{f} \right) = S(r, f)$. Therefore, \[
\frac{F}{F-1} = \frac{A}{G-1},
\]
i.e., $F(G-1-A) = -A$ that is $F = \frac{A}{-G + (1 + A)}$.

So, $f = \frac{A}{-(f^n)^{(k)} + (1 + A)}$. Therefore, $\overline{N} \left( r, \frac{1}{(f^n)^{(k)} + (1 + A)} \right) = \overline{N}(r, f) = S(r, f)$.

Hence $T(r, f) = T(r, (f^n)^{(k)}) = S(r, f)$, which is also not true. Thus $B = 0$.

So $\frac{1}{F-1} = \frac{A}{G-1}$, i.e., $G - 1 = A(F - 1)$.

If $A \neq 1$ then $G = A \left( F - 1 + \frac{1}{A} \right)$. So, $\overline{N} \left( r, \frac{1}{G} \right) = \overline{N} \left( r, \frac{1}{F - 1 + \frac{1}{A}} \right)$.

By the second fundamental theorem, we have,

\[
T(r, F) \leq \overline{N}(r, F) + \overline{N} \left( r, \frac{1}{F} \right) + \overline{N} \left( r, \frac{1}{F - 1 + \frac{1}{A}} \right) + S(r, f).
\]
i.e.,

\[
T(r, f) \leq \overline{N} \left( r, \frac{1}{F} \right) + \overline{N} \left( r, \frac{1}{f} \right) + S(r, f).
\]

\[
= \overline{N} \left( r, \frac{1}{f} \right) + \overline{N} \left( r, \frac{1}{(f^n)^{(k)}} \right) + S(r, f).
\]

\[
\leq \overline{N} \left( r, \frac{1}{f} \right) + N_{k+1} \left( r, \frac{1}{f} \right) + k \overline{N}(r, f) + S(r, f)
\]

\[
= \overline{N} \left( r, \frac{1}{f} \right) + N_{k+1} \left( r, \frac{1}{f} \right) + S(r, f)
\]

So,

\[
\Theta(0, f) + \delta_{k+1}(0, f) \leq 1 \tag{7}
\]

Now by the given condition of the theorem and by (7) we have, $\Theta(0, f) > 2$. This is not possible.

So, $A = 1$ and hence $F = G$ i.e., $f = (f^n)^{(k)}$.

This proves the theorem.

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